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Quadratic reflected BSDEs with unbounded obstacles

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Abstract

In this paper, we analyze a real-valued reflected backward stochastic differential equation (RBSDE) with an unbounded obstacle and an unbounded terminal condition when its generator f has quadratic growth in the z-variable. In particular, we obtain existence, uniqueness, and stability results, and consider the optimal stopping for quadratic g-evaluations. As an application of our results we analyze the obstacle problem for semi-linear parabolic PDEs in which the non-linearity appears as the square of the gradient. Finally, we prove a comparison theorem for these obstacle problems when the generator is concave in the z-variable.
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1. Introduction

We consider a reflected backward stochastic differential equation (RBSDE) with generator f, terminal condition ξ and obstacle L

$$L_t \le Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \in [0, T], \tag{1.1}$$

where the solution (Y, Z, K) satisfies the so-called *flat-off* condition:

$$\int_0^T (Y_t - L_t) dK_t = 0, \tag{1.2}$$

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and K is an increasing process. We will consider the case when f is allowed to have quadratic growth in the z-variable. Moreover, we will allow L and ξ to be unbounded.

The theory of RBSDEs is closely related to the theory of optimal stopping in that the snell-envelope can be represented as a solution of an RBSDE. These equations were first introduced by El Karoui et al. [10]. The authors provided the existence and uniqueness of an adapted solution for a real-valued RBSDE with square-integrable terminal condition under the Lipschitz hypothesis on the generator. There has been a few developments after this seminal result. Some generalizations were obtained for backward stochastic differential equations (BSDEs) without an obstacle and later they were generalized to RBSDEs:

(1) Lepeltier and San Martín [16] showed the existence of a maximal and a minimal solution for real-valued BSDEs, with square-integrable terminal condition when the generator f is only continuous and has linear growth in variables y and z. Then [20] adapted this result to the case of RBSDEs.
(2) Kobylanski [14] established the existence, comparison, and stability results for real-valued quadratic BSDEs (when f is allowed to have quadratic growth in the z-variable) with bounded terminal condition. In the spirit of [22], the author gave a link between the solutions of BSDEs based on a diffusion and viscosity solutions of the corresponding semi-linear parabolic PDEs. Lepeltier and San Martín [17] extended the existence result of quadratic BSDEs with bounded terminal condition to the case that the generator f can have a superlinear growth in the y-variable. Kobylanski et al. [15] made a counterpart study for RBSDEs with bounded terminal condition and bounded obstacle when the generator f has superlinear growth in y and quadratic growth in z.
(3) With help of a localization procedure and a priori bounds, Briand and Hu [5] showed that the boundedness assumption on the terminal condition is not necessary for the existence of an adapted solution to a real-valued quadratic BSDE: One only needs to require the terminal condition has exponential moment of certain order. Correspondingly, Lepeltier and Xu [18] derived the existence result for quadratic RBSDEs with such an unbounded terminal condition, but still with a bounded obstacle.

Recently, [6], under the assumption that the generator f is additionally concave in the z-variable, used a so-called "θ-difference" method to obtain comparison (thus uniqueness) and stability results for quadratic BSDEs with solutions having every exponential moment. Moreover, [9] proved that uniqueness holds among solutions having a given exponential moment by using a verification theorem that relies on the Fenchel–Legendre dual of the generator. With these results they also showed that the solutions of BSDEs are viscosity solutions of PDEs which are quadratic in the gradient. On the other hand, [8] showed that these PDEs have unique solutions.

In the current paper, we extend the results of [6,9,8] to RBSDEs. Alternatively, our results can be seen as an extension of [15,18] to the unbounded obstacles. We start by establishing two a priori estimates which will serve as our basic tools; see Section 2. The first one shows that any bounded Y has an upper bound in term of the terminal condition ξ and the obstacle L. The second estimate is on the $\mathbb{L}^p$ norms of Z and K. With the help of these two estimates, we can establish a monotone stability result (see Theorem 3.1) in the spirit of [14]. Then the existence follows as a direct consequence; see Theorem 3.2.

When the generator f is additionally concave in the z-variable, we prove a uniqueness result for RBSDEs using an argument that involves the Fenchel–Legendre dual of the generator, see Theorem 4.1. As opposed to [9] (or [1]), we are not relying on a verification argument but directly compare two solutions. Since it only requires a given exponential moment on solutions, this uniqueness result is more general than the one that would be implied by the above comparison theorem. We develop an alternative representation of the unique solution in Section 5, where we improve the results of Theorem 5.3 of [3] on optimal stopping for quadratic g-evaluations. Moreover, the concavity assumption on generator f in the z-variable as well as the aforementioned θ-difference method are used in deducing the stability result (see Theorem 6.1), which is crucial for the continuity property of the solutions of forward backward stochastic differential equations with respect to their initial conditions; see Proposition 7.1. This result together with the stability result gives a new proof of the flow property; see Proposition 7.2. A Picard-iteration procedure was introduced to show this property for BSDEs with Lipschitz generators, see e.g., Theorem 4.1 of [11]. However, it is not appropriate to apply such a Picard-iteration procedure to derive the flow property for quadratic RBSDEs.

Thanks to the flow property, the solution of the RBSDE is a viscosity solution of an associated obstacle problem for a semi-linear parabolic PDE, in which the non-linearity appears as the square of the gradient; see Theorem 7.1. It is worth pointing out that [9] shows the existence of a viscosity solution to a similar PDE (with a quadratic gradient term) without obstacle by approximating the generator f from below by a sequence of Lipschitz generators under a strong assumption that f^- has a linear growth in variables y and z. However, such a strong assumption is not necessary if we directly use the flow property to prove Theorem 7.1. Finally, we prove that in fact this obstacle problem has a unique solution, which is a direct consequence of Theorem 7.2, a comparison principle between a viscosity subsolution and a viscosity supersolution. Although inspired by Theorem 3.1 of [8], we prove Theorem 7.2 in a quite different way because there are two gaps in the proof of Theorem 3.1 of [8], see Remark 9.1 and Appendix A.3 of [2].

1.1. Notation and preliminaries

Throughout this paper we let B be a d-dimensional standard Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and consider the augmented filtration generated by it: $\mathbf{F} = \Big\{\mathcal{F}_t \triangleq \sigma\Big(\sigma\big(B_s; s \in [0,t]\big) \cup \mathcal{N}\Big)\Big\}_{t\ge 0}$, where $\mathcal{N}$ is the collection of all P-null sets in $\mathcal{F}$. We fix a finite time horizon $T > 0$ and let $\mathcal{S}_{0,T}$ denote the set of all $\mathbf{F}$-stopping times ν such that $0 \le \nu \le T$, P-a.s. For any $\nu \in \mathcal{S}_{0,T}$, we define $\mathcal{S}_{\nu,T} \triangleq \{\tau \in \mathcal{S}_{0,T} \mid \nu \le \tau \le T, P\text{-a.s.}\}$. Moreover, we will use the convention $\inf \emptyset \triangleq \infty$.

The following spaces of functions will be used in the sequel:

(1) Let $\mathbb{C}[0,T]$ denote the set of all $\mathbb{R}$-valued continuous functions on $[0,T]$, and let $\mathbb{K}[0,T]$ collect all increasing functions in $\mathbb{C}[0,T]$. For any $\{\ell_t\}_{t\in[0,T]} \in \mathbb{C}[0,T]$, we define $\ell_*^\pm \triangleq \sup_{t\in[0,T]}(\ell_t)^\pm$.

(2) For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, let $\mathbb{L}^0(\mathcal{G})$ be the space of all $\mathbb{R}$-valued, $\mathcal{G}$-measurable random variables and let
- $\mathbb{L}^p(\mathcal{G}) \triangleq \Big\{\xi \in \mathbb{L}^0(\mathcal{G}) : \|\xi\|_{\mathbb{L}^p(\mathcal{G})} \triangleq \Big\{E\big[|\xi|^p\big]\Big\}^{\frac{1}{p}} < \infty\Big\}$ for all $p \in [1,\infty)$;
- $\mathbb{L}^\infty(\mathcal{G}) \triangleq \Big\{\xi \in \mathbb{L}^0(\mathcal{G}) : \|\xi\|_{\mathbb{L}^\infty(\mathcal{G})} \triangleq \operatorname{esssup}_{\omega\in\Omega}|\xi(\omega)| < \infty\Big\}$;
- $\mathbb{L}^e(\mathcal{G}) \triangleq \big\{\xi \in \mathbb{L}^0(\mathcal{G}) : E\big[e^{p|\xi|}\big] < \infty,\ \forall\, p \in (1,\infty)\big\}$.

(3) Let $\mathbb{B}$ be a generic Banach space with norm $|\cdot|_{\mathbb{B}}$. For any $p, q \in [1,\infty)$, we define three Banach spaces:

- $\mathbb{L}^{p,q}_{\mathbf{F}}([0,T];\mathbb{B})$ denotes the space of all $\mathbb{B}$-valued, measurable, $\mathbf{F}$-adapted processes X with
$$\|X\|_{\mathbb{L}^{p,q}_{\mathbf{F}}([0,T];\mathbb{B})} \triangleq \left\{ E\left[\left(\int_0^T |X_t|^p_{\mathbb{B}}\, dt \right)^{q/p} \right] \right\}^{1/q} < \infty;$$
- $\mathbb{H}^{p,q}_{\mathbf{F}}([0,T];\mathbb{B})$ (resp. $\widehat{\mathbb{H}}^{p,q}_{\mathbf{F}}([0,T];\mathbb{B})$) $\triangleq \big\{X \in \mathbb{L}^{p,q}_{\mathbf{F}}([0,T];\mathbb{B}) : X$ is $\mathbf{F}$-predictable (resp. $\mathbf{F}$-progressively measurable)$\big\}$.
When $p = q$, we simply write $\mathbb{L}^p_{\mathbf{F}}, \mathbb{H}^p_{\mathbf{F}}$ and $\widehat{\mathbb{H}}^p_{\mathbf{F}}$ for $\mathbb{L}^{p,p}_{\mathbf{F}}, \mathbb{H}^{p,p}_{\mathbf{F}}$ and $\widehat{\mathbb{H}}^{p,p}_{\mathbf{F}}$ respectively. Moreover we let
- $\mathbb{H}^{p,\mathrm{loc}}_{\mathbf{F}}([0,T];\mathbb{B})$ (resp. $\widehat{\mathbb{H}}^{p,\mathrm{loc}}_{\mathbf{F}}([0,T];\mathbb{B})$) denote the space of all $\mathbb{B}$-valued, $\mathbf{F}$-predictable (resp. $\mathbf{F}$-progressively measurable) processes X with $\int_0^T |X_t|^p_{\mathbb{B}} dt < \infty$, P-a.s. for any $p \in [1,\infty)$.

(4) Let $\mathbb{C}^0_{\mathbf{F}}[0,T]$ be the space of all $\mathbb{R}$-valued, $\mathbf{F}$-adapted continuous processes, we need its following subspaces:
- $\mathbb{C}^\infty_{\mathbf{F}}[0,T] \triangleq \Big\{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : \|X\|_{\mathbb{C}^\infty_{\mathbf{F}}[0,T]} \triangleq \operatorname{esssup}_{\omega\in\Omega}\Big(\sup_{t\in[0,T]}|X_t(\omega)|\Big) < \infty\Big\}$;
- $\mathbb{C}^p_{\mathbf{F}}[0,T] \triangleq \Big\{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : \|X\|_{\mathbb{C}^p_{\mathbf{F}}[0,T]} \triangleq \Big\{E\Big[\sup_{t\in[0,T]}|X_t|^p\Big]\Big\}^{\frac{1}{p}} < \infty\Big\}$ for all $p \in [1,\infty)$;
- $\mathbb{V}_{\mathbf{F}}[0,T] \triangleq \big\{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : X$ has finite variation$\big\}$;
- $\mathbb{K}_{\mathbf{F}}[0,T] \triangleq \big\{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : X$ is an increasing process with $X_0 = 0\big\} \subset \mathbb{V}_{\mathbf{F}}[0,T]$;
- $\mathbb{K}^p_{\mathbf{F}}[0,T] \triangleq \big\{X \in \mathbb{K}_{\mathbf{F}}[0,T] : X_T \in \mathbb{L}^p(\mathcal{F}_T)\big\}$ for all $p \in [1,\infty)$;
- $\mathbb{E}^{\lambda,\lambda'}_{\mathbf{F}}[0,T] \triangleq \Big\{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : E\Big[e^{\lambda X_*^-} + e^{\lambda' X_*^+}\Big] < \infty\Big\} \subset \cap_{p\in[1,\infty)} \mathbb{C}^p_{\mathbf{F}}[0,T]$ for all $\lambda, \lambda' \in (0,\infty)$.

For any $\lambda \in (0,\infty)$, we set $\mathbb{E}^\lambda_{\mathbf{F}}[0,T] \triangleq \mathbb{E}^{\lambda,\lambda}_{\mathbf{F}}[0,T]$. For any $X \in \mathbb{C}^0_{\mathbf{F}}[0,T]$, one can deduce that
$$E\Big[e^{\lambda X_*}\Big] = E\Big[e^{\lambda(X_*^- \vee X_*^+)}\Big] = E\Big[e^{\lambda X_*^-} \vee e^{\lambda X_*^+}\Big] \le E\Big[e^{\lambda X_*^-} + e^{\lambda X_*^+}\Big] \le 2E\Big[e^{\lambda X_*}\Big], \quad (1.3)$$
which implies that $\mathbb{E}^\lambda_{\mathbf{F}}[0,T] = \big\{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : E\big[e^{\lambda X_*}\big] < \infty\big\}$. Moreover, for any $p \in [1,\infty)$, we set $\mathbb{S}^p_{\mathbf{F}}[0,T] \triangleq \mathbb{E}^p_{\mathbf{F}}[0,T] \times \mathbb{H}^{2,2p}_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}^p_{\mathbf{F}}[0,T]$.

1.2. Reflected BSDEs

Let $\mathscr{P}$ denote the $\mathbf{F}$-progressively measurable σ-field on $[0,T]\times\Omega$. A *parameter set* (ξ, f, L) consists of a random variable $\xi \in \mathbb{L}^0(\mathcal{F}_T)$, a function $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and a process $L \in \mathbb{C}^0_{\mathbf{F}}[0,T]$ such that f is $\mathscr{P}\times\mathscr{B}(\mathbb{R})\times\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable and that $L_T \le \xi$, P-a.s.

Definition 1.1. Given a parameter set (ξ, f, L), a triplet $(Y, Z, K) \in \mathbb{C}^0_{\mathbf{F}}[0,T] \times \widehat{\mathbb{H}}^{2,\mathrm{loc}}_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ is called a solution of the *reflected backward stochastic differential equation* with terminal condition ξ, generator f, and obstacle L (RBSDE (ξ, f, L) for short), if (1.1) and (1.2) hold P-a.s.

A function $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is said to be *Lipschitz* in (y,z) if for some $\lambda > 0$, it holds $dt \otimes dP$-a.e. that
$$|f(t,\omega,y_1,z_1) - f(t,\omega,y_2,z_2)| \le \lambda\big(|y_1-y_2| + |z_1-z_2|\big),$$
$$\forall\, y_1, y_2 \in \mathbb{R},\ \forall\, z_1, z_2 \in \mathbb{R}^d.$$

The theory of RBSDEs with Lipschitz generators was well developed in the seminal paper [10]. In this paper, we are interested in *quadratic* RBSDEs in the following sense:

(H1) For three constants $\alpha, \beta \ge 0$ and $\gamma > 0$, it holds $dt \otimes dP$-a.e. that

$$|f(t,\omega,y,z)| \le \alpha + \beta|y| + \frac{\gamma}{2}|z|^2, \quad \forall\, (y,z) \in \mathbb{R} \times \mathbb{R}^d.$$

In what follows, for any $\lambda \ge 0$ we let c_λ denote a generic constant depending on $\lambda, \alpha, \beta, \gamma$ and T (in particular, c_0 stands for a generic constant depending on α, β, γ and T), whose form may vary from line to line.

2. Two a priori estimates

We first present an a priori estimate, which is an extension of Lemma 3.1 of [18].

Proposition 2.1. *Let* (ξ, f, L) *be a parameter set such that* f *satisfies* (H1). *If* (Y, Z, K) *is a solution of the quadratic* RBSDE(ξ, f, L) *such that* $Y^+ \in \mathbb{C}^\infty_{\mathbf{F}}[0,T]$, *then it holds* P*-a.s. that*

$$Y_t \le c_0 + \frac{1}{\gamma} \ln E\left[e^{\gamma e^{\beta T}(\xi^+ \vee L_*^+)} | \mathcal{F}_t\right], \quad t \in [0,T]. \tag{2.1}$$

Proof. In light of Itô's formula, $(Y,Z,K) \in \mathbb{C}^0_{\mathbf{F}}[0,T] \times \widehat{\mathbb{H}}^2_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ with $Y^+ \in \mathbb{C}^\infty_{\mathbf{F}}[0,T]$ is a solution of the RBSDE(ξ, f, L) if and only if $\big(\widetilde{Y}, \widetilde{Z}, \widetilde{K}\big) \triangleq \big(e^{\gamma Y}, \gamma e^{\gamma Y} Z, \gamma \int_0^\cdot e^{\gamma Y_s} dK_s\big) \in \mathbb{C}^\infty_{\mathbf{F}}[0,T] \times \widehat{\mathbb{H}}^2_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ is a solution of the RBSDE$(e^{\gamma\xi}, \widetilde{f}, e^{\gamma L})$ with

$$\widetilde{f}(t,\omega,y,z) \triangleq \mathbf{1}_{\{y>0\}} \left\{ \gamma y f\left(t,\omega,\frac{\ln y}{\gamma}, \frac{z}{\gamma y}\right) - \frac{1}{2}\frac{|z|^2}{y} \right\},$$
$$\forall\, (t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d.$$

Let $\mu \triangleq \alpha\gamma \vee \beta \vee 1$. One can deduce from (H1) that $dt \otimes dP$-a.e.

$$\widetilde{f}(t,\omega,y,z) \le H(y) \triangleq y\big(\mu + \beta \ln y\big)\mathbf{1}_{\{y \ge 1\}} + \mu \mathbf{1}_{\{y<1\}}, \quad \forall\, (y,z) \in \mathbb{R} \times \mathbb{R}^d. \tag{2.2}$$

Clearly, $H(\cdot)$ is a strictly positive, increasing, continuous and convex function satisfying $\int_0^\infty \frac{1}{H(y)} dy = \infty$.

For any $x \in \mathbb{R}$ and $\widetilde{T} \in [0,T]$, the ordinary differential equation (ODE)

$$\phi(t) = e^{\gamma x} + \int_t^{\widetilde{T}} H\big(\phi(s)\big) ds, \quad t \in \big[0, \widetilde{T}\big]$$

can be solved as follows (cf. [5]):

(i) When $x \ge 0$: $\phi_t^{\widetilde{T}}(x) = \exp\left\{\mu\varphi\big(\widetilde{T} - t\big) + \gamma x e^{\beta(\widetilde{T}-t)}\right\}$, where $\varphi(s) \triangleq \frac{e^{\beta s}-1}{\beta}\mathbf{1}_{\{\beta>0\}} + s\mathbf{1}_{\{\beta=0\}}$, $\forall\, s \in [0,T]$;

(ii) When $x < 0$: $\phi_t^{\widetilde{T}}(x) = \begin{cases} e^{\gamma x} + \mu(\widetilde{T}-t) < 1 + \mu(\widetilde{T}-t) \le e^{\mu(\widetilde{T}-t)} \le e^{\mu\varphi(\widetilde{T}-t)}, & \text{if } e^{\gamma x} + \mu(\widetilde{T}-t) < 1, \\ \exp\left\{\mu\varphi\left(\widetilde{T} - t + \dfrac{e^{\gamma x}-1}{\mu}\right)\right\} \le e^{\mu\varphi(\widetilde{T}-t)} & \text{if } e^{\gamma x} + \mu(\widetilde{T}-t) \ge 1. \end{cases}$

One can check that

(ϕ1) For any $x \in \mathbb{R}$ and $\widetilde{T} \in [0,T]$, $t \to \phi_t^{\widetilde{T}}(x)$ is a decreasing and continuous function on $\big[0, \widetilde{T}\big]$;

(ϕ2) For any $x \in \mathbb{R}$ and $t \in [0,T]$, $\widetilde{T} \to \phi_t^{\widetilde{T}}(x)$ is an increasing and continuous function on $[t,T]$;
(ϕ3) For any $0 \le t \le \widetilde{T} \le T$, $x \to \phi_t^{\widetilde{T}}(x)$ is an increasing and continuous function on $\mathbb{R}$;
(ϕ4) For any $x \in \mathbb{R}$ and $0 \le t \le \widetilde{T} \le T$, $\phi_t^{\widetilde{T}}(x) \le \exp\{\mu\varphi(T) + \gamma x^+ e^{\beta T}\}$.

Let $\widetilde{\Omega} \triangleq \{\omega \in \Omega : L_T(\omega) \le \xi(\omega)$ and the path $t \to L_t(\omega)$ is continuous$\} \in \mathcal{F}$, which defines a measurable set with probability 1. Fix $\omega \in \widetilde{\Omega}$. Theorem 6.2 of [18] shows that the following reflected backward ODE

$$\begin{cases} e^{\gamma L_t(\omega)} \le \Lambda_t(\omega) = e^{\gamma\xi(\omega)} + \int_t^T H\big(\Lambda_s(\omega)\big)\,ds + k_T(\omega) - k_t(\omega), \quad t \in [0,T], \\ \int_0^T \Big(\Lambda_s(\omega) - e^{\gamma L_s(\omega)}\Big)\,dk_s(\omega) = 0 \end{cases}$$

admits a unique solution $\big(\Lambda_\cdot(\omega), k_\cdot(\omega)\big) \in \mathbb{C}[0,T] \times \mathbb{K}[0,T]$, which satisfies

$$\begin{aligned} \Lambda_t(\omega) &= \sup_{s\in[t,T]} \left(\int_t^s H\big(\Lambda_r(\omega)\big)dr + e^{\gamma\xi(\omega)}\,\mathbf{1}_{\{s=T\}} + e^{\gamma L_s(\omega)}\mathbf{1}_{\{s<T\}} \right) \\ &= \sup_{s\in[t,T]} u_t^s(\omega), \quad t \in [0,T], \end{aligned} \tag{2.3}$$

where $\{u_r^s(\omega)\}_{r\in[0,s]}$ is the unique solution of the following ODE

$$u_r^s(\omega) = e^{\gamma\xi(\omega)}\,\mathbf{1}_{\{s=T\}} + e^{\gamma L_s(\omega)}\,\mathbf{1}_{\{s<T\}} + \int_r^s H\big(u_a^s(\omega)\big)da, \quad r \in [0,s].$$

To wit, $u_r^s(\omega) = \phi_r^s\,\big(\xi(\omega)\mathbf{1}_{\{s=T\}} + L_s(\omega)\,\mathbf{1}_{\{s<T\}}\big)$. Then it follows from (2.3) and (ϕ4) that

$$\begin{aligned} &0 < e^{\gamma L_t(\omega)} \le \Lambda_t(\omega) = \sup_{s\in[t,T]} u_t^s(\omega) \le \exp\Big\{\mu\varphi(T) + \gamma e^{\beta T}\big(\xi^+(\omega) \vee L_*^+(\omega)\big)\Big\}, \\ &t \in [0,T]. \end{aligned} \tag{2.4}$$

For any $0 \le t_1 < t_2 \le T$, one can deduce from (2.3) and (ϕ1) that

$$\Lambda_{t_1}(\omega) = \sup_{s\in[t_1,T]} u_{t_1}^s(\omega) \ge \sup_{s\in[t_2,T]} u_{t_1}^s(\omega) \ge \sup_{s\in[t_2,T]} u_{t_2}^s(\omega) = \Lambda_{t_2}(\omega). \tag{2.5}$$

Thus $t \to \Lambda_t(\omega)$ is a decreasing and continuous path. Moreover, for any $t \in [0,T]$ (2.3) and (ϕ2) show that

$$\Lambda_t(\omega) = \sup_{s\in[t,T]} u_t^s(\omega) = \sup\Big\{u_t^s(\omega) : s \in \Big([t,T) \bigcap \mathbb{Q}\Big) \cup \{T\}\Big\}. \tag{2.6}$$

For any $0 \le t \le s \le T$, the continuity of $\phi_t^s(\cdot)$ by (ϕ3) implies that the random variable $\big\{u_t^s(\omega)\big\}_{\omega\in\Omega} = \phi_t^s\big(\xi\mathbf{1}_{\{s=T\}} + L_s\mathbf{1}_{\{s<T\}}\big)$ is $\mathcal{F}_s$-measurable. Then we can deduce from (2.6) that for any $t \in [0,T]$, the random variable Λ_t is $\mathcal{F}_T$-measurable (however, not necessarily $\mathcal{F}_t$-measurable).

Now, let us introduce an $\mathbf{F}$-adapted process $\mathfrak{f}_t \triangleq E[H(\Lambda_t)|\mathcal{F}_t]$, $t \in [0,T]$. Since Λ is a decreasing process by (2.5), and since $H(\cdot)$ is an increasing function, it holds for any $0 \le t < s \le T$ that $E[\mathfrak{f}_s|\mathcal{F}_t] = E[H(\Lambda_s)|\mathcal{F}_t] \le E[H(\Lambda_t)|\mathcal{F}_t] = \mathfrak{f}_t$, P-a.s. Thus $\mathfrak{f}$ is a supermartingale. As $Y^+ \in \mathbb{C}^\infty_{\mathbf{F}}[0,T]$, it follows that $(\xi^+, L^+) \in \mathbb{L}^\infty(\mathcal{F}_T) \times \mathbb{C}^\infty_{\mathbf{F}}[0,T]$. Then the continuity of process $H(\Lambda_\cdot)$, (2.4) and the Bounded Convergence Theorem imply that

$$E[\mathfrak{f}_t] = E[H(\Lambda_t)] = \lim_{s\downarrow t} E[H(\Lambda_s)] = \lim_{s\downarrow t} E[\mathfrak{f}_s], \quad t \in [0,T].$$

Thanks to Theorem 1.3.13 of [13], $\mathfrak{f}$ has a right-continuous modification $\widetilde{\mathfrak{f}}$. Hence, we can regard $\widetilde{\mathfrak{f}}$ as a generator that is independent of (y, z). It follows from Fubini's Theorem, Jensen's inequality as well as (2.4) that

$$E\int_0^T |\widetilde{\mathfrak{f}}_s|^2 ds = \int_0^T E\Big[|\widetilde{\mathfrak{f}}_s|^2\Big] ds = \int_0^T E\Big[|\mathfrak{f}_s|^2\Big] ds \le \int_0^T E\Big[E\big[|H(\Lambda_s)|^2|\mathcal{F}_s\big]\Big]ds$$
$$= \int_0^T E\big[|H(\Lambda_s)|^2\big]ds < \infty.$$

Since $e^{\gamma\xi} \in \mathbb{L}^\infty(\mathcal{F}_T)$ and $e^{\gamma L} \in \mathbb{C}^\infty_{\mathbf{F}}[0, T]$, Theorem 5.2 and Proposition 2.3 of [10] show that the RBSDE$(e^{\gamma\xi}, \widetilde{\mathfrak{f}}, e^{\gamma L})$ admits a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{K}) \in \mathbb{C}^2_{\mathbf{F}}[0, T] \times \mathbb{H}^2_{\mathbf{F}}\big([0, T]; \mathbb{R}^d\big) \times \mathbb{K}^2_{\mathbf{F}}[0, T]$ and that for any $t \in [0, T]$

$$\mathcal{Y}_t = \operatorname*{esssup}_{\tau\in\mathcal{S}_{t,T}} E\left[\int_t^\tau \widetilde{\mathfrak{f}}_s\, ds + e^{\gamma\xi}\, \mathbf{1}_{\{\tau=T\}} + e^{\gamma L_\tau}\mathbf{1}_{\{\tau<T\}}\middle| \mathcal{F}_t\right], \quad P\text{-a.s.} \tag{2.7}$$

For any $t \in [0, T]$ and $\tau \in \mathcal{S}_{t,T}$, Fubini's Theorem implies that for any $A \in \mathcal{F}_t$

$$E\left[\mathbf{1}_A \int_t^\tau \widetilde{\mathfrak{f}}_s\, ds\right] = \int_t^T E\left[\mathbf{1}_A\mathbf{1}_{\{s\le\tau\}}\widetilde{\mathfrak{f}}_s\right] ds = \int_t^T E\left[\mathbf{1}_A\mathbf{1}_{\{s\le\tau\}}E[H(\Lambda_s)|\mathcal{F}_s]\right] ds$$
$$= \int_t^T E\left[\mathbf{1}_A\mathbf{1}_{\{s\le\tau\}}H(\Lambda_s)\right] ds = E\left[\mathbf{1}_A \int_t^\tau H(\Lambda_s)ds\right].$$

Thus $E\left[\int_t^\tau \widetilde{\mathfrak{f}}_s\, ds|\mathcal{F}_t\right] = E\left[\int_t^\tau H(\Lambda_s)ds|\mathcal{F}_t\right]$, P-a.s. Then (2.7), (2.3) and (2.4) imply that for any $t \in [0, T]$

$$\mathcal{Y}_t = \operatorname*{esssup}_{\tau\in\mathcal{S}_{t,T}} E\left[\int_t^\tau H(\Lambda_s)ds + e^{\gamma\xi}\, \mathbf{1}_{\{\tau=T\}} + e^{\gamma L_\tau}\mathbf{1}_{\{\tau<T\}}\middle| \mathcal{F}_t\right]$$
$$\le E[\Lambda_t|\mathcal{F}_t] \le e^{\mu\varphi(T)}E\left[e^{\gamma e^{\beta T}(\xi^+\vee L^+_*)}|\mathcal{F}_t\right] \le C_*, \quad P\text{-a.s.}, \tag{2.8}$$

with $C_* \triangleq \exp\left\{\mu\varphi(T) + \gamma e^{\beta T}\Big(\|\xi^+\|_{\mathbb{L}^\infty(\mathcal{F}_T)} \vee \|L^+\|_{\mathbb{C}^\infty_{\mathbf{F}}[0,T]}\Big)\right\}$. By the continuity of process $\mathcal{Y}$, it holds P-a.s. that

$$0 < e^{\gamma L_t} \le \mathcal{Y}_t \le e^{\mu\varphi(T)}E\left[e^{\gamma e^{\beta T}(\xi^+\vee L^+_*)}|\mathcal{F}_t\right] \le C_*, \quad t \in [0, T], \tag{2.9}$$

which shows that $\mathcal{Y} \in \mathbb{C}^\infty_{\mathbf{F}}[0, T]$ with $\|\mathcal{Y}\|_{\mathbb{C}^\infty_{\mathbf{F}}[0,T]} \le C_*$.

To finalize the proof, it suffices to show that $P\Big(\widetilde{Y}_t \le \mathcal{Y}_t,\ \forall\, t \in [0, T]\Big) = 1$. To see this, we fix $n \in \mathbb{N}$ and define the $\mathbf{F}$-stopping time $\tau_n \triangleq \inf\left\{t \in [0, T] : \int_0^t |Z_s|^2 ds > n\right\} \wedge T$. Clearly, $\lim_{n\to\infty} \uparrow \tau_n = T$, P-a.s. Applying Tanaka's formula to the process $(\widetilde{Y} - \mathcal{Y})^+$ yields that

$$(\widetilde{Y}_{\tau_n\wedge t} - \mathcal{Y}_{\tau_n\wedge t})^+ = (\widetilde{Y}_{\tau_n} - \mathcal{Y}_{\tau_n})^+ + \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}\big(\widetilde{f}(s, \widetilde{Y}_s, \widetilde{Z}_s) - \widetilde{\mathfrak{f}}_s\big)ds$$
$$+ \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}\big(d\widetilde{K}_s - d\mathcal{K}_s\big) - \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(\widetilde{Z}_s - \mathcal{Z}_s)\, dB_s$$
$$- \frac{1}{2}\int_{\tau_n\wedge t}^{\tau_n} d\mathfrak{L}_s, \quad t \in [0, T], \tag{2.10}$$

where $\mathfrak{L}$ is a real-valued, $\mathbf{F}$-adapted, increasing and continuous process known as "local time".

Since the function $H(\cdot)$ is increasing, continuous and convex, Jensen's inequality and (2.8) show that

$$H(\widetilde{Y}_s) - \mathfrak{f}_s \le H(\widetilde{Y}_s) - H\big(E[\Lambda_s|\mathcal{F}_s]\big) \le H(\widetilde{Y}_s) - H(\mathcal{Y}_s) \le C_H|\widetilde{Y}_s - \mathcal{Y}_s|, \quad s \in [0,T], \tag{2.11}$$

where C_H is the Lipschitz coefficient of function $H(\cdot)$ over $\Big\{x \in \mathbb{R} : |x| \le \|\widetilde{Y}\|_{\mathbb{C}^\infty_{\mathbf{F}}[0,T]} \vee \|\mathcal{Y}\|_{\mathbb{C}^\infty_{\mathbf{F}}[0,T]}\Big\}$. Moreover, the flat-off condition of $\big(\widetilde{Y}, \widetilde{Z}, \widetilde{K}\big)$ implies that

$$\int_0^T \mathbf{1}_{\{\widetilde{Y}_s > \mathcal{Y}_s\}} d\widetilde{K}_s = \int_0^T \mathbf{1}_{\{e^{\gamma L_s} = \widetilde{Y}_s > \mathcal{Y}_s\}} d\widetilde{K}_s = 0, \quad P\text{-a.s.} \tag{2.12}$$

Taking the expectation in (2.10), we can deduce from (2.2), Fubini's Theorem, (2.11) and (2.12) that

$$\begin{aligned} E\left[(\widetilde{Y}_{\tau_n\wedge t} - \mathcal{Y}_{\tau_n\wedge t})^+\right] - E\left[(\widetilde{Y}_{\tau_n} - \mathcal{Y}_{\tau_n})^+\right] \quad &\le \int_t^T E\left[\mathbf{1}_{\{s\le\tau_n\}}\mathbf{1}_{\{\widetilde{Y}_s > \mathcal{Y}_s\}}\big(H(\widetilde{Y}_s) - \mathfrak{f}_s\big)\right] ds \\ \le C_H \int_t^T E\left[\mathbf{1}_{\{s\le\tau_n\}}\mathbf{1}_{\{\widetilde{Y}_s > \mathcal{Y}_s\}}(\widetilde{Y}_s - \mathcal{Y}_s)^+\right] ds & \\ \le C_H \int_t^T E\left[(\widetilde{Y}_{\tau_n\wedge s} - \mathcal{Y}_{\tau_n\wedge s})^+\right] ds, \quad t \in [0,T]. & \end{aligned}$$

Then Gronwall's inequality shows that $E\left[(\widetilde{Y}_{\tau_n\wedge t} - \mathcal{Y}_{\tau_n\wedge t})^+\right] \le e^{C_H T} E\left[(\widetilde{Y}_{\tau_n} - \mathcal{Y}_{\tau_n})^+\right]$, $\forall t \in [0,T]$. As $n \to \infty$, the continuity of processes $\widetilde{Y}, \mathcal{Y}$ and the Bounded Convergence Theorem imply that for any $t \in [0,T]$

$$E\Big[(\widetilde{Y}_t - \mathcal{Y}_t)^+\Big] = 0, \quad \text{thus } \widetilde{Y}_t \le \mathcal{Y}_t,\ P\text{-a.s.}$$

Using the continuity of processes $\widetilde{Y}$ and $\mathcal{Y}$ again, we obtain $P\Big(\widetilde{Y}_t \le \mathcal{Y}_t,\ \forall t \in [0,T]\Big) = 1$, which together with (2.9) leads to (2.1). □

For a solution (Y, Z, K) of a quadratic RBSDE(ξ, f, L) such that L_*^- and Y_*^+ have exponential moments of certain orders, the next result estimates the norms of (Z, K) in $\mathbb{H}^{2,2p}_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}^p_{\mathbf{F}}[0,T]$ for some $p > 1$.

Proposition 2.2. *Let* (ξ, f, L) *be a parameter set such that* f *satisfies* (H1). *If* (Y, Z, K) *is a solution of the quadratic* RBSDE(ξ, f, L) *such that* $Y \in \mathbb{E}^{\lambda\gamma,\lambda'\gamma}_{\mathbf{F}}[0,T]$ *for some* $\lambda, \lambda' > 1$ *with* $\frac{1}{\lambda} + \frac{1}{\lambda'} < 1$, *then*

$$E\left[\left(\int_0^T |Z_s|^2 ds\right)^p + K_T^p\right] \le c_{\lambda,\lambda',p}\, E\left[e^{\lambda\gamma Y_*^-} + e^{\lambda'\gamma Y_*^+}\right] < \infty, \quad \forall p \in \left(1, \frac{\lambda\lambda'}{\lambda+\lambda'}\right).$$

Proof. We set $p_o \triangleq \sqrt{\frac{\lambda\lambda'}{p\,(\lambda+\lambda')}} \wedge 2 > 1$ and define stopping times $\tau_n \triangleq \inf\big\{t \in [0,T] : \int_0^t e^{-p_o\gamma Y_s}|Z_s|^2 ds > n\big\} \wedge T$, $\forall n \in \mathbb{N}$. Since $E\big[e^{\lambda\gamma Y_*^-}\big] < \infty$ and $Z \in \widehat{\mathbb{H}}^{2,\text{loc}}_{\mathbf{F}}([0,T];\mathbb{R}^d)$, it holds P-a.s. that $Y_*^- + \int_0^T |Z_s|^2 ds < \infty$. Then it follows that $\int_0^T e^{-p_o\gamma Y_s}|Z_s|^2 ds \le e^{p_o\gamma Y_*^-}\int_0^T |Z_s|^2 ds < \infty$, P-a.s. Hence, for P-a.s. $\omega \in \Omega$, there exists an $n(\omega) \in \mathbb{N}$ such that $\tau_{n(\omega)}(\omega) = T$. For any $n \in \mathbb{N}$, applying Itô's formula to process $e^{-p_o\gamma Y}$ and using the fact that $\alpha + \beta x \le \left(\alpha \vee \frac{\beta}{(p_o^2-p_o)\gamma}\right) e^{(p_o^2-p_o)\gamma x}$, $\forall\, x \ge 0$, we obtain that P-a.s.

$$\begin{aligned}\frac{1}{2}p_o\gamma \int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds &\le \frac{1}{p_o\gamma}e^{-p_o\gamma Y_{\tau_n}} - \int_0^{\tau_n} e^{-p_o\gamma Y_s} f(s, Y_s, Z_s)ds \\ &\quad - \int_0^{\tau_n} e^{-p_o\gamma Y_s} dK_s + \int_0^{\tau_n} e^{-p_o\gamma Y_s} Z_s dB_s \\ &\le \frac{1}{p_o\gamma}e^{p_o\gamma Y_*^-} + \left(\alpha \vee \frac{\beta}{(p_o^2-p_o)\gamma}\right) \\ &\quad \times \int_0^{\tau_n} e^{-p_o\gamma Y_s + (p_o^2-p_o)\gamma|Y_s|} ds \\ &\quad + \frac{\gamma}{2}\int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds + \left|\int_0^{\tau_n} e^{-p_o\gamma Y_s} Z_s dB_s\right|. \end{aligned} \tag{2.13}$$

Observe that $\int_0^{\tau_n} e^{-p_o\gamma Y_s + (p_o^2-p_o)\gamma|Y_s|} ds \le \int_0^{\tau_n} e^{-p_o^2\gamma \mathbf{1}_{\{Y_s<0\}} Y_s} ds \le T e^{p_o^2\gamma Y_*^-}$, P-a.s., which together with the Burkholder–Davis–Gundy inequality and (2.13) implies that

$$\begin{aligned} & E\left[\left(\int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds\right)^{\lambda p_o^{-2}}\right] \\ &\le c_{\lambda,\lambda',p}\, E\left[e^{\lambda\gamma Y_*^-} + e^{\frac{\lambda}{2p_o}\gamma Y_*^-}\left(\int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds\right)^{\frac{1}{2}\lambda p_o^{-2}}\right] \\ &\le c_{\lambda,\lambda',p}\, E\left[e^{\lambda\gamma Y_*^-}\right] + \frac{1}{2}E\left[\left(\int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds\right)^{\lambda p_o^{-2}}\right]. \end{aligned}$$

Since $E\left[\left(\int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds\right)^{\lambda p_o^{-2}}\right] \le n^{\lambda p_o^{-2}} < \infty$, it follows that $E\Big[\Big(\int_0^{\tau_n} e^{-p_o\gamma Y_s}|Z_s|^2 ds\Big)^{\lambda p_o^{-2}}\Big] \le c_{\lambda,\lambda',p}\, E\left[e^{\lambda\gamma Y_*^-}\right]$. As $n \to \infty$, the Monotone Convergence Theorem gives that $E\left[\left(\int_0^T e^{-p_o\gamma Y_s}|Z_s|^2 ds\right)^{\lambda p_o^{-2}}\right] \le c_{\lambda,\lambda',p}\, E\left[e^{\lambda\gamma Y_*^-}\right]$. Since $\frac{\lambda p_o p}{\lambda - p_o^2 p} < \frac{\lambda p_o^2 p}{\lambda - p_o^2 p} \le \lambda'$, applying Young's inequality with $\widetilde{p} = \frac{\lambda}{\lambda - p_o^2 p}$ and $\widetilde{q} = \frac{\lambda}{p_o^2 p}$ yields that

$$\begin{aligned} E\left[\left(\int_0^T |Z_s|^2 ds\right)^p\right] &\le E\left[e^{p_o p\gamma Y_*^+}\left(\int_0^T e^{-p_o\gamma Y_s}|Z_s|^2 ds\right)^p\right] \\ &\le c_{\lambda,\lambda',p}\, E\left[e^{\lambda'\gamma Y_*^+} + e^{\lambda\gamma Y_*^-}\right] < \infty. \end{aligned} \tag{2.14}$$

On the other hand, since $Y_* \le Y_*^- + Y_*^+$, it holds P-a.s. that

$$\begin{aligned} K_T &= Y_0 - \xi - \int_0^T f(s, Y_s, Z_s)ds + \int_0^T Z_s dB_s \\ &\le \alpha T + (2+\beta T)(Y_*^- + Y_*^+) + \frac{\gamma}{2}\int_0^T |Z_s|^2 ds + \left|\int_0^T Z_s dB_s\right|. \end{aligned}$$

Then the Burkholder–Davis–Gundy inequality and (2.14) imply that

$$E\big[K_T^p\big] \le c_p\, E\left[1 + \big(Y_*^-\big)^p + \big(Y_*^+\big)^p + \left(\int_0^T |Z_s|^2 ds\right)^p\right] \le c_{\lambda,\lambda',p}\, E\left[e^{\lambda\gamma Y_*^-} + e^{\lambda'\gamma Y_*^+}\right] < \infty. \quad \square$$

3. Existence

As usually, the existence is based on the following monotone stability result.

Theorem 3.1. *For any $n \in \mathbb{N}$, let $\big\{(\xi_n, f_n, L^n)\big\}_{n\in\mathbb{N}}$ be a parameter set and let $(Y^n, Z^n, K^n) \in \mathbb{C}^0_{\mathbf{F}}[0,T] \times \mathbb{H}^{2,\mathrm{loc}}_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ be a solution of the* RBSDE (ξ_n, f_n, L^n) *such that*

- (M1) *All generators f_n, $n \in \mathbb{N}$ satisfy* (H1) *with the same constants $\alpha, \beta \ge 0$ and $\gamma > 0$;*
- (M2) *There exists a function $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that for $dt \otimes dP$-a.e. $(t,\omega) \in [0,T] \times \Omega$, the mapping $f(t,\omega,\cdot,\cdot)$ is continuous and $f_n(t,\omega,y,z)$ converges to $f(t,\omega,y,z)$ locally uniformly in (y,z);*
  and that for some $L \in \mathbb{C}^0_{\mathbf{F}}[0,T]$ and some real-valued, $\mathbf{F}$-adapted process Y, either of the following two holds:
- (M3a) *It holds P-a.s. that for any $t \in [0,T]$, $\{L^n_t\}_{n\in\mathbb{N}}$ and $\{Y^n_t\}_{n\in\mathbb{N}}$ are both increasing sequences in n with $\lim_{n\to\infty} \uparrow L^n_t = L_t$ and $\lim_{n\to\infty} \uparrow Y^n_t = Y_t$ respectively;*
- (M3b) *It holds P-a.s. that for any $t \in [0,T]$, $\{L^n_t\}_{n\in\mathbb{N}}$ and $\{Y^n_t\}_{n\in\mathbb{N}}$ are both decreasing sequences in n with $\lim_{n\to\infty} \downarrow L^n_t = L_t$ and $\lim_{n\to\infty} \downarrow Y^n_t = Y_t$ respectively.*

Denote $\mathscr{L}_t \triangleq (L^1_t)^- \vee L^-_t$ and $\mathscr{Y}_t \triangleq (Y^1_t)^+ \vee Y^+_t$, $\forall\, t \in [0,T]$. If $\Xi \triangleq E\Big[e^{\lambda\gamma\mathscr{L}_*} + e^{\lambda'\gamma\mathscr{Y}_*}\Big] < \infty$ for some $\lambda, \lambda' > 6$ with $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6}$, then $Y \in \mathbb{E}^{\lambda\gamma,\lambda'\gamma}_{\mathbf{F}}[0,T]$ and there exist $(Z,K) \in \cap_{p\in\left(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\right)} \mathbb{H}^{2,2p}_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}^p_{\mathbf{F}}[0,T]$ such that the triplet (Y,Z,K) is a solution of the RBSDE (ξ, f, L) *with $\xi \triangleq Y_T$.*

Proof. Since it holds P-a.s. that

$$-\mathscr{L}_t \le L^1_t \wedge L_t \le L^n_t \le Y^n_t \le Y^1_t \vee Y_t \le \mathscr{Y}_t, \quad t \in [0,T],\ \forall\, n \in \mathbb{N}. \tag{3.1}$$

(1) Let $\lambda_o \triangleq 5 + \frac{1}{2}\left(\frac{\lambda\lambda'}{\lambda+\lambda'} - 6\right) < \frac{\lambda\lambda'}{\lambda+\lambda'} - 1$. It follows that $p_o \triangleq \frac{\lambda\lambda'}{\lambda\lambda' - \lambda_o(\lambda+\lambda')} \in \left(1, \frac{\lambda\lambda'}{\lambda+\lambda'}\right)$. For any $n \in \mathbb{N}$, since $E\left[e^{\lambda\gamma(Y^n)^-_*} + e^{\lambda'\gamma(Y^n)^+_*}\right] \le E\left[e^{\lambda\gamma\mathscr{L}_*} + e^{\lambda'\gamma\mathscr{Y}_*}\right] < \infty$ by (3.1), applying Proposition 2.2 with $p = p_o$ yields that

$$E\left[\left(\int_0^T |Z^n_s|^2 ds\right)^{p_o} + \big(K^n_T\big)^{p_o}\right] \le c_{\lambda,\lambda'} E\left[e^{\lambda\gamma(Y^n)^-_*} + e^{\lambda'\gamma(Y^n)^+_*}\right] \le c_{\lambda,\lambda'}\Xi < \infty, \tag{3.2}$$

which shows that $\{Z^n\}_{n\in\mathbb{N}}$ is a bounded subset of the reflexive Banach space $\mathbb{H}^{2,2p_o}_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Then Theorem 5.2.1 of [26] implies that $\{Z^n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence (we still denote it by $\{Z^n\}_{n\in\mathbb{N}}$) with limit $Z \in \mathbb{H}^{2,2p_o}_{\mathbf{F}}([0,T];\mathbb{R}^d)$.

(2) In this step, we will show that $\{Z^n\}_{n\in\mathbb{N}}$ strongly converges to Z in $\mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$.

We define $\phi(x) \triangleq \frac{1}{\lambda_o\gamma}\left(e^{\lambda_o\gamma|x|} - \lambda_o\gamma|x| - 1\right) \ge 0, \ \forall\, x \in \mathbb{R}$. Fix $n \in \mathbb{N}$. For any $m \ge n$, we set $\xi_{m,n} \triangleq \xi_m - \xi_n$ and $\Theta^{m,n} \triangleq \Theta^m - \Theta^n$ for $\Theta = Y, Z, K, L$. Applying Itô's formula to the process $\phi\big(Y^{m,n}_\cdot\big)$ yields that

$$\begin{aligned}
&\phi\left(Y^{m,n}_t\right) + \frac{1}{2}\int_t^T \phi''\left(Y^{m,n}_s\right)|Z^{m,n}_s|^2 ds \\
&\quad = \phi\left(\xi_{m,n}\right) + \int_t^T \phi'\left(Y^{m,n}_s\right)\left(f_m(s, Y^m_s, Z^m_s) - f_n(s, Y^n_s, Z^n_s)\right)ds \\
&\qquad + \int_t^T \phi'\left(Y^{m,n}_s\right) dK^{m,n}_s - \int_t^T \phi'\left(Y^{m,n}_s\right) Z^{m,n}_s dB_s, \quad t \in [0, T]. \qquad (3.3)
\end{aligned}$$

Since $|\phi'(x)| = e^{\lambda_o\gamma|x|} - 1, \ x \in \mathbb{R}$, applying Young's inequality with

$$\begin{aligned}
&p_1 = \frac{\lambda}{\lambda_o}, \qquad p_2 = \frac{\lambda'}{\lambda_o} \quad \text{and} \\
&p_3 = \left(1 - \frac{1}{p_1} - \frac{1}{p_2}\right)^{-1} = \frac{\lambda\lambda'}{\lambda\lambda' - \lambda_o(\lambda + \lambda')} = p_o,
\end{aligned} \qquad (3.4)$$

we can deduce from (3.1), the Burkholder–Davis–Gundy inequality, and (3.2) that

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|\int_0^t \phi'\left(Y^{m,n}_s\right) Z^{m,n}_s dB_s\right|\right] \\
&\quad \le c_0 E\left[e^{\lambda_o\gamma(\mathscr{L}_* + \mathscr{Y}_*)}\left(1 + \int_0^T |Z^{m,n}_s|^2 ds\right)\right] \\
&\quad \le c_{\lambda,\lambda'}\, E\left[e^{\lambda_o p_1\gamma\mathscr{L}_*} + e^{\lambda_o p_2\gamma\mathscr{Y}_*} + \left(1 + \int_0^T |Z^{m,n}_s|^2 ds\right)^{p_o}\right] \\
&\quad \le c_{\lambda,\lambda'}\big(1 + \Xi\big) < \infty. \qquad (3.5)
\end{aligned}$$

Thus $\int_0^\cdot \phi'\left(Y^{m,n}_s\right) Z^{m,n}_s dB_s$ is a uniformly integrable martingale. Letting $t = 0$, taking expectation in (3.3), and using (H1) we obtain

$$\begin{aligned}
&E\left[\phi\big(Y^{m,n}_0\big)\right] + \frac{1}{2}E\int_0^T \phi''\big(Y^{m,n}_s\big)|Z^{m,n}_s|^2 ds \\
&\quad \le E\left[\phi\big(\xi_{m,n}\big)\right] + E\int_0^T \phi'\big(Y^{m,n}_s\big)dK^{m,n}_s + E\int_0^T |\phi'(Y^{m,n}_s)|\Bigg(2\alpha + \beta|Y^m_s| + \beta|Y^n_s| \\
&\qquad + \frac{1}{2}\gamma\left(2|Z^{m,n}_s|^2 + (\lambda_o - 2)|Z_s - Z^n_s|^2 + \left(3 + \frac{9}{\lambda_o - 5}\right)|Z_s|^2\right)\Bigg)ds, \qquad (3.6)
\end{aligned}$$

where we used the fact that $|Z^m_s|^2 + |Z^n_s|^2 \le 2|Z^{m,n}_s|^2 + 3|Z^n_s|^2$ and $|Z^n_s|^2 \le \big(1 + \frac{\lambda_o - 5}{3}\big)|Z_s - Z^n_s|^2 + \big(1 + \frac{3}{\lambda_o - 5}\big)|Z_s|^2$.

Since $P\big(|Y^{m,n}_t| \le |Y_t - Y^n_t| \le |Y_t - Y^1_t|, \ \forall\, t \in [0, T]\big) = 1$, the monotonicity of ϕ and $|\phi'|$ implies that P-a.s.

$$\begin{aligned}
&\phi(\xi_{m,n}) \le \phi(\xi - \xi_n) \quad \text{and} \quad |\phi'(Y^{m,n}_t)| \le |\phi'(Y_t - Y^n_t)| \le |\phi'(Y_t - Y^1_t)|, \\
&\quad t \in [0, T]. \qquad (3.7)
\end{aligned}$$

Similar, it holds P-a.s. that

$$|\phi'(L_t^{m,n})| \le |\phi'(L_t - L_t^n)| \le |\phi'(L_t - L_t^1)|, \quad t \in [0,T]. \tag{3.8}$$

We also see from (3.5) that

$$E\int_0^T |\phi'(Y_s^{m,n})|\,|Z_s^{m,n}|^2 ds \le E\left[\sup_{s\in[0,T]} |\phi'(Y_s^{m,n})| \int_0^T |Z_s^{m,n}|^2 ds\right] < \infty, \tag{3.9}$$

which together with (3.6), (3.7) and (3.1) implies that

$$\begin{aligned} E\int_0^T \left(\phi'' - 2\gamma|\phi'|\right)\left(Y_s^{m,n}\right)|Z_s^{m,n}|^2 ds &\le 2E\left[\phi\left(\xi - \xi_n\right)\right] + 2E\int_0^T \phi'\left(Y_s^{m,n}\right) dK_s^{m,n} \\ &+ E\int_0^T |\phi'(Y_s - Y_s^n)| \Bigg(4\alpha + 2\beta(\mathscr{L}_s + \mathscr{Y}_s) + (\lambda_o - 2)\gamma|Z_s - Z_s^n|^2 \\ &+ \left(3 + \frac{9}{\lambda_o - 5}\right)\gamma|Z_s|^2\Bigg) ds. \end{aligned} \tag{3.10}$$

Now we estimate the second term on the right-hand-side of (3.10) by two cases of assumption (M3). Assume (M3a) first. Since ϕ' is an increasing and continuous function on $\mathbb{R}$, the flat-off condition of (Y^m, Z^m, K^m), (3.2) and (3.8) imply that

$$\begin{aligned} E\int_0^T \phi'\left(Y_s^{m,n}\right) dK_s^{m,n} &\le E\int_0^T \phi'\left(Y_s^{m,n}\right) dK_s^m \le E\int_0^T \phi'\left(Y_s^m - L_s^n\right) dK_s^m \\ &= E\int_0^T \mathbf{1}_{\{Y_s^m = L_s^m\}} \phi'\left(L_s^{m,n}\right) dK_s^m \\ &\le \left\|K_T^m\right\|_{\mathbb{L}^{p_o}(\mathcal{F}_T)} \left\|\phi'\left(L^{m,n}\right)\right\|_{\mathbb{C}_{\mathbf{F}}^{\frac{p_o}{p_o-1}}[0,T]} \\ &\le c_{\lambda,\lambda'}\, \Xi^{\frac{1}{p_o}} \left\|\phi'\left(L - L^n\right)\right\|_{\mathbb{C}_{\mathbf{F}}^{\frac{p_o}{p_o-1}}[0,T]}. \end{aligned} \tag{3.11}$$

On the other hand, it holds for the case of (M3b) that

$$\begin{aligned} E\int_0^T \phi'\left(Y_s^{m,n}\right) dK_s^{m,n} &\le -E\int_0^T \mathbf{1}_{\{Y_s^n = L_s^n\}} \phi'\left(L_s^{m,n}\right) dK_s^n \\ &\le c_{\lambda,\lambda'}\, \Xi^{\frac{1}{p_o}} \left\|\phi'\left(L - L^n\right)\right\|_{\mathbb{C}_{\mathbf{F}}^{\frac{p_o}{p_o-1}}[0,T]}. \end{aligned}$$

Since $\left\{\sqrt{|\phi'(Y^{m,n})|} Z^{m,n}\right\}_{m\ge n}$ weakly converges to

$$\sqrt{|\phi'(Y - Y^n)|}\left(Z - Z^n\right) \text{ in } \mathbb{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d), \tag{3.12}$$

which is proved in Appendix A.1, Theorem 5.1.1(ii) of [26] shows that

$$E\int_0^T |\phi'(Y_s - Y_s^n)|\,|Z_s - Z_s^n|^2 ds \le \varliminf_{m\to\infty} E\int_0^T |\phi'(Y_s^{m,n})|\,|Z_s^{m,n}|^2 ds. \tag{3.13}$$

As $\mathbb{H}_{\mathbf{F}}^{2,2p_o}([0,T];\mathbb{R}^d) \subset \mathbb{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d)$, the sequence $\{Z^m\}_{m\ge n}$ also weakly converges to Z in $\mathbb{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d)$. Applying Theorem 5.1.1(ii) of [26] once again, we can deduce from (3.10), (3.11) and (3.13) that

$$\lambda_o\gamma E\int_0^T |Z_s-Z^n_s|^2ds \le \varlimsup_{m\to\infty} E\int_0^T \big(\phi''-\lambda_o\gamma|\phi'|\big)\big(Y^{m,n}_s\big)|Z^{m,n}_s|^2ds$$
$$\big(\because \phi''(x)-\lambda_o\gamma|\phi'(x)|=\lambda_o\gamma,\ \forall\, x\in\mathbb{R}\big)$$
$$=\varlimsup_{m\to\infty} E\int_0^T \left(\phi''-2\gamma|\phi'|\right)\left(Y^{m,n}_s\right)|Z^{m,n}_s|^2ds-(\lambda_o-2)\gamma$$
$$\times \varliminf_{m\to\infty} E\int_0^T |\phi'(Y^{m,n}_s)||Z^{m,n}_s|^2ds$$
$$\le 2E\big[\phi(\xi-\xi_n)\big]+c_{\lambda,\lambda'}\,\Xi^{\frac{1}{p_o}}\|\phi'(L-L^n)\|_{\mathbb{C}_{\mathbf{F}}^{\frac{p_o}{p_o-1}}[0,T]}$$
$$+c_{\lambda,\lambda'}E\int_0^T |\phi'(Y_s-Y^n_s)|\big(1+\mathscr{L}_s+\mathscr{Y}_s+|Z_s|^2\big)ds. \tag{3.14}$$

Since $\lambda_o<\frac{\lambda\lambda'}{\lambda+\lambda'}$, it follows that $\lambda'>\frac{\lambda_o\lambda}{\lambda-\lambda_o}$. Applying Young's inequality with $\widetilde{p}=\frac{\lambda}{\lambda_o}$ and $\widetilde{q}=\frac{\lambda}{\lambda-\lambda_o}$, we can deduce from (3.1) that P-a.s., $0\le\phi(\xi-\xi_n)\le\frac{1}{\lambda_o\gamma}e^{\lambda_o\gamma(\mathscr{L}_*+\mathscr{Y}_*)}\le c_{\lambda,\lambda'}\big(e^{\lambda\gamma\mathscr{L}_*}+e^{\lambda'\gamma\mathscr{Y}_*}\big)$, $\forall\, n\in\mathbb{N}$. As $E\left[e^{\lambda\gamma\mathscr{L}_*}+e^{\lambda'\gamma\mathscr{Y}_*}\right]<\infty$, the continuity of ϕ and the Dominated Convergence Theorem imply that

$$\lim_{n\to\infty}\downarrow\ E\left[\phi\left(\xi-\xi_n\right)\right]=0. \tag{3.15}$$

In light of Dini's Theorem and (M3), it holds P-a.s. that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|L^n_t-L_t|=\lim_{n\to\infty}\sup_{t\in[0,T]}|Y^n_t-Y_t|=0. \tag{3.16}$$

The continuity of ϕ' implies that

$$0=\lim_{n\to\infty}|\phi'(\sup_{t\in[0,T]}|L_t-L^n_t|)|=\lim_{n\to\infty}\sup_{t\in[0,T]}\exp\left\{\lambda_o\gamma|L_t-L^n_t|\right\}-1$$
$$=\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\phi'\big(L_t-L^n_t\big)\right|,\qquad P\text{-a.s.}$$

It follows from (3.8) that P-a.s. $\sup_{t\in[0,T]}\left|\phi'\left(L_t-L^n_t\right)\right|^{\frac{p_o}{p_o-1}}\le\sup_{t\in[0,T]}\left|\phi'\left(L_t-L^1_t\right)\right|^{\frac{p_o}{p_o-1}}$, $\forall\, n\in\mathbb{N}$. Applying Young's inequality with $\widetilde{p}=\frac{\lambda+\lambda'}{\lambda'}$ and $\widetilde{q}=\frac{\lambda+\lambda'}{\lambda}$, one can deduce from (3.1) that

$$E\left[\sup_{t\in[0,T]}\left|\phi'\left(L_t-L^1_t\right)\right|^{\frac{p_o}{p_o-1}}\right]\le E\left[e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_*+\mathscr{Y}_*)}\right]$$
$$\le c_{\lambda,\lambda'}E\left[e^{\lambda\gamma\mathscr{L}_*}+e^{\lambda'\gamma\mathscr{Y}_*}\right]<\infty. \tag{3.17}$$

The Dominated Convergence Theorem then implies that

$$\lim_{n\to\infty}\downarrow\ E\left[\sup_{t\in[0,T]}\left|\phi'\left(L_t-L^n_t\right)\right|^{\frac{p_o}{p_o-1}}\right]=0. \tag{3.18}$$

Next, we can deduce from (3.1) and (3.7) that P-a.s.

$$|\phi'(Y_t-Y^n_t)|\big(1+\mathscr{L}_t+\mathscr{Y}_t+|Z_t|^2\big)$$
$$\le c_{\lambda,\lambda'}e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_t+\mathscr{Y}_t)}+|\phi'(Y_t-Y^1_t)|\,|Z_t|^2,\quad \forall\, t\in[0,T],\ \forall\, n\in\mathbb{N}.$$

Similar to (3.17), one has $E\Big[\sup_{t\in[0,T]}\big|\phi'\big(Y_t-Y^1_t\big)\big|^{\frac{p_o}{p_o-1}}\Big]\le c_{\lambda,\lambda'}\Xi<\infty$, which together with Young's inequality and (3.17) shows that

$$E\int_0^T\Big(e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_t+\mathscr{Y}_t)}+|\phi'(Y_t-Y^1_t)|\,|Z_t|^2\Big)dt$$
$$\le c_{\lambda,\lambda'}E\left[e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_*+\mathscr{Y}_*)}+\sup_{t\in[0,T]}|\phi'(Y_t-Y^1_t)|^{\frac{p_o}{p_o-1}}+\left(\int_0^T|Z_t|^2dt\right)^{p_o}\right]<\infty.$$

Then the continuity of ϕ' and the Dominated Convergence Theorem imply that $\lim_{n\to\infty}E\int_0^T|\phi'(Y_s-Y^n_s)|\big(1+\mathscr{L}_t+\mathscr{Y}_t+|Z_t|^2\big)ds=0$, which together with (3.15), (3.18) and Doob's martingale inequality leads to that

$$\lim_{n\to\infty}E\int_0^T|Z_s-Z^n_s|^2ds=0\quad\text{and}\quad\lim_{n\to\infty}E\left[\sup_{t\in[0,T]}\left|\int_0^t(Z_s-Z^n_s)dB_s\right|^2\right]=0\tag{3.19}$$

(3) Next, we show that $Y\in\mathbb{E}^{\lambda\gamma,\lambda'\gamma}_{\mathbf{F}}[0,T]$.

By (3.19), we can extract a subsequence of $\{Z^n\}_{n\in\mathbb{N}}$ (we still denote it by $\{Z^n\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty}Z^n_t=Z_t$, $dt\otimes dP$-a.e. In fact, we can choose this subsequence so that $Z^*\triangleq\sup_{n\in\mathbb{N}}|Z^n|\in\mathbb{H}^2_{\mathbf{F}}[0,T]$; see [16] or [14, Lemma 2.5]. By (M2), it holds $dt\otimes dP$-a.e. that

$$f\big(t,\omega,y,z\big)=\lim_{n\to\infty}f_n\big(t,\omega,y,z\big),\quad\forall\,(y,z)\in\mathbb{R}\times\mathbb{R}^d,\tag{3.20}$$

which together with the measurability of $f_n,n\in\mathbb{N}$ implies that f is also $\mathscr{P}\times\mathscr{B}(\mathbb{R})\times\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable. Moreover, we see from (3.20) and (M1) that f also satisfies (H1). For $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$, the continuity of mapping $f(t,\omega,\cdot,\cdot)$ shows that

$$\lim_{n\to\infty}\big|f\big(t,\omega,Y^n_t(\omega),Z^n_t(\omega)\big)-f\big(t,\omega,Y_t(\omega),Z_t(\omega)\big)\big|=0.\tag{3.21}$$

On the other hand, (M2) implies that for $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$,

$$0\le\varlimsup_{n\to\infty}\big|f_n\big(t,\omega,Y^n_t(\omega),Z^n_t(\omega)\big)-f\big(t,\omega,Y^n_t(\omega),Z^n_t(\omega)\big)\big|$$
$$\le\lim_{n\to\infty}\Big(\sup\Big\{\big|f_n\big(t,\omega,y,z\big)-f\big(t,\omega,y,z\big)\big|:|y|\le|Y^1_t(\omega)|\vee|Y_t(\omega)|<\infty,$$
$$|z|\le Z^*_t(\omega)<\infty\Big\}\Big)=0,$$

which together with (3.21) yields that $dt\otimes dP$-a.e.

$$\lim_{n\to\infty}\big|f_n\big(t,\omega,Y^n_t(\omega),Z^n_t(\omega)\big)-f\big(t,\omega,Y_t(\omega),Z_t(\omega)\big)\big|=0.\tag{3.22}$$

Moreover, (H1) and (3.1) show that $dt\otimes dP$-a.e.

$$\big|f_n(t,Y^n_t,Z^n_t)-f\big(t,Y_t,Z_t\big)\big|\le2\alpha+2\beta(\mathscr{L}_*+\mathscr{Y}_*)+\frac{\gamma}{2}\big(|Z^*_t|^2+|Z_t|^2\big),$$
$$\forall\,n\in\mathbb{N}.\tag{3.23}$$

Let us assume that except on a P-null set $\mathscr{N}$, (3.22), (3.23) hold for a.e. $t\in[0,T]$ and $\mathscr{L}_*+\mathscr{Y}_*+\int_0^T\big(|Z^*_t|^2+|Z_t|^2\big)dt<\infty$. For any $\omega\in\mathscr{N}^c$, the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty}\int_0^T \big|f_n\big(t,\omega,Y_t^n(\omega),Z_t^n(\omega)\big)-f\big(t,\omega,Y_t(\omega),Z_t(\omega)\big)\big|\,dt=0. \tag{3.24}$$

For any $n\in\mathbb{N}$, integrating with respect to t in (3.23) yields that

$$\int_0^T \big|f_n\big(t,\omega,Y_t^n(\omega),Z_t^n(\omega)\big)-f\big(t,\omega,Y_t(\omega),Z_t(\omega)\big)\big|\,dt \le c_{\lambda,\lambda'}e^{\frac{\lambda\lambda'\gamma}{(\lambda+\lambda')\,p_o}(\mathscr{L}_*(\omega)+\mathscr{Y}_*(\omega))}+\frac{\gamma}{2}\int_0^T\Big(|Z_t^n(\omega)|^2+|Z_t(\omega)|^2\Big)dt.$$

Then it follows from (3.17) and (3.2) that

$$E\Bigg[\bigg(\int_0^T \big|f_n\big(t,,Y_t^n,Z_t^n\big)-f\big(t,Y_t,Z_t\big)\big|\,dt\bigg)^{p_o}\Bigg]\le c_{\lambda,\lambda'}\Xi+c_{\lambda,\lambda'}E\Bigg[\bigg(\int_0^T|Z_t|^2dt\bigg)^{p_o}\Bigg] < \infty,\quad \forall n\in\mathbb{N},$$

which implies that $\Big\{\Big(\int_0^T \big|f_n\big(t,,Y_t^n,Z_t^n\big)-f\big(t,Y_t,Z_t\big)\big|\,dt\Big)^{\frac{1+p_o}{2}}\Big\}_{n\in\mathbb{N}}$ is uniformly integrable sequence in $\mathbb{L}^1(\mathcal{F}_T)$. Hence, one can deduce from (3.24) that

$$\lim_{n\to\infty}E\Bigg[\bigg(\int_0^T \big|f_n\big(t,Y_t^n,Z_t^n\big)-f\big(t,Y_t,Z_t\big)\big|\,dt\bigg)^{\frac{1+p_o}{2}}\Bigg]=0. \tag{3.25}$$

Since $\phi(x)\ge\frac{\lambda_o\gamma}{2}|x|^2$ and $|\phi'(x)|\ge\lambda_o\gamma|x|,\ x\in\mathbb{R}$, we can deduce from (3.15) and (3.18) that

$$\lim_{n\to\infty}\downarrow\ E\Big[\big(\xi-\xi_n\big)^2\Big]=0\quad\text{and}\quad \lim_{n\to\infty}\downarrow\ \|L-L^n\|_{\mathbb{C}_{\mathbf{F}}^{\frac{p_o}{p_o-1}}[0,T]}=0. \tag{3.26}$$

Moreover, for any $p\in[1,\infty)$, (3.1) and (3.17) imply that

$$\|Y^n\|^p_{\mathbb{C}^p_{\mathbf{F}}[0,T]}\le E\Big[\big(\mathscr{L}_*+\mathscr{Y}_*\big)^p\Big]\le c_{\lambda,\lambda',p}\,E\Big[e^{\frac{\lambda\lambda'}{\lambda+\lambda'}\gamma(\mathscr{L}_*+\mathscr{Y}_*)}\Big]\le c_{\lambda,\lambda',p}\,\Xi,\quad \forall n\in\mathbb{N}. \tag{3.27}$$

Now for any $m,n\in\mathbb{N}$ with $m\ge n$, applying Itô's formula to the process $\big(Y^{m,n}_\cdot\big)^2$ yields that

$$(Y_t^{m,n})^2\le\xi_{m,n}^2+2\int_t^T Y_s^{m,n}\big(f_m(s,Y_s^m,Z_s^m)-f_n(s,Y_s^n,Z_s^n)\big)ds+2\int_t^T Y_s^{m,n}dK_s^{m,n} -2\int_t^T Y_s^{m,n}Z_s^{m,n}dB_s,\quad t\in[0,T].$$

The flat-off condition of (Y^m,Z^m,K^m) implies that P-a.s.

$$\int_t^T Y_s^{m,n}dK_s^{m,n} \le\begin{cases}\displaystyle\int_t^T(Y_s^m-L_s^n)dK_s^m=\int_t^T L_s^{m,n}dK_s^m\le K_T^m\sup_{s\in[0,T]}|L_s^{m,n}|, & t\in[0,T],\quad\text{in case (M3a)};\\ \displaystyle\int_t^T(Y_s^n-L_s^m)dK_s^n=-\int_t^T L_s^{m,n}dK_s^n\le K_T^n\sup_{s\in[0,T]}|L_s^{m,n}|, & t\in[0,T],\quad\text{in case (M3b)}.\end{cases}$$

Then Hölder's inequality, (3.2), the Burkholder–Davis–Gundy inequality and (3.27) imply that

$$E\Big[\sup_{t\in[0,T]}|Y^{m,n}_t|^2\Big] \le E\Big[\xi^2_{m,n}\Big] + c_{\lambda,\lambda'}\,\Xi^{\frac{p_o-1}{p_o+1}}\,\big\|f_m(\cdot,Y^m_\cdot,Z^m_\cdot) - f_n(\cdot,Y^n_\cdot,Z^n_\cdot)\big\|_{\widehat{\mathbb{H}}^{1,\frac{1+p_o}{2}}_{\mathbf{F}}([0,T];\mathbb{R})} + c_{\lambda,\lambda'}\,\Xi^{\frac{1}{p_o}}\,\|L^{m,n}\|_{\mathbb{C}^{\frac{p_o}{p_o-1}}_{\mathbf{F}}[0,T]} + c_{\lambda,\lambda'}\,\Xi^{\frac{1}{2}}\,\big\|Z^{m,n}\big\|_{\mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)}.$$

Hence, we can deduce from (3.26), (3.25) and (3.19) that $\{Y^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}^2_{\mathbf{F}}[0,T]$. Let $\widetilde{Y}$ be its limit in $\mathbb{C}^2_{\mathbf{F}}[0,T]$. There is a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{i\to\infty}\downarrow\sup_{t\in[0,T]}|Y^{n_i}_t-\widetilde{Y}_t|=0$, P-a.s., which together with (M3) implies that $P\big(\widetilde{Y}_t = Y_t,\ \forall\, t\in[0,T]\big)=1$. So Y is a continuous process satisfying

$$\lim_{n\to\infty}\downarrow\sup_{t\in[0,T]}|Y^n_t-Y_t|=0,\qquad P\text{-a.s.} \tag{3.28}$$

Since $E\Big[e^{\lambda\gamma Y^-_*}+e^{\lambda'\gamma Y^+_*}\Big]\le E\Big[e^{\lambda\gamma\mathscr{L}_*}+e^{\lambda'\gamma\mathscr{Y}_*}\Big]<\infty$ by (3.1), we see that $Y\in\mathbb{E}^{\lambda\gamma,\lambda'\gamma}_{\mathbf{F}}[0,T]$.

(4) Now let us define an $\mathbf{F}$-adapted, continuous process $K_t \triangleq Y_0 - Y_t - \int_0^t f(s,Y_s,Z_s)\,ds + \int_0^t Z_s dB_s$, $t\in[0,T]$. By (3.25) and (3.19), $\big\{(Y^n,Z^n)\big\}_{n\in\mathbb{N}}$ has a subsequence (we still denote it by $\big\{(Y^n,Z^n)\big\}_{n\in\mathbb{N}}$) such that P-a.s.

$$\lim_{n\to\infty}\left\{\int_0^T \big|f_n(t,Y^n_t,Z^n_t)-f\big(t,Y_t,Z_t\big)\big|\,dt + \sup_{t\in[0,T]}\left|\int_0^t\big(Z^n_s-Z_s\big)dB_s\right|\right\}=0.$$

This together with (3.28) leads to that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|K^n_t-K_t|=0,\qquad P\text{-a.s.}, \tag{3.29}$$

which implies that K is an increasing process. To wit, $K\in\mathbb{K}_{\mathbf{F}}[0,T]$. Letting $n\to\infty$ in (3.1) yields that P-a.s.

$$L_t\le Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds + K_T-K_t-\int_t^T Z_s dB_s,\qquad t\in[0,T].$$

(5) For P-a.s. $\omega\in\Omega$, since $\big\{\big(Y^n_\cdot(\omega),L^n_\cdot(\omega),K^n_\cdot(\omega)\big)\big\}_{n\in\mathbb{N}}$ uniformly converges $\big(Y_\cdot(\omega),L_\cdot(\omega),K_\cdot(\omega)\big)$ in t by (3.16) and (3.29), one can deduce from standard arguments and the flat-off condition of each $\big(Y^n_t,L^n_t,K^n_t\big)$ that

$$\int_0^T\big(Y_t(\omega)-L_t(\omega)\big)dK_t(\omega)=\lim_{n\to 0}\int_0^T\big(Y^n_t(\omega)-L^n_t(\omega)\big)dK^n_t(\omega)=0,$$

which together with the previous steps show that (Y,Z,K) is a solution of the quadratic RBSDE(ξ,f,L). Since $Y\in\mathbb{E}^{\lambda\gamma,\lambda'\gamma}_{\mathbf{F}}[0,T]$, Proposition 2.2 shows that $(Z,K)\in\mathbb{H}^{2,2p}_{\mathbf{F}}([0,T];\mathbb{R}^d)\times\mathbb{K}^p_{\mathbf{F}}[0,T]$ for any $p\in\Big(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\Big)$. □

As a consequence of Theorem 3.1, we have the following existence result.

Theorem 3.2. *Let* (ξ,f,L) *be a parameter set such that* f *satisfies* (H1) *and that*

$$\textit{For } dt\otimes dP\textit{-a.e. } (t,\omega)\in[0,T]\times\Omega,\ \textit{the mapping } f(t,\omega,\cdot,\cdot)\ \textit{is continuous.} \tag{3.30}$$

If $E\Big[e^{\lambda\gamma L_*^-}+e^{\lambda'\gamma e^{\beta T}(\xi^+\vee L_*^+)}\Big]<\infty$ *for some* $\lambda,\lambda'>6$ *with* $\frac{1}{\lambda}+\frac{1}{\lambda'}<\frac{1}{6}$*, then the quadratic* RBSDE (ξ,f,L) *admits a solution* $(Y,Z,K)\in\cap_{p\in\left(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\right)}\mathbb{E}_{\mathbf{F}}^{\lambda\gamma,\lambda'\gamma}[0,T]\times\mathbb{H}_{\mathbf{F}}^{2,2p}([0,T];\mathbb{R}^d)\times\mathbb{K}_{\mathbf{F}}^p[0,T]$ *that satisfies* (2.1).

In addition, if $\xi^+\vee L_*\in\mathbb{L}^e(\mathcal{F}_T)$*, then this solution* (Y,Z,K) *belongs to* $\mathbb{S}_{\mathbf{F}}^p[0,T]$ *for all* $p\in[1,\infty)$. *More precisely, for any* $p\in(1,\infty)$ *we have*

$$E\left[e^{p\gamma Y_*}\right]\le E\left[e^{p\gamma L_*^-}\right]+c_p\,E\left[e^{p\gamma e^{\beta T}(\xi^+\vee L_*^+)}\right]<\infty;$$
$$E\left[\left(\int_0^T|Z_s|^2ds\right)^p+K_T^p\right]\le c_p\,E\left[e^{3p\gamma Y_*}\right]<\infty. \tag{3.31}$$

Sketch of the proof. For any $i,n\in\mathbb{N}$, we set $\xi_{i,n}\triangleq\big(\xi\vee(-i)\big)\wedge n$ and $L^{i,n}\triangleq\big(L\vee(-i)\big)\wedge n$. Theorem 1 of [15] shows that the quadratic RBSDE$\big(\xi^{i,n},f,L^{i,n}\big)$ admits a maximal bounded solution $\big(Y^{i,n},Z^{i,n},K^{i,n}\big)\in\mathbb{C}_{\mathbf{F}}^\infty[0,T]\times\mathbb{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d)\times\mathbb{K}_{\mathbf{F}}[0,T]$. Letting $n\to\infty$ and then letting $i\to\infty$, we can deduce from Theorem 3.1 as well as Proposition 2.1 that the RBSDE (ξ,f,L) has such a solution (Y,Z,K). If $\xi^+\vee L_*\in\mathbb{L}^e(\mathcal{F}_T)$, one can use Proposition 2.2 and Doob's martingale inequality to obtain (3.31). See [2] for details. □

4. Uniqueness

In the rest of the paper, we impose two more hypotheses on generator f which together imply (3.30).

(H2) f is *Lipschitz* in y: For some $\kappa\ge 0$, it holds $dt\otimes dP$-a.e. that

$$|f(t,\omega,y_1,z)-f(t,\omega,y_2,z)|\le\kappa|y_1-y_2|,\quad\forall\,y_1,y_2\in\mathbb{R},\ \forall\,z\in\mathbb{R}^d. \tag{4.1}$$

(H3) f is *concave* in z: i.e., it holds $dt\otimes dP$-a.e. that

$$f\big(t,\omega,y,\theta z_1+(1-\theta)z_2\big)\ge\theta f(t,\omega,y,z_1)+(1-\theta)f(t,\omega,y,z_2),$$
$$\forall\,(\theta,y)\in(0,1)\times\mathbb{R},\ \forall\,z_1,z_2\in\mathbb{R}^d. \tag{4.2}$$

From now on, for any $\lambda\ge 0$ the generic constant c_λ also depends on κ implicitly. The following uniqueness result derives from a Legendre–Fenchel transformation argument, which was used in [9], [10, Section 7] and [1, Section 4].

Theorem 4.1. *Let* (ξ,f,L) *be a parameter set such that* f *satisfies* (H2) *and* (H3). *Assume that for three constants* $\alpha,\beta\ge 0$ *and* $\gamma>0$*, it holds* $dt\otimes dP$*-a.e. that*

$$f(t,\omega,y,z)\ge-\alpha-\beta|y|-\frac{\gamma}{2}|z|^2,\quad\forall\,(y,z)\in\mathbb{R}\times\mathbb{R}^d. \tag{4.3}$$

Then the RBSDE(ξ,f,L) *has at most one solution* $(Y,Z,K)\in\mathbb{E}_{\mathbf{F}}^{\lambda,\lambda'}[0,T]\times\widehat{\mathbb{H}}_{\mathbf{F}}^{2,\mathrm{loc}}([0,T];\mathbb{R}^d)\times\mathbb{K}_{\mathbf{F}}[0,T]$ *with* $\lambda\in(\gamma,\infty)$ *and* $\lambda'\in(0,\infty)$.

Proof. Suppose that the RBSDE(ξ,f,L) has two solutions $\big\{(Y^i,Z^i,K^i)\big\}_{i=1,2}\subset\mathbb{E}_{\mathbf{F}}^{\lambda_i,\lambda_i'}[0,T]\times\widehat{\mathbb{H}}_{\mathbf{F}}^{2,\mathrm{loc}}([0,T];\mathbb{R}^d)\times\mathbb{K}_{\mathbf{F}}[0,T]$ with $\lambda_i\in(\gamma,\infty)$ and $\lambda_i'\in(0,\infty)$. We set $\lambda\triangleq\lambda_1\wedge\lambda_2$ and $\lambda'\triangleq\lambda_1'\wedge\lambda_2'$. Clearly, $-f$ is convex in z. For any $(t,\omega,y)\in[0,T]\times\Omega\times\mathbb{R}$, it is well-known that the Legendre–Fenchel transformation of $f(t,\omega,y,\cdot)$: $\widehat{f}(t,\omega,y,\mathfrak{q}) \triangleq \sup_{z\in\mathbb{R}^d}\big(\langle \mathfrak{q}, z\rangle + f(t,\omega,y,z)\big)$, $\forall\, \mathfrak{q} \in \mathbb{R}^d$ is an $\mathbb{R}\cup\{\infty\}$-valued, convex and lower semicontinuous function. Let $\mathfrak{N}$ be the $dt\otimes dP$-null set except on which (4.1)–(4.3) hold. Given $(t,\omega)\in\mathfrak{N}^c$, $\widehat{f}$ has the following properties:

$$\text{(1) By (4.3),}\quad \widehat{f}(t,\omega,y,\mathfrak{q}) \ge -\alpha - \beta|y| + \frac{1}{2\gamma}|\mathfrak{q}|^2, \quad \forall\,(y,z)\in\mathbb{R}\times\mathbb{R}^d. \tag{4.4}$$

(2) For any $\mathfrak{q}\in\mathbb{R}^d$, if $\widehat{f}(t,\omega,y,\mathfrak{q})<\infty$ for some $y\in\mathbb{R}$, then (H2) implies that for any $y'\in\mathbb{R}$,

$$\widehat{f}(t,\omega,y',\mathfrak{q})<\infty \quad \text{and} \quad |\widehat{f}(t,\omega,y,\mathfrak{q}) - \widehat{f}(t,\omega,y',\mathfrak{q})| \le \kappa|y-y'|. \tag{4.5}$$

(3) For any $y\in\mathbb{R}$, since $-f(t,\omega,y,\cdot)$ is convex on $\mathbb{R}^d$, the conjugacy relation shows that

$$-f(t,\omega,y,z) = \sup_{\mathfrak{q}\in\mathbb{R}^d}\big(\langle z,\mathfrak{q}\rangle - \widehat{f}(t,\omega,y,\mathfrak{q})\big), \quad \forall\, z\in\mathbb{R}^d. \tag{4.6}$$

Moreover, the convexity of $-f(t,\omega,y,\cdot)$ on $\mathbb{R}^d$ implies its continuity on $\mathbb{R}^d$, thus $\widehat{f}(t,\omega,y,\mathfrak{q}) = \sup_{z\in\mathbb{Q}^d}\big(\langle\mathfrak{q},z\rangle + f(t,\omega,y,z)\big)$, $\forall\,\mathfrak{q}\in\mathbb{R}^d$, which implies that $\widehat{f}$ is $\mathscr{P}\times\mathscr{B}(\mathbb{R})\times\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable.

(4) For any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, let $\partial(-f)(t,\omega,y,z)$ denote the subdifferential of the function $-f(t,\omega,y,\cdot)$ at z (see e.g., [25]). It is a non-empty convex compact subset of $\mathfrak{q}\in\mathbb{R}^d$ such that $-f(t,\omega,y,z') + f(t,\omega,y,z) \ge \langle\mathfrak{q}, z'-z\rangle$ for any $z'\in\mathbb{R}^d$, to wit,

$$\langle\mathfrak{q},z\rangle + f(t,\omega,y,z) = \sup_{z'\in\mathbb{R}^d}\big(\langle\mathfrak{q},z'\rangle + f(t,\omega,y,z')\big) = \widehat{f}(t,\omega,y,\mathfrak{q}). \tag{4.7}$$

Let $i=1,2$. For any $(t,\omega)\in\mathfrak{N}^c$, we choose a $\mathfrak{q}^i(t,\omega)\in\partial(-f)\big(t,\omega,Y^i_t(\omega),Z^i_t(\omega)\big)$. By (4.7),

$$\widehat{f}\big(t,\omega,Y^i_t(\omega),\mathfrak{q}^i(t,\omega)\big) = \langle Z^i_t(\omega),\mathfrak{q}^i(t,\omega)\rangle + f\big(t,\omega,Y^i_t(\omega),Z^i_t(\omega)\big) < \infty. \tag{4.8}$$

Thanks to the Measurable Selection Theorem (see e.g., Lemma 1 of [4] or Lemma 16.34 of [12]), there exists an $\mathbf{F}$-progressively measurable process $\widetilde{\mathfrak{q}}^i$ such that

$$f\big(t,\omega,Y^i_t(\omega),Z^i_t(\omega)\big) = \widehat{f}\big(t,\omega,Y^i_t(\omega),\widetilde{\mathfrak{q}}^i_t(\omega)\big) - \langle Z^i_t(\omega),\widetilde{\mathfrak{q}}^i_t(\omega)\rangle, \quad \forall\,(t,\omega)\in\mathfrak{N}^c, \tag{4.9}$$

which together with (4.4) leads to that

$$\begin{aligned} f\big(t,\omega,Y^i_t(\omega),Z^i_t(\omega)\big) &\ge -\alpha - \beta|Y^i_t(\omega)| + \frac{1}{2\gamma}|\widetilde{\mathfrak{q}}^i_t(\omega)|^2 \\ &\quad - \frac{1}{2}\left(2\gamma|Z^i_t(\omega)|^2 + \frac{1}{2\gamma}|\widetilde{\mathfrak{q}}^i_t(\omega)|^2\right), \quad \forall\,(t,\omega)\in\mathfrak{N}^c. \end{aligned} \tag{4.10}$$

Since $(Y^i,Z^i,K^i)\in\mathbb{E}^{\lambda_i,\lambda'_i}_{\mathbf{F}}[0,T]\times\widehat{\mathbb{H}}^{2,\mathrm{loc}}_{\mathbf{F}}([0,T];\mathbb{R}^d)\times\mathbb{K}_{\mathbf{F}}[0,T]$ solves the RBSDE(ξ,f,L), it holds P-a.s. that $Y^i_* + \int_0^T |Z^i_t|^2 dt + |\int_0^T f(t,Y^i_t,Z^i_t)dt| < \infty$. Then it follows from (4.10) that

$$\frac{1}{4\gamma}\int_0^T |\widetilde{\mathfrak{q}}^i_t|^2 dt \le \int_0^T f\big(t,Y^i_t,Z^i_t\big)dt + \big(\alpha+\beta Y^i_*\big)T + \gamma\int_0^T |Z^i_t|^2 dt < \infty, \quad P\text{-a.s.} \tag{4.11}$$

Next, let us pick up an $N \in \mathbb{N}$ such that $\frac{T}{N} \le \frac{\lambda\lambda'}{2\beta(\lambda+\lambda')}\Big(\frac{1}{\gamma} - \frac{1}{\lambda}\Big)$. Let $t_0 \triangleq 0$. For $j \in \{1, \dots, N\}$, we set $t_j \triangleq \frac{jT}{N}$ and define the process $M^{i,j}_t \triangleq \exp\Big(-\int_0^t \mathbf{1}_{\{s \ge t_{j-1}\}} \widetilde{\mathfrak{q}}^i_s dB_s - \frac{1}{2}\int_0^t \mathbf{1}_{\{s \ge t_{j-1}\}} |\widetilde{\mathfrak{q}}^i_s|^2 ds\Big)$, $t \in [0, t_j]$.

Given $n \in \mathbb{N}$, we define the $\mathbf{F}$-stopping time $\tau^j_n \triangleq \inf\Big\{t \in [t_{j-1}, t_j] : \int_{t_{j-1}}^t \sum_{i=1}^2 \big(|Z^i_s|^2 + |\widetilde{\mathfrak{q}}^i_s|^2\big) ds > n\Big\} \wedge t_j$. Clearly, $\lim_{n\to\infty} \uparrow \tau^j_n = t_j$, P-a.s. by (4.11), and $\Big\{M^{i,j}_{\tau^j_n \wedge t}\Big\}_{t\in[0,t_j]}$ is a uniformly integrable martingale thanks to Novikov's Criterion. Hence, $\frac{dQ^{i,j}_n}{dP} \triangleq M^{i,j}_{\tau^j_n}$ induces a probability $Q^{i,j}_n$ that is equivalent to P. Girsanov Theorem shows that $\big\{B^{i,j,n}_t \triangleq B_t + \int_0^t \mathbf{1}_{\{t_{j-1} \le s \le \tau^j_n\}} \widetilde{\mathfrak{q}}^i_s ds\big\}_{t\in[0,t_j]}$ is a Brownian Motion under $Q^{i,j}_n$ and

$$
\begin{aligned}
E\left[M^{i,j}_{\tau^j_n} \ln M^{i,j}_{\tau^j_n}\right] &= E_{Q^{i,j}_n}\left[\ln M^{i,j}_{\tau^j_n}\right] = E_{Q^{i,j}_n}\left[-\int_{t_{j-1}}^{\tau^j_n} \widetilde{\mathfrak{q}}^i_s dB^{i,j,n}_s + \frac{1}{2}\int_{t_{j-1}}^{\tau^j_n} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right] \\
&= \frac{1}{2} E_{Q^{i,j}_n}\left[\int_{t_{j-1}}^{\tau^j_n} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right]. \qquad (4.12)
\end{aligned}
$$

It is well-known that for any $(x, \mu) \in \mathbb{R} \times (0, \infty)$, $x\mu \le e^x + \mu(\ln\mu - 1) \le e^x + \mu\ln\mu$, thus $x\mu = \lambda x \frac{\mu}{\lambda} \le e^{\lambda x} + \frac{\mu}{\lambda}\big(\ln\mu - \ln\lambda\big)$, which together with (4.12) implies that for $k = 1, 2$

$$
\begin{aligned}
E_{Q^{i,j}_n}\Big[\sup_{t\in[0,t_j]} \big(Y^k_t\big)^-\Big] &= E\Big[\sup_{t\in[0,t_j]} \big(Y^k_t\big)^- M^{i,j}_{\tau^j_n}\Big] \\
&\le E\left[e^{\lambda (Y^k)^-_*}\right] + \frac{1}{\lambda} E_{Q^{i,j}_n}\left[\ln M^{i,j}_{\tau^j_n} - \ln\lambda\right] \\
&\le \widetilde{c}^{\,k}_\lambda + \frac{1}{2\lambda} E_{Q^{i,j}_n}\left[\int_{t_{j-1}}^{\tau^j_n} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right]. \qquad (4.13)
\end{aligned}
$$

where $\widetilde{c}^{\,k}_\lambda \triangleq E\big[e^{\lambda (Y^k)^-_*}\big] + \frac{(\ln\lambda)^-}{\lambda}$. Similarly, $E_{Q^{i,j}_n}\Big[\sup_{t\in[0,t_j]}\big(Y^k_t\big)^+\Big] \le \widetilde{c}^{\,k}_{\lambda'} + \frac{1}{2\lambda'} E_{Q^{i,j}_n}\Big[\int_{t_{j-1}}^{\tau^j_n} |\widetilde{\mathfrak{q}}^i_s|^2 ds\Big]$.

We can deduce from (4.9), (4.4) and Girsanov Theorem that

$$
\begin{aligned}
Y^i_{t_{j-1}} - Y^i_{\tau^j_n} &= \int_{t_{j-1}}^{\tau^j_n} f(s, Y^i_s, Z^i_s) ds + K^i_{\tau^j_n} - K^i_{t_{j-1}} - \int_{t_{j-1}}^{\tau^j_n} Z^i_s dB_s \\
&\ge \int_{t_{j-1}}^{\tau^j_n} \Big(\widehat{f}\big(s, Y^i_s, \widetilde{\mathfrak{q}}^i_s\big) - \langle Z^i_s, \widetilde{\mathfrak{q}}^i_s\rangle\Big) ds - \int_{t_{j-1}}^{\tau^j_n} Z^i_s dB_s \\
&\ge \int_{t_{j-1}}^{\tau^j_n} \left(-\alpha - \beta|Y^i_s| + \frac{1}{2\gamma}|\widetilde{\mathfrak{q}}^i_s|^2\right) ds - \int_{t_{j-1}}^{\tau^j_n} Z^i_s dB^{i,j,n}_s, \quad P\text{-a.s.} \qquad (4.14)
\end{aligned}
$$

By Bayes' rule (see e.g., [13, Lemma 3.5.3]), $E_{Q_n^{i,j}}[Y^i_{t_{j-1}}] = E[Y^i_{t_{j-1}} M^{i,j}_{t_{j-1}}] = E[Y^i_{t_{j-1}}]$. Then taking $E_{Q_n^{i,j}}$ in (4.14), one can deduce from (4.13) that

$$\frac{1}{2\gamma} E_{Q_n^{i,j}}\left[\int_{t_{j-1}}^{\tau_n^j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right] \le E\Big[\big(Y^i\big)^+_*\Big] + \frac{\alpha T}{N} + \left(1+\frac{\beta T}{N}\right) E_{Q_n^{i,j}}\Big[\sup_{t\in[0,t_j]} \big(Y^i_t\big)^-\Big]$$
$$+ \frac{\beta T}{N} E_{Q_n^{i,j}}\Big[\sup_{t\in[0,t_j]} \big(Y^i_t\big)^+\Big]$$
$$\le \Xi + \left(\frac{1}{2\lambda} + \frac{\beta T}{N}\left(\frac{1}{2\lambda}+\frac{1}{2\lambda'}\right)\right) E_{Q_n^{i,j}}\left[\int_{t_{j-1}}^{\tau_n^j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right],$$

where $\Xi \triangleq \frac{1}{\lambda'} E\Big[e^{\lambda'(Y^i)^+_*}\Big] + \frac{\alpha T}{N} + (1+\beta T)\widetilde{c}^{\,i}_\lambda + \beta T \widetilde{c}^{\,i}_{\lambda'}$. It follows from (4.12) and the setting of N that $\frac{1}{2}\big(\frac{1}{\gamma} - \frac{1}{\lambda}\big) E\big[M^{i,j}_{\tau_n^j} \ln M^{i,j}_{\tau_n^j}\big] = \frac{1}{4}\big(\frac{1}{\gamma}-\frac{1}{\lambda}\big) E_{Q_n^{i,j}}\Big[\int_{t_{j-1}}^{\tau_n^j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\Big] \le \Xi$. In light of de la Vallée–Poussin's lemma, $\Big\{M^{i,j}_{\tau_n^j}\Big\}_{n\in\mathbb{N}}$ is uniformly integrable. Hence, $E\Big[M^{i,j}_{t_j}\Big] = \lim_{n\to\infty} E\Big[M^{i,j}_{\tau_n^j}\Big] = 1$, which shows that $M^{i,j}$ is a martingale. Thus $\frac{dQ^{i,j}}{dP} \triangleq M^{i,j}_{t_j}$ induces a probability $Q^{i,j}$ that is equivalent to P, and $\big\{B^{i,j}_t \triangleq B_t + \int_0^t \mathbf{1}_{\{s\ge t_{j-1}\}} \widetilde{\mathfrak{q}}^i_s ds\big\}_{t\in[0,t_j]}$ is a Brownian Motion under $Q^{i,j}$. Then Fatou's lemma implies that

$$E_{Q^{i,j}}\left[\int_{t_{j-1}}^{t_j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right] = E\left[M^{i,j}_{t_j} \int_{t_{j-1}}^{t_j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right] \le \varliminf_{n\to\infty} E\left[M^{i,j}_{\tau_n^j} \int_{t_{j-1}}^{\tau_n^j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right]$$
$$= \varliminf_{n\to\infty} E_{Q_n^{i,j}}\left[\int_{t_{j-1}}^{\tau_n^j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right] \le \frac{4\lambda\gamma\Xi}{\lambda-\gamma}.$$

And an analogy to (4.12) shows that

$$E_{Q^{i,j}}\Big[\ln M^{i,j}_{t_j}\Big] = \frac{1}{2} E_{Q^{i,j}}\left[\int_{t_{j-1}}^{t_j} |\widetilde{\mathfrak{q}}^i_s|^2 ds\right] \le \frac{2\lambda\gamma\Xi}{\lambda-\gamma}. \tag{4.15}$$

Now for any $n\in\mathbb{N}$, applying Tanaka's formula to the process $\big(Y^1-Y^2\big)^+$, we can deduce from (4.6), (4.9), the flat-off condition of (Y^1, Z^1, K^1), (4.8), (4.5) as well as Girsanov Theorem that

$$\Big(Y^1_{\tau_j^n\wedge t} - Y^2_{\tau_j^n\wedge t}\Big)^+ = \Big(Y^1_{\tau_j^n} - Y^2_{\tau_j^n}\Big)^+ + \int_{\tau_j^n\wedge t}^{\tau_j^n} \mathbf{1}_{\{Y^1_s > Y^2_s\}}\big(f(s,Y^1_s,Z^1_s) - f(s,Y^2_s,Z^2_s)\big) ds$$
$$+ \int_{\tau_j^n\wedge t}^{\tau_j^n} \mathbf{1}_{\{Y^1_s > Y^2_s\}} (dK^1_s - dK^2_s) - \int_{\tau_j^n\wedge t}^{\tau_j^n} \mathbf{1}_{\{Y^1_s > Y^2_s\}} (Z^1_s - Z^2_s)\, dB_s$$
$$- \frac{1}{2}\int_{\tau_j^n\wedge t}^{\tau_j^n} d\mathfrak{L}_s$$
$$\le \Big(Y^1_{\tau_j^n} - Y^2_{\tau_j^n}\Big)^+ + \int_{\tau_j^n\wedge t}^{\tau_j^n} \mathbf{1}_{\{Y^1_s > Y^2_s\}}\Big(\widehat{f}\big(s, Y^1_s, \widetilde{\mathfrak{q}}^2_s\big)$$

$$- \langle Z^1_s, \widetilde{\mathfrak{q}}^2_s \rangle - \widehat{f}\big(s, Y^2_s, \widetilde{\mathfrak{q}}^2_s\big) + \langle Z^2_s, \widetilde{\mathfrak{q}}^2_s \rangle\Big) ds + \int_{\tau^n_j \wedge t}^{\tau^n_j} \mathbf{1}_{\{L_s = Y^1_s > Y^2_s\}} dK^1_s$$

$$- \int_{\tau^n_j \wedge t}^{\tau^n_j} \mathbf{1}_{\{Y^1_s > Y^2_s\}} (Z^1_s - Z^2_s)\, dB_s$$

$$\le \Big(Y^1_{\tau^n_j} - Y^2_{\tau^n_j}\Big)^+ + \kappa \int_{\tau^n_j \wedge t}^{\tau^n_j} \mathbf{1}_{\{Y^1_s > Y^2_s\}} \big(Y^1_s - Y^2_s\big)^+ ds$$

$$- \int_{\tau^n_j \wedge t}^{\tau^n_j} \mathbf{1}_{\{Y^1_s > Y^2_s\}} (Z^1_s - Z^2_s)\, dB^{2,j}_s, \quad t \in [t_{j-1}, t_j], \tag{4.16}$$

where $\mathfrak{L}$ is a real-valued, $\mathbf{F}$-adapted, increasing and continuous process known as "local time". Taking the expectation $E_{Q^{2,j}}$ and using Fubini's Theorem, we obtain

$$E_{Q^{2,j}}\Big[\Big(Y^1_{\tau^n_j \wedge t} - Y^2_{\tau^n_j \wedge t}\Big)^+\Big] \le E_{Q^{2,j}}\Big[\Big(Y^1_{\tau^n_j} - Y^2_{\tau^n_j}\Big)^+\Big] + \kappa \int_t^{t_j} E_{Q^{2,j}}\Big[\Big(Y^1_s - Y^2_s\Big)^+\Big] ds,$$
$$t \in [t_{j-1}, t_j].$$

Then an application of Gronwall's inequality yields that

$$E_{Q^{2,j}}\Big[\Big(Y^1_{\tau^n_j \wedge t} - Y^2_{\tau^n_j \wedge t}\Big)^+\Big] \le e^{\kappa T} E_{Q^{2,j}}\Big[\Big(Y^1_{\tau^n_j} - Y^2_{\tau^n_j}\Big)^+\Big], \quad t \in [t_{j-1}, t_j]. \tag{4.17}$$

Similar to (4.13), one can deduce from (4.15) that

$$E_{Q^{2,j}}\Big[\sup_{t \in [0,t_j]} \Big(Y^1_t - Y^2_t\Big)^+\Big] \le E_{Q^{2,j}}\Big[\sup_{t \in [0,t_j]} \big(Y^1_t\big)^+ + \sup_{t \in [0,t_j]} \big(Y^2_t\big)^-\Big]$$
$$\le \widetilde{c}^{\,1}_{\lambda'} + \widetilde{c}^{\,2}_{\lambda} + \left(\frac{1}{2\lambda'} + \frac{1}{2\lambda}\right) E_{Q^{2,j}}\left[\int_{t_{j-1}}^{t_j} |\widetilde{\mathfrak{q}}^2_s|^2 ds\right] < \infty.$$

If $Y^1_{t_j} \le Y^2_{t_j}$, P-a.s., as $n \to \infty$ in (4.17), dominated convergence theorem implies that for any $t \in [t_{j-1}, t_j]$

$$E_{Q^{2,j}}\Big[\big(Y^1_t - Y^2_t\big)^+\Big] = 0, \quad \text{thus } Y^1_t \le Y^2_t,\ P\text{-a.s.} \tag{4.18}$$

In particular, $Y^1_{t_{j-1}} \le Y^2_{t_{j-1}}$, P-a.s. On the other hand, if $Y^2_{t_j} \le Y^1_{t_j}$, P-a.s., interchanging (Y^1, Z^1, Z^1) with (Y^2, Z^2, Z^2) and estimating under $Q^{1,j}$ in the above arguments (from (4.16) to (4.18)) give that for any $t \in [t_{j-1}, t_j]$, $Y^2_t \le Y^1_t$, P-a.s. Therefore, starting from $Y^1_T = Y^2_T = \xi$, P-a.s., we can use backward induction to conclude that for any $t \in [0, T]$, $Y^1_t = Y^2_t$, P-a.s. Then the continuity of processes Y^1 and Y^2 shows that Y^1 and Y^2 are indistinguishable, which implies that

$$0 = Y^1_0 - Y^1_t - (Y^2_0 - Y^2_t)$$
$$= \int_0^t \big(f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s)\big)\, ds + K^1_t - K^2_t - \int_0^t (Z^1_s - Z^2_s) dB_s$$
$$= \int_0^t \big(f(s, Y^1_s, Z^1_s) - f(s, Y^1_s, Z^2_s)\big)\, ds + K^1_t - K^2_t - \int_0^t (Z^1_s - Z^2_s) dB_s,$$
$$t \in [0, T]. \tag{4.19}$$

Since the set of continuous martingales and that of finite variation processes only intersect at constants, one can deduce that $Z_t^1 = Z_t^2$, $dt \otimes dP$-a.e. Putting it back into (4.19) shows that K^1 and K^2 are indistinguishable. □

5. An optimal stopping problem for quadratic g-evaluations

In this section, we will solve an optimal stopping problem in which the objective of the stopper is to determine an optimal stopping time τ_* that satisfies

$$\sup_{\tau \in \mathcal{S}_{0,T}} \mathcal{E}^g_{0,\tau}[\mathcal{R}_\tau] = \mathcal{E}^g_{0,\tau_*}[\mathcal{R}_{\tau_*}], \tag{5.1}$$

where $\mathcal{E}^g$ is a "quadratic g-evaluation" (a type of non-linear expectation to be defined below), and $\mathcal{R}$ is a reward process that we will specify shortly.

Let $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathscr{P} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function that satisfies (H1)–(H3). For any $\tau \in \mathcal{S}_{0,T}$, It is clear that $g_\tau(t,\omega,y,z) \triangleq \mathbf{1}_{\{t<\tau\}} g(t,\omega,y,z)$, $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ is also a $\mathscr{P} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function that satisfies (H1)–(H3). Thus, we know from Corollary 6 of [6] that for any $\xi \in \mathbb{L}^e(\mathcal{F}_T)$, the following quadratic BSDE

$$Y_t = \xi + \int_t^T \mathbf{1}_{\{s<\tau\}} g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0,T] \tag{5.2}$$

admits a unique solution $(Y^{\tau,\xi}, Z^{\tau,\xi})$ in $\cap_{p\in(1,\infty)} \mathbb{E}^p_{\mathbf{F}}[0,T] \times \mathbb{H}^{2,2p}_{\mathbf{F}}([0,T];\mathbb{R}^d)$. If $\xi \in \mathbb{L}^e(\mathcal{F}_\tau)$, one can deduce that

$$P\Big(Y_t^{\tau,\xi} = Y_{\tau\wedge t}^{\tau,\xi},\ \forall\, t \in [0,T]\Big) = 1 \quad \text{and} \quad Z_t^{\tau,\xi} = \mathbf{1}_{\{t<\tau\}} Z_t^{\tau,\xi},\ dt \otimes dP\text{-a.e.} \tag{5.3}$$

Definition 5.1. A "quadratic g-evaluation" with domain $\mathbb{L}^e(\mathcal{F}_T)$ is a family of operators $\big\{\mathcal{E}^g_{\nu,\tau} : \mathbb{L}^e(\mathcal{F}_\tau) \mapsto \mathbb{L}^e(\mathcal{F}_\nu)\big\}_{\nu\in\mathcal{S}_{0,T},\tau\in\mathcal{S}_{\nu,T}}$ such that $\mathcal{E}^g_{\nu,\tau}[\xi] \triangleq Y_\nu^{\tau,\xi}$, $\forall\, \xi \in \mathbb{L}^e(\mathcal{F}_\tau)$. In particular, for any $\xi \in \mathbb{L}^e(\mathcal{F}_T)$, we can define the "quadratic g-expectation" of ξ at a stopping time $\nu \in \mathcal{S}_{0,T}$ by $\mathcal{E}^g[\xi|\mathcal{F}_\nu] \triangleq \mathcal{E}^g_{\nu,T}[\xi]$.

The g-evaluation was introduced by Peng [24] for Lipschitz generators over $\mathbb{L}^2(\mathcal{F}_T)$. Then [19] extended the notion for quadratic generators, however, on $\mathbb{L}^\infty(\mathcal{F}_T)$. Thanks to Theorem 5 of [6] and the uniqueness of the solution $(Y^{\tau,\xi}, Z^{\tau,\xi})$, one can show that the quadratic g-evaluation $\mathcal{E}^g_{\nu,\tau}$ inherit the basic properties of g-evaluations with Lipschitz generators such as: *Monotonicity*, *Time-Consistency*, *Constant-Preserving*, *Zero-one Law* and *Translation Invariance* (see [2]).

Now, we assume that the reward process $\mathcal{R}$ is in the form of

$$\mathcal{R}_t \triangleq \mathbf{1}_{\{t<T\}} \mathcal{L}_t + \mathbf{1}_{\{t=T\}} \xi, \quad t \in [0,T], \tag{5.4}$$

for some $\mathcal{L} \in \mathbb{C}^0_{\mathbf{F}}[0,T]$ and $\xi \in \mathbb{L}^0(\mathcal{F}_T)$ with $\mathcal{L}_T \le \xi$, P-a.s. One can regard $\mathcal{L}$ as the running reward and ξ as the final reward with a possible bonus.

When $\xi^+ \vee \mathcal{L}_* \in \mathbb{L}^e(\mathcal{F}_T)$, the quadratic RBSDE $(\xi, g, \mathcal{L})$ admits a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{K})$ in $\cap_{p\in[1,\infty)} \mathbb{S}^p_{\mathbf{F}}[0,T]$ thanks to Theorems 3.2 and 4.1. In fact, the continuous process $\mathcal{Y}$ is the snell envelope of the reward process $\mathcal{R}$ under the quadratic g-evaluation, and the first time process $\mathcal{Y}$

meets process $\mathcal{R}$ after time $t = 0$ is an optimal stopping time for (5.1). More precisely, we have the following result.

Theorem 5.1. *Let* $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *be a* $\mathscr{P} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$*-measurable function that satisfies* (H1)–(H3)*, and let* $\mathcal{R}$ *be a reward process in the form of* (5.4)*. If* $\xi^+ \vee \mathcal{L}_* \in \mathbb{L}^e(\mathcal{F}_T)$*, then for any* $\nu \in \mathcal{S}_{0,T}$*,*

$$\mathcal{Y}_\nu = \operatorname*{esssup}_{\tau \in \mathcal{S}_{\nu,T}} \mathcal{E}^g_{\nu,\tau}\big[\mathcal{R}_\tau\big] = \mathcal{E}^g_{\nu,\tau_*(\nu)}\big[\mathcal{R}_{\tau_*(\nu)}\big], \quad P\text{-a.s.},$$

where $\mathcal{Y}$ *is of the unique solution to the quadratic* RBSDE $(\xi, g, \mathcal{L})$ *and* $\tau_*(\nu) \triangleq \inf\big\{t \in [\nu, T] : \mathcal{Y}_t = \mathcal{R}_t\big\} \in \mathcal{S}_{\nu,T}$.

This theorem extends Section 3 of [21], it also extends Theorem 5.3 of [3] except that the continuity condition on the reward process $\mathcal{R}$ is strengthened. The proof of Theorem 5.1 depends on the following comparison theorem for quadratic BSDEs, which generalizes Theorem 5 of [6].

Proposition 5.1. *For* $i = 1, 2$*, let* $f_i : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *be a* $\mathscr{P} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$*-measurable function, and let* $(Y^i, Z^i, V^i) \in \mathbb{C}^0_{\mathbf{F}}[0, T] \times \widehat{\mathbb{H}}^{2,\mathrm{loc}}_{\mathbf{F}}([0, T]; \mathbb{R}^d) \times \mathbb{V}_{\mathbf{F}}[0, T]$ *solves the following BSDE*

$$Y^i_t = Y^i_T + \int_t^T f_i(s, Y^i_s, Z^i_s)\,ds + V^i_T - V^i_t - \int_t^T Z^i_s dB_s, \quad t \in [0, T] \tag{5.5}$$

such that $Y^1_T \le Y^2_T$*, P-a.s., that* $E\left[e^{\lambda (Y^1)^+_*} + e^{\lambda (Y^2)^-_*}\right] < \infty$ *for all* $\lambda \in (1, \infty)$*, and that for some* $\theta_0 \in (0, 1)$*,* $\theta V^1 - V^2$ *is a decreasing process for any* $\theta \in (\theta_0, 1)$*. If either of the following two holds:*

(i) f_1 *satisfies* (H1′), (H2)*;* f_1 *is concave in* z*; and* $\Delta f(t) \triangleq f_1(t, Y^2_t, Z^2_t) - f_2(t, Y^2_t, Z^2_t) \le 0$*,* $dt \otimes dP$*-a.e.;*
(ii) f_2 *satisfies* (H1′), (H2)*;* f_2 *is concave in* z*; and* $\Delta f(t) \triangleq f_1(t, Y^1_t, Z^1_t) - f_2(t, Y^1_t, Z^1_t) \le 0$*,* $dt \otimes dP$*-a.e.;*

(where (H1′) *is an extension of* (H1) *in that the constant* α *is replaced by an* **F***-progressively measurable, non-negative process* $\{\alpha_t\}_{t\in[0,T]}$ *such that* $E\big[\exp\big\{p \int_0^T \alpha_r dr\big\}\big] < \infty$ *for some* $p > \gamma e^{2\kappa T}$*) then it holds P-a.s. that* $Y^1_t \le Y^2_t$ *for any* $t \in [0, T]$*.*

In addition, if $Y^1_\tau = Y^2_\tau$*, P-a.s. for some* $\tau \in \mathcal{S}_{0,T}$*, then*

$$P\left(Y^1_T = Y^2_T, \int_\tau^T \Delta f(s)ds = 0\right) > 0. \tag{5.6}$$

Proof of Theorem 5.1. Fix $\nu \in \mathcal{S}_{0,T}$. For any $\tau \in \mathcal{S}_{\nu,T}$, it holds P-a.s. that

$$\begin{aligned}\mathcal{Y}_{\tau\wedge t} &= \mathcal{Y}_\tau + \int_{\tau\wedge t}^\tau g\big(s, \mathcal{Y}_s, \mathcal{Z}_s\big)ds + \mathcal{K}_\tau - \mathcal{K}_{\tau\wedge t} - \int_{\tau\wedge t}^\tau \mathcal{Z}_s dB_s \\ &= \mathcal{Y}_\tau + \int_t^T \mathbf{1}_{\{s<\tau\}} g\big(s, \mathcal{Y}_{\tau\wedge s}, \mathbf{1}_{\{s<\tau\}}\mathcal{Z}_s\big)ds + \mathcal{K}_\tau - \mathcal{K}_{\tau\wedge t} \\ &\quad - \int_t^T \mathbf{1}_{\{s<\tau\}} \mathcal{Z}_s dB_s, \quad t \in [0, T].\end{aligned} \tag{5.7}$$

Since $\mathcal{Y}_\tau \ge \mathbf{1}_{\{\tau<T\}}\mathcal{L}_\tau + \mathbf{1}_{\{\tau=T\}}\xi = \mathcal{R}_\tau$, P-a.s., applying Proposition 5.1 with $(Y^1, Z^1, V^1) = \big(Y^{\tau,\mathcal{R}_\tau}, Z^{\tau,\mathcal{R}_\tau}, 0\big)$ and $(Y^2, Z^2, V^2) = \big\{\big(\mathcal{Y}_{\tau\wedge t}, \mathbf{1}_{\{t<\tau\}}\mathcal{Z}_t, \mathcal{K}_{\tau\wedge t}\big)\big\}_{t\in[0,T]}$ yields that P-a.s., $\mathcal{Y}_{\tau\wedge t} \ge Y_t^{\tau,\mathcal{R}_\tau}$ for any $t \in [0, T]$. In particular, we have $\mathcal{Y}_\nu \ge Y_\nu^{\tau,\mathcal{R}_\tau} = \mathcal{E}^g_{\nu,\tau}[\mathcal{R}_\tau]$, P-a.s. So it remains to show that $\mathcal{Y}_\nu = \mathcal{E}^g_{\nu,\tau_*(\nu)}\big[\mathcal{R}_{\tau_*(\nu)}\big]$, P-a.s. To see this, we define

$$\widetilde{\mathcal{Y}}_t \triangleq \mathbf{1}_{\{t<\nu\}} Y_t^{\nu,\mathcal{Y}_\nu} + \mathbf{1}_{\{t\ge\nu\}}\mathcal{Y}_{\tau_*(\nu)\wedge t} \quad \text{and} \quad \widetilde{\mathcal{Z}}_t \triangleq \mathbf{1}_{\{t<\nu\}} Z_t^{\nu,\mathcal{Y}_\nu} + \mathbf{1}_{\{\nu\le t<\tau_*(\nu)\}}\mathcal{Z}_t, \quad \forall\, t \in [0, T].$$

Clearly, $\big(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}\big) \in \cap_{p\in(1,\infty)} \mathbb{E}^p_{\mathbf{F}}[0, T] \times \mathbb{H}^{2,2p}_{\mathbf{F}}([0, T]; \mathbb{R}^d)$. The flat-off condition of $(\mathcal{Y}, \mathcal{Z}, \mathcal{K})$ and the continuity of $\mathcal{K}$ imply that P-a.s.

$$0 = \int_{[\nu,\tau_*(\nu))} \mathbf{1}_{\{\mathcal{Y}_s>\mathcal{L}_s\}} d\mathcal{K}_s = \int_{[\nu,\tau_*(\nu))} \mathbf{1}_{\{\mathcal{Y}_s>\mathcal{R}_s\}} d\mathcal{K}_s = \int_{[\nu,\tau_*(\nu))} d\mathcal{K}_s = \lim_{s\nearrow\tau_*(\nu)} \mathcal{K}_s - \mathcal{K}_\nu = \mathcal{K}_{\tau_*(\nu)} - \mathcal{K}_\nu.$$

Hence, taking $\tau = \tau_*(\nu)$ and $t = \nu \vee t$ in (5.7), we can deduce that P-a.s.

$$\mathcal{Y}_{(\nu\vee t)\wedge\tau_*(\nu)} = \mathcal{R}_{\tau_*(\nu)} + \int_{\nu\vee t}^T \mathbf{1}_{\{s<\tau_*(\nu)\}} g\big(s, \widetilde{\mathcal{Y}}_s, \widetilde{\mathcal{Z}}_s\big) ds - \int_{\nu\vee t}^T \widetilde{\mathcal{Z}}_s dB_s, \quad t \in [0, T]. \tag{5.8}$$

In particular, we have

$$\mathcal{Y}_\nu = \mathcal{R}_{\tau_*(\nu)} + \int_\nu^T \mathbf{1}_{\{s<\tau_*(\nu)\}} g\big(s, \widetilde{\mathcal{Y}}_s, \widetilde{\mathcal{Z}}_s\big) ds - \int_\nu^T \widetilde{\mathcal{Z}}_s dB_s, \quad P\text{-a.s.} \tag{5.9}$$

Fix $t \in [0, T]$. One can deduce from (5.3) and (5.9) that

$$\begin{aligned}\mathbf{1}_{\{t<\nu\}} Y_t^{\nu,\mathcal{Y}_\nu} &= \mathbf{1}_{\{t<\nu\}}\mathcal{Y}_\nu + \mathbf{1}_{\{t<\nu\}} \int_t^\nu g\Big(s, Y_s^{\nu,\mathcal{Y}_\nu}, Z_s^{\nu,\mathcal{Y}_\nu}\Big) ds - \mathbf{1}_{\{t<\nu\}} \int_t^\nu Z_s^{\nu,\mathcal{Y}_\nu} dB_s \\ &= \mathbf{1}_{\{t<\nu\}}\mathcal{Y}_\nu + \mathbf{1}_{\{t<\nu\}} \int_t^\nu g\big(s, \widetilde{\mathcal{Y}}_s, \widetilde{\mathcal{Z}}_s\big) ds - \mathbf{1}_{\{t<\nu\}} \int_t^\nu \widetilde{\mathcal{Z}}_s dB_s \\ &= \mathbf{1}_{\{t<\nu\}}\mathcal{R}_{\tau_*(\nu)} + \mathbf{1}_{\{t<\nu\}} \int_t^T \mathbf{1}_{\{s<\tau_*(\nu)\}} g\big(s, \widetilde{\mathcal{Y}}_s, \widetilde{\mathcal{Z}}_s\big) ds - \mathbf{1}_{\{t<\nu\}} \int_t^T \widetilde{\mathcal{Z}}_s dB_s,\end{aligned}$$

which together with (5.8) implies that P-a.s.

$$\widetilde{\mathcal{Y}}_t = \mathcal{R}_{\tau_*(\nu)} + \int_t^T \mathbf{1}_{\{s<\tau_*(\nu)\}} g\big(s, \widetilde{\mathcal{Y}}_s, \widetilde{\mathcal{Z}}_s\big) ds - \int_t^T \widetilde{\mathcal{Z}}_s dB_s. \tag{5.10}$$

The continuity of process $\widetilde{\mathcal{Y}}_t$ further shows that P-a.s., (5.10) holds for any $t \in [0, T]$. To wit, $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) \in \cap_{p\in(1,\infty)} \mathbb{E}^p_{\mathbf{F}}[0, T] \times \mathbb{H}^{2,2p}_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ is the unique solution of the BSDE (5.2) with $(\tau, \xi) = \big(\tau_*(\nu), \mathcal{R}_{\tau_*(\nu)}\big)$. Therefore, it follows that $\mathcal{Y}_\nu = \widetilde{\mathcal{Y}}_\nu = \mathcal{E}^g_{\nu,\tau_*(\nu)}\big[\mathcal{R}_{\tau_*(\nu)}\big]$. □

6. Stability

Inspired by the "θ-difference" method introduced in [6], we obtain the following stability result.

Theorem 6.1. *Let $\{(\xi_m, f_m, L^m)\}_{m\in\mathbb{N}_0}$ be a sequence of parameter sets such that*

(S1) *With the same constants $\alpha, \beta, \kappa \ge 0$ and $\gamma > 0$, f_0 satisfies* (H1) *and $\{f_n\}_{n\in\mathbb{N}}$ satisfy* (H1)–(H3)*;*

(S2) *It holds P-a.s. that ξ_n converges to ξ_0 and that L^n_t converges to L^0_t uniformly in $t \in [0,T]$;*
(S3) $\Xi(p) \triangleq \sup_{m\in\mathbb{N}_0} E\Big[e^{p\,(\xi_m^+ \vee L^m_*)}\Big] < \infty$ *for all* $p \in (1,\infty)$.

We let $(Y^0, Z^0, K^0) \in \cap_{p\in[1,\infty)} \mathbb{S}^p_{\mathbf{F}}[0,T]$ be a solution of the quadratic RBSDE(ξ_0, f_0, L^0), *and for any $n \in \mathbb{N}$ we let (Y^n, Z^n, K^n) be the unique solution of the quadratic* RBSDE(ξ_n, f_n, L^n) *in $\cap_{p\in[1,\infty)} \mathbb{S}^p_{\mathbf{F}}[0,T]$. If $f_n\big(t, Y^0_t, Z^0_t\big)$ converges $dt \otimes dP$-a.e. to $f_0\big(t, Y^0_t, Z^0_t\big)$, then for any $p \in [1,\infty)$, $\Big\{\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p\Big\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence in $\mathbb{L}^1(\mathcal{F}_T)$ and*

$$\lim_{n\to\infty} E\bigg[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p\bigg] = \lim_{n\to\infty} E\Bigg[\bigg(\int_0^T |Z^n_s - Z^0_s|^2 ds\bigg)^p\Bigg] = 0.$$

Moreover, if it holds $dt \otimes dP$-a.e. that $f_n(t,\omega,y,z)$ converges to $f_0(t,\omega,y,z)$ locally uniformly in (y,z), then up to a subsequence, we further have $\lim_{n\to\infty} E\Big[\sup_{t\in[0,T]} |K^n_t - K^0_t|^p\Big] = 0,\ \forall\, p \in [1,\infty)$.

Proof. (1) Fix $n \in \mathbb{N}$, $\theta \in (0,1)$ and $\varepsilon > 0$. We first show that P-a.s.

$$|Y^n_t - Y^0_t| \le (1-\theta)\big(|Y^0_t| + |Y^n_t|\big) + \frac{1-\theta}{\gamma} \ln\Bigg(\sum_{i=1}^4 I^{n,i}_t\Bigg), \quad t \in [0,T], \tag{6.1}$$

where $I^{n,i}_t \triangleq E\big[I^{n,i}_T | \mathcal{F}_t\big]$ for $i = 1,2,3,4$ such that

$$\begin{aligned}
I^{n,1}_T &\triangleq D_T \eta_n \quad \text{with } D_t \triangleq \exp\Bigg\{\gamma e^{2\kappa T} \int_0^t \big(\alpha + (\beta+\kappa)|Y^0_s|\big) ds\Bigg\}, \quad t \in [0,T] \text{ and} \\
&\qquad \eta_n \triangleq \exp\big\{\zeta_\theta e^{\kappa T}\big(|\xi_n - \theta\xi_0| \vee |\xi_0 - \theta\xi_n|\big)\big\}; \\
I^{n,2}_T &\triangleq \zeta_\theta e^{\kappa T} D_T \Upsilon_n \int_0^T |\Delta_n f(s)| ds \quad \text{with } \zeta_\theta \triangleq \frac{\gamma e^{\kappa T}}{1-\theta},\ \Upsilon_n \triangleq \exp\big\{\zeta_\theta e^{\kappa T}\big(Y^n_* + Y^0_*\big)\big\} \\
&\qquad \text{and } \Delta_n f(t) \triangleq f_n\big(t, Y^0_t, Z^0_t\big) - f_0\big(t, Y^0_t, Z^0_t\big),\ t \in [0,T]; \\
I^{n,3}_T &\triangleq \Big(1 + \zeta_\theta \exp\big\{\kappa T + \varepsilon\zeta_\theta e^{\kappa T}\big\}\Big)\Big(1 + D_T \exp\Big\{\gamma e^{2\kappa T}\big(Y^0_* + Y^n_*\big)\Big\}\big(K^0_T + K^n_T\big)\Big); \\
I^{n,4}_T &\triangleq \frac{\zeta_\theta}{\varepsilon} e^{\kappa T} D_T \Upsilon_n \Big(\sup_{t\in[0,T]} |L^n_t - L^0_t|\Big)\big(K^0_T + K^n_T\big).
\end{aligned}$$

We set $U^n \triangleq \theta Y^0 - Y^n$, $V^n \triangleq \theta Z^0 - Z^n$ and define two processes

$$a^n_t \triangleq \mathbf{1}_{\{U^n_t \neq 0\}} \frac{f_n(t, \theta Y^0_t, Z^n_t) - f_n(t, Y^n_t, Z^n_t)}{U^n_t} - \kappa \mathbf{1}_{\{U^n_t = 0\}}, \qquad A^n_t \triangleq \int_0^t a^n_s ds,$$
$$t \in [0,T].$$

Applying Itô's formula to the process $\Gamma^n_t \triangleq \exp\big\{\zeta_\theta e^{A^n_t} U^n_t\big\},\ t \in [0,T]$ yields that

$$\Gamma^n_t = \Gamma^n_T + \int_t^T G^n_s ds + \zeta_\theta \int_t^T \Gamma^n_s e^{A^n_s}(\theta dK^0_s - dK^n_s) - \zeta_\theta \int_t^T \Gamma^n_s e^{A^n_s} V^n_s dB_s,$$
$$t \in [0,T],$$

where $G^n_t = \zeta_\theta \Gamma^n_t e^{A^n_t}\Big(\theta f_0(t,Y^0_t,Z^0_t) - f_n(t,Y^n_t,Z^n_t) - a^n_t U^n_t - \frac{1}{2}\zeta_\theta e^{A^n_t}|V^n_t|^2\Big)$. In light of (H1) and the concavity of f_n in z, it holds $dt\otimes dP$-a.e. that for any $y\in\mathbb{R}$

$$f_n(t,y,Z^n_t) \ge \theta f_n\big(t,y,Z^0_t\big) + (1-\theta) f_n\left(t,y,\frac{-V^n_t}{1-\theta}\right) \ge \theta f_n\big(t,y,Z^0_t\big) \\ -(1-\theta)\,(\alpha+\beta|y|) - \frac{\gamma}{2(1-\theta)}|V^n_t|^2, \tag{6.2}$$

which together with (H2) implies that $dt\otimes dP$-a.e.

$$\begin{aligned} G^n_t &= \zeta_\theta \Gamma^n_t e^{A^n_t}\left(-\theta\Delta_n f(t) + \theta f_n(t,Y^0_t,Z^0_t) - f_n(t,\theta Y^0_t,Z^n_t) - \frac{1}{2}\zeta_\theta e^{A^n_t}|V^n_t|^2\right)\\ &\le \zeta_\theta \Gamma^n_t e^{A^n_t}\Bigg(|\Delta_n f(t)| + \theta f_n(t,Y^0_t,Z^0_t) - f_n(t,Y^0_t,Z^n_t)\\ &\quad + |f_n(t,Y^0_t,Z^n_t) - f_n(t,\theta Y^0_t,Z^n_t)| - \frac{\gamma}{2(1-\theta)}|V^n_t|^2\Bigg)\\ &\le \gamma e^{2\kappa T}\Gamma^n_t\big(\alpha+(\beta+\kappa)|Y^0_t|\big) + \zeta_\theta e^{\kappa T}\Gamma^n_t|\Delta_n f(t)|. \end{aligned}$$

Integration by parts gives that

$$\begin{aligned} \Gamma^n_t &\le D_t\Gamma^n_t \le D_T\Gamma^n_T + \zeta_\theta e^{\kappa T}\int_t^T D_s\Gamma^n_s|\Delta_n f(s)|ds + \zeta_\theta\int_t^T D_s\Gamma^n_s e^{A^n_s}dK^0_s\\ &\quad - \zeta_\theta\int_t^T D_s\Gamma^n_s e^{A^n_s}V^n_s dB_s\\ &\le I^{n,1}_T + I^{n,2}_T + \zeta_\theta e^{\kappa T}D_T\int_0^T \Gamma^n_s dK^0_s - \zeta_\theta\int_t^T D_s\Gamma^n_s e^{A^n_s}V^n_s dB_s, \quad t\in[0,T]. \end{aligned}\tag{6.3}$$

The flat-off condition of (Y^0,Z^0,K^0) implies that

$$\begin{aligned} \int_0^T \Gamma^n_s dK^0_s &= \int_0^T \mathbf{1}_{\{Y^0_s=L^0_s\}}\Gamma^n_s dK^0_s\\ &= \int_0^T \mathbf{1}_{\{Y^0_s=L^0_s\le L^n_s+\varepsilon\}}\Gamma^n_s dK^0_s + \int_0^T \mathbf{1}_{\{Y^0_s=L^0_s> L^n_s+\varepsilon\}}\Gamma^n_s dK^0_s\\ &\le \int_0^T \mathbf{1}_{\{Y^0_s\le Y^n_s+\varepsilon\}}\exp\left\{\gamma e^{2\kappa T}|Y^n_s| + \varepsilon\zeta_\theta e^{\kappa T}\right\}dK^0_s\\ &\quad + \Upsilon_n\int_0^T \mathbf{1}_{\{|L^n_s-L^0_s|>\varepsilon\}}dK^0_s\\ &\le \exp\left\{\gamma e^{2\kappa T}Y^n_* + \varepsilon\zeta_\theta e^{\kappa T}\right\}K^0_T + \frac{1}{\varepsilon}\Upsilon_n\Big(\sup_{t\in[0,T]}|L^n_t - L^0_t|\Big)K^0_T, \quad P\text{-a.s.} \end{aligned}\tag{6.4}$$

For each $p\in(1,\infty)$, (3.31) and (S3) imply that

$$\begin{aligned} \sup_{n'\in\mathbb{N}} E\left[e^{p\gamma Y^{n'}_*} + \left(\int_0^T |Z^{n'}_s|^2 ds\right)^p + \big(K^{n'}_T\big)^p\right] &\le c_p \sup_{n'\in\mathbb{N}} E\Big[e^{3p\gamma e^{\beta T}\left(\xi^+_{n'}\vee L^{n'}_*\right)}\Big]\\ &\le c_p\,\Xi\Big(3p\gamma e^{\beta T}\Big). \end{aligned}$$

Thus, it follows that

$$\sup_{m\in\mathbb{N}_0} E\bigg[e^{p\gamma Y^m_*}+\bigg(\int_0^T |Z^m_s|^2 ds\bigg)^p+\big(K^m_T\big)^p\bigg]$$
$$\le c_p\,\Xi\Big(3p\gamma e^{\beta T}\Big)+E\bigg[e^{p\gamma Y^0_*}+\bigg(\int_0^T |Z^0_s|^2 ds\bigg)^p+\big(K^0_T\big)^p\bigg]\triangleq\widetilde{\Xi}(p), \tag{6.5}$$

which together with (S1) implies that

$$E[\eta_n^p]\ \le\ E\Big[e^{p\ \zeta_\theta e^{\kappa T}(|\xi_n|+|\xi_0|)}\Big]\le\frac{1}{2}E\Big[e^{2p\ \zeta_\theta e^{\kappa T}(\xi_n^+\vee L^n_*)}+e^{2p\ \zeta_\theta e^{\kappa T}(\xi_0^+\vee L^0_*)}\Big]$$
$$\le\ \Xi\Big(2p\ \zeta_\theta e^{\kappa T}\Big), \tag{6.6}$$

$$E[\Upsilon_n^p]\le\frac{1}{2}E\Big[e^{2p\ \zeta_\theta e^{\kappa T}Y^n_*}+e^{2p\ \zeta_\theta e^{\kappa T}Y^0_*}\Big]\le\widetilde{\Xi}\bigg(\frac{2p}{1-\theta}e^{2\kappa T}\bigg), \tag{6.7}$$

$$E\bigg[\bigg(\int_0^T |\Delta_n f(s)|ds\bigg)^p\bigg]\ \le\ E\bigg[\bigg(2T(\alpha+\beta Y^0_*)+\gamma\int_0^T |Z^0_s|^2 ds\bigg)^p\bigg]$$
$$\le\ c_p E\bigg[e^{p\gamma Y^0_*}+\bigg(\int_0^T |Z^0_s|^2 ds\bigg)^p\bigg], \tag{6.8}$$

$$E\bigg[\sup_{t\in[0,T]}|L^n_t-L^0_t|^p\bigg]\le c_p E\Big[\big(L^n_*\big)^p+\big(L^0_*\big)^p\Big]\le c_p E\Big[e^{pL^n_*}+e^{pL^0_*}\Big]\le c_p\ \Xi(p). \tag{6.9}$$

Since $D_T\ \le\ c_0\exp\Big\{\gamma(\beta+\kappa)Te^{2\kappa T}\ Y^0_*\Big\}$, P-a.s., we also see that $D_T\in\mathbb{L}^p(\mathcal{F}_T)$. Thus, one can deduce from Young's inequality and (6.5)–(6.9) that random variables $I^{n,i}_T$, $i=1,2,3,4$ are all integrable. Moreover, the Burkholder–Davis–Gundy inequality and Hölder's inequality imply that

$$E\bigg[\sup_{t\in[0,T]}\bigg|\int_0^t D_s\Gamma^n_s e^{A^n_s}V^n_s dB_s\bigg|\bigg]\le c_0 E\bigg[\bigg(\int_0^T \big(D_s\Gamma^n_s\big)^2 e^{2A^n_s}|V^n_s|^2 ds\bigg)^{1/2}\bigg]$$
$$\le c_0 E\bigg[D_T\,\Upsilon_n\bigg(\int_0^T |V^n_s|^2 ds\bigg)^{1/2}\bigg]$$
$$\le c_0\,\|D_T\|_{\mathbb{L}^4(\mathcal{F}_T)}\,\|\Upsilon_n\|_{\mathbb{L}^4(\mathcal{F}_T)}\,\big\|V^n\big\|_{\mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)}<\infty, \tag{6.10}$$

thus $\int_0^\cdot D_s\Gamma^n_s e^{A^n_s}V^n_s dB_s$ is a uniformly integrable martingale.

For any $t\in[0,T]$, taking $E[\cdot|\mathcal{F}_t]$ in (6.3) and (6.4) yields that $\Gamma^n_t\le\sum_{i=1}^4 I^{n,i}_t$, P-a.s. It then follows that

$$Y^0_t-Y^n_t\le(1-\theta)|Y^0_t|+\theta Y^0_t-Y^n_t\le(1-\theta)|Y^0_t|+\frac{1-\theta}{\gamma}\ln\bigg(\sum_{i=1}^4 I^{n,i}_t\bigg),\quad P\text{-a.s.} \tag{6.11}$$

To show the other half of (6.1), we set $\widetilde{U}^n \triangleq \theta Y^n - Y^0$, $\widetilde{V}^n \triangleq \theta Z^n - Z^0$ and define two processes

$$\tilde{a}^n_t \triangleq \mathbf{1}_{\{Y^0_t \neq Y^n_t\}} \frac{f_n(t, Y^0_t, Z^n_t) - f_n(t, Y^n_t, Z^n_t)}{Y^0_t - Y^n_t} - \kappa \mathbf{1}_{\{Y^0_t = Y^n_t\}}, \qquad \widetilde{A}^n_t \triangleq \int_0^t \tilde{a}^n_s ds, \quad t \in [0, T].$$

Applying Itô's formula to the process $\widetilde{\Gamma}^n_t \triangleq \exp\{\zeta_\theta e^{\widetilde{A}^n_t} \widetilde{U}^n_t\}$, $t \in [0, T]$, yields that

$$\widetilde{\Gamma}^n_t = \widetilde{\Gamma}^n_T + \int_t^T \widetilde{G}^n_s ds + \zeta_\theta \int_t^T \widetilde{\Gamma}^n_s e^{\widetilde{A}^n_s}(\theta dK^n_s - dK^0_s) - \zeta_\theta \int_t^T \widetilde{\Gamma}^n_s e^{\widetilde{A}^n_s} \widetilde{V}^n_s dB_s, \quad t \in [0, T],$$

where $\widetilde{G}^n_t = \zeta_\theta \widetilde{\Gamma}^n_t e^{\widetilde{A}^n_t} \Big(\theta f_n(t, Y^n_t, Z^n_t) - f_0(t, Y^0_t, Z^0_t) - \tilde{a}^n_t \widetilde{U}^n_t - \frac{1}{2}\zeta_\theta e^{\widetilde{A}^n_t} |\widetilde{V}^n_t|^2\Big)$. Similar to (6.2), (H1) and the concavity of f_n in z show that $dt \otimes dP$-a.e.

$$f_n(t, y, Z^0_t) \ge \theta f_n\big(t, y, Z^n_t\big) - (1-\theta)\,(\alpha + \beta|y|) - \frac{\gamma}{2(1-\theta)} |\widetilde{V}^n_t|^2, \quad \forall\, y \in \mathbb{R},$$

which together with (H2) implies that $dt \otimes dP$-a.e.

$$\begin{aligned}
\widetilde{G}^n_t &\le \zeta_\theta \widetilde{\Gamma}^n_t e^{\widetilde{A}^n_t} \left(\theta f_n\big(t, Y^n_t, Z^n_t\big) - f_n(t, Y^0_t, Z^0_t) + \Delta_n f(t) - \tilde{a}^n_t \widetilde{U}^n_t - \frac{\gamma}{2(1-\theta)} |\widetilde{V}^n_t|^2\right) \\
&\le \zeta_\theta \widetilde{\Gamma}^n_t e^{\widetilde{A}^n_t} \Big(\theta f_n\big(t, Y^n_t, Z^n_t\big) - \theta f_n\big(t, Y^0_t, Z^n_t\big) \\
&\qquad + |\Delta_n f(t)| - \tilde{a}^n_t \widetilde{U}^n_t + (1-\theta)\left(\alpha + \beta|Y^0_t|\right)\Big) \\
&= \zeta_\theta \widetilde{\Gamma}^n_t e^{\widetilde{A}^n_t} \Big((1-\theta)\tilde{a}^n_t Y^0_t + |\Delta_n f(t)| + (1-\theta)\left(\alpha + \beta|Y^0_t|\right)\Big) \\
&\le \gamma e^{2\kappa T} \widetilde{\Gamma}^n_t \big(\alpha + (\beta + \kappa)|Y^0_t|\big) + \zeta_\theta e^{\kappa T} \widetilde{\Gamma}^n_t |\Delta_n f(t)|.
\end{aligned}$$

Similarly to (6.3), integration by parts gives that

$$\widetilde{\Gamma}^n_t \le I^{n,1}_T + I^{n,2}_T + \zeta_\theta e^{\kappa T} D_T \int_0^T \widetilde{\Gamma}^n_s dK^n_s - \zeta_\theta \int_t^T D_s \widetilde{\Gamma}^n_s e^{\widetilde{A}^n_s} \widetilde{V}^n_s dB_s, \quad t \in [0, T], \tag{6.12}$$

where $\int_0^\cdot D_s \widetilde{\Gamma}^n_s e^{\widetilde{A}^n_s} \widetilde{V}^n_s dB_s$ is a uniformly integrable martingale, which can be shown by using similar arguments to those lead to (6.10). And similar to (6.4), the flat-off condition of (Y^n, Z^n, K^n) implies that

$$\int_0^T \widetilde{\Gamma}^n_s dK^n_s \le \exp\left\{\gamma e^{2\kappa T} Y^0_* + \varepsilon \zeta_\theta e^{\kappa T}\right\} K^n_T + \frac{1}{\varepsilon} \Upsilon_n\Big(\sup_{t \in [0,T]} |L^n_t - L^0_t|\Big) K^n_T, \quad P\text{-a.s.} \tag{6.13}$$

For any $t \in [0, T]$, taking $E[\cdot|\mathcal{F}_t]$ in (6.12) and (6.13) yields that $\widetilde{\Gamma}^n_t \le \sum_{i=1}^4 I^{n,i}_t$, P-a.s. It then follows that

$$Y^n_t - Y^0_t \le (1-\theta)|Y^n_t| + \theta Y^n_t - Y^0_t \le (1-\theta)|Y^0_t| + \frac{1-\theta}{\gamma} \ln\left(\sum_{i=1}^4 I^{n,i}_t\right), \quad P\text{-a.s.},$$

which together with (6.11) as well as the continuity of processes Y^n, Y^0 and $\sum_{i=1}^4 I^{n,i}$ implies (6.1).

(2) For any $\delta > 0$, (6.1), (6.5), (6.7), Doob's martingale inequality and Hölder's inequality imply that

$$
\begin{aligned}
& P\Big(\sup_{t\in[0,T]} |Y_t^n - Y_t^0| \ge \delta\Big) \\
& \le P\Big((1-\theta)\big(Y_*^0 + Y_*^n\big) \ge \delta/2\Big) + P\left(\frac{1-\theta}{\gamma}\ln\Big(\sum_{i=1}^4 I_*^{n,i}\Big) \ge \delta/2\right) \\
& \le 2\frac{1-\theta}{\delta} E\big[Y_*^0 + Y_*^n\big] + \sum_{i=1}^4 P\left(I_*^{n,i} \ge \frac{1}{4} e^{\frac{\delta\gamma}{2(1-\theta)}}\right) \\
& \le \frac{1-\theta}{\delta\gamma} E\Big[e^{2\gamma Y_*^0} + e^{2\gamma Y_*^n}\Big] + 4e^{\frac{-\delta\gamma}{2(1-\theta)}} \sum_{i=1}^4 E\Big[I_T^{n,i}\Big] \\
& \le 2\frac{1-\theta}{\delta\gamma}\widetilde{\Xi}(2) + 4e^{\kappa T} e^{\frac{-\delta\gamma}{2(1-\theta)}} C\left(\|\eta_n\|_{\mathbb{L}^2(\mathcal{F}_T)} + \zeta_\theta\left\{\widetilde{\Xi}\left(\frac{8}{1-\theta}e^{2\kappa T}\right)\right\}^{\frac{1}{4}}\right. \\
& \quad \times \left\|\int_0^T |\Delta_n f(s)| ds\right\|_{\mathbb{L}^4(\mathcal{F}_T)} + 1 + \zeta_\theta e^{\varepsilon\zeta_\theta e^{\kappa T}} \\
& \quad \left. + \frac{\zeta_\theta}{\varepsilon}\left\{\widetilde{\Xi}\left(\frac{8}{1-\theta}e^{2\kappa T}\right)\right\}^{\frac{1}{4}} \|L^n - L^0\|_{\mathbb{C}^4_{\mathbf{F}}[0,T]}\right),
\end{aligned} \tag{6.14}
$$

with $C = 1 + \|D_T\|_{\mathbb{L}^2(\mathcal{F}_T)} + \sup_{n\in\mathbb{N}}\Big(E\Big[D_T e^{\gamma e^{2\kappa T}(Y_*^0+Y_*^n)}\big(K_T^0 + K_T^n\big)\Big] + \|D_T(K_T^0 + K_T^n)\|_{\mathbb{L}^2(\mathcal{F}_T)}\Big)$. Hölder's inequality and (6.5) show that C is a finite constant. The convergence of $\Delta_n f$ to 0 and (S1) imply that $dt \otimes dP$-a.e.

$$
\lim_{n\to\infty} \Delta_n f(t,\omega) = 0 \quad \text{and} \quad |\Delta_n f(t,\omega)| \le 2\alpha + 2\beta Y_*^0(\omega) + \gamma |Z_t^0(\omega)|^2, \quad \forall n \in \mathbb{N}. \tag{6.15}
$$

Hence, for P-a.s. $\omega \in \Omega$ we may assume that (6.15) holds for a.e. $t \in [0,T]$, and that $Y_*^0(\omega) + \int_0^T |Z_s^0(\omega)|^2 ds < \infty$. The Dominated convergence theorem then yields that $\lim_{n\to\infty}\int_0^T |\Delta_n f(s,\omega)| ds = 0$. By (S2), it also holds P-a.s. that $\lim_{n\to\infty}\eta_n = e^{\gamma e^{2\kappa T}|\xi_0|}$ and $\lim_{n\to\infty}\Big(\sup_{t\in[0,T]}|L_t^n - L_t^0|\Big) = 0$. Applying (6.6), (6.8) and (6.9) with any $p > 4$ shows that $\{\eta_n^2\}_{n\in\mathbb{N}}$, $\Big\{\big(\int_0^T |\Delta_n f(s)| ds\big)^4\Big\}_{n\in\mathbb{N}}$ and $\Big\{\sup_{t\in[0,T]}|L_t^n - L_t^0|^4\Big\}_{n\in\mathbb{N}}$ are all uniformly integrable sequences in $\mathbb{L}^1(\mathcal{F}_T)$, which leads to that $\lim_{n\to\infty} E\big[\eta_n^2\big] = E\Big[e^{2\gamma e^{2\kappa T}|\xi_0|}\Big]$ and $\lim_{n\to\infty} E\Big[\big(\int_0^T |\Delta_n f(s)| ds\big)^4 + \sup_{t\in[0,T]}|L_t^n - L_t^0|^4\Big] = 0$. Hence, letting $n \to \infty$ in (6.14) and then letting $\varepsilon \to 0$ yield that

$$
\begin{aligned}
& \overline{\lim_{n\to\infty}} P\Big(\sup_{t\in[0,T]} |Y_t^n - Y_t^0| \ge \delta\Big) \\
& \le 2\frac{1-\theta}{\delta\gamma}\widetilde{\Xi}(2) + 4e^{\kappa T} e^{\frac{-\delta\gamma}{2(1-\theta)}} C\left(1 + \|e^{\gamma e^{2\kappa T}|\xi_0|}\|_{\mathbb{L}^2(\mathcal{F}_T)} + \frac{\gamma e^{\kappa T}}{1-\theta}\right).
\end{aligned}
$$

As $\theta \to 1$, we obtain $\lim_{n\to\infty} P\Big(\sup_{t\in[0,T]} |Y^n_t - Y^0_t| \ge \delta\Big) = 0$, which implies that for any $p \in [1,\infty)$, $\exp\Big\{p\gamma \cdot \sup_{t\in[0,T]} |Y^n_t - Y^0_t|\Big\}$ converges to 1 in probability.

(3) Fix $p \in [1,\infty)$. Since $E\Big[\exp\Big\{2p\gamma \cdot \sup_{t\in[0,T]} |Y^n_t - Y^0_t|\Big\}\Big] \le \frac{1}{2}E\Big[e^{4p\gamma Y^n_*} + e^{4p\gamma Y^0_*}\Big] \le \widetilde{\Xi}(4p)$ holds for any $n \in \mathbb{N}$ by (6.5), we see that $\Big\{\exp\Big\{p\gamma \cdot \sup_{t\in[0,T]} |Y^n_t - Y^0_t|\Big\}\Big\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence in $\mathbb{L}^1(\mathcal{F}_T)$, which implies that $\lim_{n\to\infty} E\Big[\exp\Big\{p\gamma \cdot \sup_{t\in[0,T]} |Y^n_t - Y^0_t|\Big\}\Big] = 1$. In particular, it follows that $\Big\{\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p\Big\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence in $\mathbb{L}^1(\mathcal{F}_T)$ and that

$$\lim_{n\to\infty} E\Big[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p\Big] = 0. \tag{6.16}$$

For any $n \in \mathbb{N}$, applying Itô's formula to the process $|Y^n - Y^0|^2$, we can deduce from (S1) that

$$\begin{aligned}
&\int_0^T |Z^n_s - Z^0_s|^2 ds \\
&= |\xi_n - \xi_0|^2 - |Y^n_0 - Y^0_0|^2 + 2\int_0^T (Y^n_s - Y^0_s)\big(f_n(s, Y^n_s, Z^n_s) - f_0(s, Y^0_s, Z^0_s)\big)\, ds \\
&\quad + 2\int_0^T (Y^n_s - Y^0_s)(dK^n_s - dK^0_s) - 2\int_0^T (Y^n_s - Y^0_s)(Z^n_s - Z^0_s)\, dB_s \\
&\le 2\sup_{t\in[0,T]} |Y^n_t - Y^0_t| \Bigg(2\alpha T + \beta T\big(Y^n_* + Y^0_*\big) \\
&\quad + \frac{\gamma}{2}\int_0^T \big(|Z^n_s|^2 + |Z^0_s|^2\big) ds + K^n_T + K^0_T\Bigg) \\
&\quad + \sup_{t\in[0,T]} |Y^n_t - Y^0_t|^2 + 2\left|\int_0^T (Y^n_s - Y^0_s)(Z^n_s - Z^0_s)\, dB_s\right|, \qquad P\text{-a.s.}
\end{aligned}$$

Then the Burkholder–Davis–Gundy inequality, Hölder's inequality, and (6.5) imply that

$$\begin{aligned}
&E\left[\left(\int_0^T |Z^n_s - Z^0_s|^2 ds\right)^p\right] \\
&\le c_p E\Big[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^{2p}\Big] + c_p E\left[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p \cdot \left(\int_0^T |Z^n_s - Z^0_s|^2\, ds\right)^{\frac{p}{2}}\right] \\
&\quad + c_p \left\{E\Big[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^{2p}\Big]\right\}^{\frac{1}{2}} \\
&\quad \times \left\{\sup_{m\in\mathbb{N}_0} E\left[e^{2p\gamma Y^m_*} + \left(\int_0^T |Z^m_s|^2 ds\right)^{2p} + \big(K^m_T\big)^{2p}\right]\right\}^{\frac{1}{2}}
\end{aligned}$$

$$\leq c_p E\bigg[\sup_{t\in[0,T]} |Y_t^n - Y_t^0|^{2p}\bigg] + \frac{1}{2} E\bigg[\bigg(\int_0^T |Z_s^n - Z_s^0|^2 ds\bigg)^p\bigg] + c_p \sqrt{\widetilde{\Xi}(2p)} \bigg\{E\bigg[\sup_{t\in[0,T]} |Y_t^n - Y_t^0|^{2p}\bigg]\bigg\}^{\frac{1}{2}}.$$

It is clear that $E\left[\left(\int_0^T |Z_s^n - Z_s^0|^2 ds\right)^p\right] < \infty$ as $Z^n, Z^0 \in \mathbb{H}_{\mathbf{F}}^{2,2p}([0,T];\mathbb{R}^d)$. Hence, it follows that

$$E\bigg[\bigg(\int_0^T |Z_s^n - Z_s^0|^2 ds\bigg)^p\bigg] \leq c_p E\bigg[\sup_{t\in[0,T]} |Y_t^n - Y_t^0|^{2p}\bigg] + c_p \sqrt{\widetilde{\Xi}(2p)} \bigg\{E\bigg[\sup_{t\in[0,T]} |Y_t^n - Y_t^0|^{2p}\bigg]\bigg\}^{\frac{1}{2}}.$$

As $n \to \infty$, (6.16) implies

$$\lim_{n\to\infty} E\bigg[\bigg(\int_0^T |Z_s^n - Z_s^0|^2 ds\bigg)^p\bigg] = 0. \tag{6.17}$$

(4) Let us further assume that $dt \otimes dP$-a.e., $f_n(t,\omega,y,z)$ converges to $f_0(t,\omega,y,z)$ locally uniformly in (y,z). By (6.16) and (6.17) with $p = 1$, $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$ has a subsequence (we still denote it by $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty} \sup_{t\in[0,T]} |Y_t^n - Y_t^0| = 0$, P-a.s. and $\lim_{n\to\infty} Z_t^n = Z_t^0$, $dt \otimes dP$-a.e. In fact, we can choose this subsequence so that $\sup_{n\in\mathbb{N}} |Z^n| \in \mathbb{H}_{\mathbf{F}}^2[0,T]$; see [16] or [14, Lemma 2.5]. Fix $p \in [1,\infty)$. Using similar arguments to those lead to (3.25), we can deduce from (S1) and (6.5) that

$$\lim_{n\to\infty} E\bigg[\bigg(\int_0^T |f_n(s,Y_s^n,Z_s^n) - f_0(s,Y_s^0,Z_s^0)| ds\bigg)^p\bigg] = 0. \tag{6.18}$$

For any $n \in \mathbb{N}$, it holds P-a.s. that

$$K_t^n - K_t^0 = Y_0^n - Y_0^0 - (Y_t^n - Y_t^0) - \int_0^t \big(f_n(s,Y_s^n,Z_s^n) - f_0(s,Y_s^0,Z_s^0)\big)\, ds + \int_0^t (Z_s^n - Z_s^0)\, dB_s, \quad t \in [0,T].$$

The Burkholder–Davis–Gundy inequality then implies that

$$E\bigg[\sup_{t\in[0,T]} |K_t^n - K_t^0|^p\bigg] \leq c_p\, E\bigg[\sup_{t\in[0,T]} |Y_t^n - Y_t^0|^p + \bigg(\int_0^T |f_n(s,Y_s^n,Z_s^n) - f_0(s,Y_s^0,Z_s^0)| ds\bigg)^p + \bigg(\int_0^T |Z_s^n - Z_s^0|^2 ds\bigg)^{\frac{p}{2}}\bigg],$$

where $E\left[\left(\int_0^T |Z_s^n - Z_s^0|^2 ds\right)^{\frac{p}{2}}\right] \le \left\{E\left[\left(\int_0^T |Z_s^n - Z_s^0|^2 ds\right)^p\right]\right\}^{\frac{1}{2}}$ due to Hölder's inequality. As $n \to \infty$, (6.16)–(6.18) lead to $\lim_{n\to\infty} E\left[\sup_{t\in[0,T]} |K_t^n - K_t^0|^p\right] = 0$. □

7. An obstacle problem for PDEs

In this section, we show that in the Markovian case, quadratic RBSDEs with unbounded obstacles provide a probabilistic interpretation of solutions of some obstacle problem for semi-linear parabolic PDEs, in which the non-linearity appears as the square of the gradient.

For any $t \in [0, \infty)$, $B^t = \{B_s^t \triangleq B_{t+s} - B_t\}_{s\in[0,\infty)}$ is also a d-dimensional standard Brownian Motion on the probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbf{F}^t$ be the augmented filtration generated by B^t, i.e., $\mathbf{F}^t = \left\{\mathcal{F}_s^t \triangleq \sigma\left(\sigma\left(B_r^t; r \in [0, s]\right) \cup \mathcal{N}\right)\right\}_{s\ge 0}$. Let $k \in \mathbb{N}$, $\kappa \ge 0$ and $\varpi \in [1, 2)$. We consider the following functions:

(1) $b : [0, T] \times \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{k\times d}$ are two continuous functions such that $\sigma_* \triangleq \sup_{(t,x)\in[0,T]\times\mathbb{R}^k} |\sigma(t, x)| < \infty$, and that

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \le \kappa|x - x'|, \quad \forall\, t \in [0, T],\ \forall\, x, x' \in \mathbb{R}^k. \tag{7.1}$$

(2) $h : \mathbb{R}^k \to \mathbb{R}$ and $l : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ are two continuous functions such that

$$l(T, x) \le h(x), \quad \forall\, x \in \mathbb{R}^k \quad \text{and} \quad |h(x)| \vee |l(t, x)| \le \kappa\left(1 + |x|^{\varpi}\right), \quad \forall\, (t, x) \in [0, T] \times \mathbb{R}^k. \tag{7.2}$$

(3) $f : [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a jointly continuous function that satisfies

(i) There exist $\alpha, \beta \ge 0$ and $\gamma > 0$ such that for any $(t, x, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^d$ and $y, y' \in \mathbb{R}$

$$|f(t, x, y, z)| \le \alpha + \beta|y| + \frac{\gamma}{2}|z|^2 \quad \text{and} \quad |f(t, x, y, z) - f(t, x, y', z)| \le \kappa|y - y'|; \tag{7.3}$$

(ii) The mapping $z \to f(t, x, y, z)$ is *concave* for all $(t, x, y) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}$. (7.4)

For any $\lambda \ge 0$, we let $\widetilde{c}_\lambda$ denote a generic constant, depending on $\lambda, \alpha, \beta, \gamma, \kappa, \varpi, T, \sigma_*$ and on $b_0 \triangleq \sup_{t\in[0,T]} |b(t, 0)| < \infty$, whose form may vary from line to line.

Given $(t, x) \in [0, T] \times \mathbb{R}^k$, it is well-known that the SDE

$$X_s = x + \int_t^s b(r, X_r)dr + \int_t^s \sigma(r, X_r)dB_r, \quad s \in [t, T] \tag{7.5}$$

admits a unique solution $\left\{X_s^{t,x}\right\}_{s\in[t,T]}$, an $\mathbb{R}^k$-valued continuous process, such that $X_s^{t,x} \in \mathcal{F}_{s-t}^t \subset \mathcal{F}_s$ for any $s \in [t, T]$. In addition, we set $X_s^{t,x} \triangleq x$, $\forall\, s \in [0, t]$.

We recall from [9, Section 5] the following estimate for the exponential moments of process $\left\{|X_s^{t,x}|^{\varpi}\right\}_{s\in[t,T]}$.

Lemma 7.1. *Let* $p \in [1,\infty)$. *For any* $(t,x) \in [0,T] \times \mathbb{R}^k$, *we have*

$$E\left[\exp\left\{p \sup_{s\in[t,T]} \left|X_s^{t,x}\right|^{\varpi}\right\}\right] \le \widetilde{c}_p \exp\left\{p\, 3^{\varpi-1} e^{\kappa\varpi T} |x|^{\varpi}\right\}.$$

Our objective in this section is to find a unique viscosity solution of the following obstacle problem for semi-linear parabolic PDEs:

$$\begin{cases} \min\left\{(u-l)(t,x), -\partial_t u(t,x) - \mathcal{L}u(t,x) - f\Big(t,x,u(t,x),(\sigma^T\cdot\nabla_x u)(t,x)\Big)\right\} = 0, \\ \quad \forall\,(t,x) \in (0,T)\times\mathbb{R}^k, \\ u(T,x) = h(x), \quad \forall\, x \in \mathbb{R}^k, \end{cases} \tag{7.6}$$

where σ^T denotes the transpose of σ and $\mathcal{L}u(t,x) \triangleq \frac{1}{2}\mathrm{trace}\big((\sigma\sigma^T D_x^2 u)(t,x)\big) + \langle b(t,x), \nabla_x u(t,x)\rangle$.

Definition 7.1. A function $u \in C([0,T]\times\mathbb{R}^k)$ is called a viscosity subsolution (resp. viscosity supersolution) of (7.6) if $u(T,x) \le$ (resp. $\ge$) $h(x)$, $\forall\, x \in \mathbb{R}^k$, and if for any $(t_0,x_0,\varphi) \in (0,T)\times\mathbb{R}^k\times C^{1,2}\big([0,T]\times\mathbb{R}^k\big)$ such that $u(t_0,x_0) = \varphi(t_0,x_0)$ and that $u-\varphi$ attains a local maximum (resp. local minimum) at (t_0,x_0), we have

$$\min\left\{(u-l)(t_0,x_0), -\partial_t\varphi(t_0,x_0) - f\big(t_0,x_0,u(t_0,x_0),(\sigma^T\cdot\nabla_x u)(t_0,x_0)\big)\right\} \le \text{(resp. } \ge\text{) } 0.$$

A function $u \in C([0,T]\times\mathbb{R}^k)$ is called a viscosity solution of (7.6) if it is both a viscosity subsolution and a viscosity supersolution of (7.6).

For any $(t,x) \in [0,T]\times\mathbb{R}^k$, let $\mathscr{P}^t$ denote the $\mathbf{F}^t$-progressively measurable σ-field on $[0,T-t]\times\Omega$. Since $\widetilde{X}_s^{t,x} \triangleq X_{t+s}^{t,x}$, $s \in [0,T-t]$ is an $\mathbf{F}^t$-adapted continuous process, the joint continuity of f implies that

$$\tilde{f}^{t,x}(s,\omega,y,z) \triangleq f\Big(t+s, \widetilde{X}_s^{t,x}(\omega), y, z\Big), \quad \forall\,(s,\omega,y,z) \in [0,T-t]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$$

is a $\mathscr{P}^t \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function, namely, it is a generator with respect to $\mathbf{F}^t$ over the period $[0,T-t]$. By (7.3) and (7.4), $\tilde{f}^{t,x}$ also satisfies (H1)–(H3). On the other hand, (7.2) shows that $\big\{\widetilde{L}_s^{t,x} \triangleq l\big(t+s,\widetilde{X}_s^{t,x}\big)\big\}_{s\in[0,T-t]}$ is also an $\mathbf{F}^t$-adapted continuous process such that $\widetilde{L}_{T-t}^{t,x} = l\big(T,X_T^{t,x}\big) \le h\big(X_T^{t,x}\big) \in \mathcal{F}_{T-t}^t$. For any $p \in [1,\infty)$, (7.2) and Lemma 7.1 imply that

$$\begin{aligned} E\left[\exp\left\{p\Big(|h\big(X_T^{t,x}\big)| \vee \widetilde{L}_*^{t,x}\Big)\right\}\right] &\le e^{p\kappa} E\left[\exp\Big\{(1\vee p\kappa) \sup_{s\in[t,T]} |X_s^{t,x}|^{\varpi}\Big\}\right] \\ &\le \widetilde{c}_p \exp\left\{(1\vee p\kappa)\, 3^{\varpi-1} e^{\kappa\varpi T}|x|^{\varpi}\right\}. \end{aligned} \tag{7.7}$$

Hence, Theorems 3.2 and 4.1 show that the quadratic RBSDE$\Big(h(X_T^{t,x}), \tilde{f}^{t,x}, \widetilde{L}^{t,x}\Big)$ with respect to B^t over the period $[0,T-t]$ admits a unique solution $\Big(\widetilde{Y}^{t,x}, \widetilde{Z}^{t,x}, \widetilde{K}^{t,x}\Big)$ in $\cap_{p\in[1,\infty)} \mathbb{S}_{\mathbf{F}^t}^p[0,T-t]$.

The continuity of process $\{X_s^{t,x}\}_{s\in[0,T]}$, (7.3) and (7.4) imply that

$$f^{t,x}(s,\omega,y,z) \triangleq \mathbf{1}_{\{s\ge t\}}\tilde f^{t,x}(s-t,\omega,y,z) = \mathbf{1}_{\{s\ge t\}} f\big(s, X_s^{t,x}(\omega), y, z\big),$$
$$\forall (s,\omega,y,z)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$$

is a $\mathscr{P}\times\mathscr{B}(\mathbb{R})\times\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function that satisfies (H1)–(H3) with the same constants $\alpha,\beta,\kappa\ge 0$ and $\gamma>0$ as f. Let $L_s^{t,x}\triangleq \widetilde{L}^{t,x}_{(s-t)^+} = l\big(s\vee t, X^{t,x}_{s\vee t}\big)$, $s\in[0,T]$, which is clearly an $\mathbf{F}$-adapted continuous process with $L_T^{t,x}=\widetilde{L}^{t,x}_{T-t}\le h\big(X_T^{t,x}\big)$. Then one can show that

$$\big(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}\big) \triangleq \Big(\widetilde{Y}^{t,x}_{(s-t)^+}, \mathbf{1}_{\{s\ge t\}}\widetilde{Z}^{t,x}_{s-t}, \mathbf{1}_{\{s\ge t\}}\widetilde{K}^{t,x}_{s-t}\Big),\quad s\in[0,T]$$

satisfies the quadratic RBSDE$\big(h(X_T^{t,x}), f^{t,x}, L^{t,x}\big)$ over the period $[0,T]$, and that $\big(Y^{t,x}, Z^{t,x}, K^{t,x}\big)\in\cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,T]$. Since $E\Big[\exp\Big\{p\Big(|h\big(X_T^{t,x}\big)|\vee L_*^{t,x}\Big)\Big\}\Big]<\infty$ by (7.7), Theorems 3.2 and 4.1 again show that $\big(Y^{t,x}, Z^{t,x}, K^{t,x}\big)$ is the unique solution of the quadratic RBSDE$\big(h(X_T^{t,x}), f^{t,x}, L^{t,x}\big)$ in $\cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,T]$.

The main objective of this section is to demonstrate that

$$u(t,x)\triangleq \widetilde{Y}_0^{t,x} = Y_t^{t,x},\quad \forall (t,x)\in[0,T]\times\mathbb{R}^k \tag{7.8}$$

is a viscosity solution of (7.6).

Proposition 7.1. *The function u defined in (7.8) is continuous such that $|u(t,x)|\le \widetilde{c}_0\big(1+|x|^{\varpi}\big)$ for any $(t,x)\in[0,T]\times\mathbb{R}^k$.*

Sketch of the Proof: Given $(t,x)\in[0,T]\times\mathbb{R}^k$, for each sequence $\{(t_n,x_n)\}_{n\in\mathbb{N}}\subset[0,T]\times\mathbb{R}^k$ that converges to (t,x), a standard calculation shows that $\lim_{n\to\infty}E\Big[\sup_{s\in[0,T]}|X_s^{t_n,x_n}-X_s^{t,x}|^2\Big]=0$. Thus, up to a subsequence, we can deduce from the continuity of functions h,l and f that P-a.s.

$$\lim_{n\to\infty} h\big(X_T^{t_n,x_n}\big) = h\big(X_T^{t,x}\big),\qquad \lim_{n\to\infty}\sup_{s\in[0,T]}|L_s^{t_n,x_n}-L_s^{t,x}|=0,$$
$$\text{and}\quad \lim_{n\to\infty} f^{t_n,x_n}(s,y,z) = f^{t,x}(s,y,z),\quad \forall (s,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d.$$

Then applying Theorem 6.1 yields that $\lim_{n\to\infty}\sup_{s\in[0,T]}|Y_s^{t_n,x_n}-Y_s^{t,x}|=0$, P-a.s. In particular, one has $\lim_{n\to\infty}u\big(t_n,x_n\big)=\lim_{n\to\infty}\widetilde{Y}_0^{t_n,x_n}=\lim_{n\to\infty}Y_0^{t_n,x_n}=Y_0^{t,x}=\widetilde{Y}_0^{t,x}=u(t,x)$. Moreover, Theorem 3.2, (7.2) and Lemma 7.1 imply that

$$\begin{aligned}-\kappa\big(1+|x|^{\varpi}\big) &\le l(t,x) = \widetilde{L}_0^{t,x}\le u(t,x)=\widetilde{Y}_0^{t,x}\le \widetilde{c}_0\\ &\quad + \frac{1}{\gamma}\ln E\Big[\exp\Big\{\gamma e^{\beta T}\big(|h(X_T^{t,x})|\vee\widetilde{L}_*^{t,x}\big)\Big\}\Big]\\ &\le \widetilde{c}_0+\frac{1}{\gamma}\ln E\Big[\exp\Big\{\gamma\kappa e^{\beta T}\sup_{s\in[t,T]}|X_s^{t,x}|^{\varpi}\Big\}\Big]\le \widetilde{c}_0\big(1+|x|^{\varpi}\big).\quad\square\end{aligned}$$

For any $\xi\in\mathcal{O}^{t,x}\triangleq\big\{\xi\in\mathbb{L}^0(\mathcal{F}_T):\ \xi\ge L_T^{t,x},\ P\text{-a.s. and } E\big[e^{p\,\xi^+}\big]<\infty,\ \forall p\in(1,\infty)\big\}$, Theorems 3.2 and 4.1 assure a unique solution $\big(Y^{t,x,\xi}, Z^{t,x,\xi}, K^{t,x,\xi}\big)$ of the quadratic RBSDE$\big(\xi, f^{t,x}, L^{t,x}\big)$ in $\cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,T]$. For each $s\in[0,T]$, we can regard $\mathcal{E}^{t,x}[\xi|\mathcal{F}_s]\triangleq$

$Y_s^{t,x,\xi}$, $\xi \in \mathcal{O}^{t,x}$ as a non-linear conditional expectation on $\mathcal{O}^{t,x}$ with respect to $\mathcal{F}_s$ (cf. g-expectations in the case of BSDEs, see e.g., [23,19], Subsection 5.4 of [3] and Section 5 of the current paper). Then the diffusion $X^{t,x}$ has the following Markov property under $\mathcal{E}^{t,x}$:

Proposition 7.2. *Let u be the function defined in (7.8). For any $(t,x) \in [0,T] \times \mathbb{R}^k$ it holds P-a.s. that*

$$u\big(s, X_s^{t,x}\big) = Y_s^{t,x} = \widetilde{Y}_{s-t}^{t,x}, \quad s \in [t,T]. \tag{7.9}$$

For case of quadratic BSDEs, [14] pointed out that the flow property derives from the Markovian property of the diffusion process $X^{t,x}$ and from the uniqueness of the quadratic BSDE (see line 1–5 of page 591 therein). However, the author neither proved it in details nor mentioned the role of the stability result. So we would like to provide a complete proof of the flow property (7.9):

Proof of Proposition 7.2. (1) We fix $s \in [t,T]$ and denote $\Theta_{t'}^0 \triangleq \Theta_{t'}^{t,x}$, $t' \in [s,T]$ for $\Theta = X, Y, Z, K$. Given $n \in \mathbb{N}$, there exist a finite subset $\{x_i^n\}_{i=1}^{j_n}$ of $B_{2^n}(0) \triangleq \{x \in \mathbb{R}^k : |x| < 2^n\}$ and a disjoint partition $\{\mathcal{I}_i^n\}_{i=1}^{j_n}$ of $B_{2^n}(0)$ such that $x_i^n \in \mathcal{I}_i^n \in \mathscr{B}(\mathbb{R}^k)$ and $\mathcal{I}_i^n \subset \overline{B}_{2^{-n}}(x_i^n)$ for $i = 1, \ldots, j_n$. Let $\mathcal{A}_i^n \triangleq \{X_s^0 \in \mathcal{I}_i^n\} \in \mathcal{F}_s$, $i = 1, \ldots, j_n$ and let $\mathcal{A}_0^n \triangleq \{X_s^0 \in B_{2^n}^c(0)\} \in \mathcal{F}_s$. For any $t' \in [s,T]$ and $\Theta = X, Y, Z, K$, we define $\Theta_{t'}^n \triangleq \sum_{i=0}^{j_n} \mathbf{1}_{\mathcal{A}_i^n} \Theta_{t'}^{s,x_i^n} \in \mathcal{F}_{t'}$ with $x_0^n \triangleq 0$. Then for any $i = 0, \ldots, j_n$,

$$\begin{aligned}
\mathbf{1}_{\mathcal{A}_i^n} X^{s,x_i^n} &= x_i^n \mathbf{1}_{\mathcal{A}_i^n} + \int_s^{t'} \mathbf{1}_{\mathcal{A}_i^n} b\Big(r, X_r^{s,x_i^n}\Big) dr + \int_s^{t'} \mathbf{1}_{\mathcal{A}_i^n} \sigma\Big(r, X_r^{s,x_i^n}\Big) dB_r \\
&= x_i^n \mathbf{1}_{\mathcal{A}_i^n} + \int_s^{t'} \mathbf{1}_{\mathcal{A}_i^n} b\big(r, X_r^n\big) dr + \int_s^{t'} \mathbf{1}_{\mathcal{A}_i^n} \sigma\big(r, X_r^n\big) dB_r, \quad P\text{-a.s.;}
\end{aligned}$$

and that

$$\begin{aligned}
\mathbf{1}_{\mathcal{A}_i^n} l\big(t', X_{t'}^n\big) &= \mathbf{1}_{\mathcal{A}_i^n} l\big(t', X_{t'}^{s,x_i^n}\big) = \mathbf{1}_{\mathcal{A}_i^n} L_{t'}^{s,x_i^n} \le \mathbf{1}_{\mathcal{A}_i^n} Y_{t'}^{s,x_i^n} \\
&= \mathbf{1}_{\mathcal{A}_i^n} h\big(X_T^{s,x_i^n}\big) + \int_{t'}^T \mathbf{1}_{\mathcal{A}_i^n} f\big(r, X_r^{s,x_i^n}, Y_r^{s,x_i^n}, Z_r^{s,x_i^n}\big) dr \\
&\quad + \mathbf{1}_{\mathcal{A}_i^n} K_T^{s,x_i^n} - \mathbf{1}_{\mathcal{A}_i^n} K_{t'}^{s,x_i^n} - \int_{t'}^T \mathbf{1}_{\mathcal{A}_i^n} Z_r^{s,x_i^n} dB_r \\
&= \mathbf{1}_{\mathcal{A}_i^n} h\big(X_T^n\big) + \int_{t'}^T \mathbf{1}_{\mathcal{A}_i^n} f\big(r, X_r^n, Y_r^n, Z_r^n\big) dr + \mathbf{1}_{\mathcal{A}_i^n} K_T^n - \mathbf{1}_{\mathcal{A}_i^n} K_{t'}^n \\
&\quad - \int_{t'}^T \mathbf{1}_{\mathcal{A}_i^n} Z_r^n dB_r, \quad P\text{-a.s.}
\end{aligned}$$

Summing up both expressions over $i = 0, \ldots, j_n$, one can deduce from the continuity of function l as well as the continuity of processes $\{X_{t'}^n\}_{t' \in [s,T]}$, $\{Y_{t'}^n\}_{t' \in [s,T]}$ and $\{K_{t'}^n\}_{t' \in [s,T]}$ that P-a.s.

$$X_{t'}^n = X_s^n + \int_s^{t'} b(r, X_r^n) dr + \int_s^{t'} \sigma(r, X_r^n) dB_r, \quad t' \in [s,T]; \tag{7.10}$$

$$l\big(t', X_{t'}^n\big) \le Y_{t'}^n = h(X_T^n) + \int_{t'}^T f(r, X_r^n, Y_r^n, Z_r^n) dr + K_T^n - K_{t'}^n - \int_{t'}^T Z_r^n dB_r, \quad t' \in [s,T]. \tag{7.11}$$

Moreover, we also have

$$\int_s^T \big(Y_r^n - l(r, X_r^n)\big) dK_r^n = \sum_{i=0}^{j_n} \mathbf{1}_{\mathcal{A}_i^n} \int_s^T \big(Y_r^{s,x_i^n} - L_r^{s,x_i^n}\big)\, dK_r^{s,x_i^n} = 0, \quad P\text{-a.s.} \tag{7.12}$$

By (7.5), it holds P-a.s. that $X_{t'}^0 = X_s^0 + \int_s^{t'} b(r, X_r^0)dr + \int_s^{t'} \sigma(r, X_r^0)dB_r$, $\forall\, t' \in [s, T]$. Subtracting it from (7.10), we see from (7.1) that P-a.s.

$$\sup_{s' \in [s,t']} |X_{s'}^n - X_{s'}^0| \le |X_s^n - X_s^0| + \kappa \int_s^{t'} |X_r^n - X_r^0| dr + \sup_{s' \in [s,t']} \left| \int_s^{s'} \big(\sigma(r, X_r^n) - \sigma(r, X_r^0)\big) dB_r \right|, \quad t' \in [s, T]. \tag{7.13}$$

Similar to Lemma 7.1, we can deduce that (see [2] for details)

$$E\left[\exp\left\{p \sup_{t' \in [s,T]} |X_{t'}^n - X_{t'}^0|^{\varpi}\right\}\right] \le \widetilde{c}_p \left\{E\left[\exp\left\{p\, 2^{\varpi} e^{\kappa \varpi T} |X_s^n - X_s^0|^{\varpi}\right\}\right]\right\}^{\frac{1}{2}} \le \widetilde{c}_p \left\{E\left[\exp\left\{p\, 2^{2\varpi - 1} e^{\kappa \varpi T} |X_s^0|^{\varpi}\right\}\right]\right\}^{\frac{1}{2}},$$

where we used the fact that

$$|X_s^n - X_s^0| = \mathbf{1}_{\{|X_s^0| < 2^n\}} |X_s^n - X_s^0| + \mathbf{1}_{\{|X_s^0| \ge 2^n\}} |X_s^0| \le 2^{-n} + \mathbf{1}_{\{|X_s^0| \ge 2^n\}} |X_s^0|. \tag{7.14}$$

Thus it follows that for any $p \in [1, \infty)$

$$\begin{aligned} E\left[\exp\left\{p \sup_{t' \in [s,T]} |X_{t'}^n|^{\varpi}\right\}\right] &\le \frac{1}{2} E\left[\exp\left\{p 2^{\varpi} \sup_{t' \in [s,T]} |X_{t'}^n - X_{t'}^0|^{\varpi}\right\}\right] + \frac{1}{2} E\left[\exp\left\{p 2^{\varpi} \sup_{t' \in [s,T]} |X_{t'}^0|^{\varpi}\right\}\right] \\ &\le \widetilde{c}_p + E\left[\exp\left\{p\, 2^{3\varpi - 1} e^{\kappa \varpi T} \sup_{t' \in [s,T]} |X_{t'}^0|^{\varpi}\right\}\right]. \end{aligned} \tag{7.15}$$

As $\left\{\big(Y^{s,x_i^n}, Z^{s,x_i^n}, K^{s,x_i^n}\big)\right\}_{i=0,\dots,j_n} \subset \cap_{p \in [1,\infty)} \mathbb{S}_{\mathbf{F}}^p[0, T]$, it holds for any $p \in [1, \infty)$ that

$$\begin{aligned} &E\left[\exp\left\{p \sup_{t' \in [s,T]} |Y_{t'}^n|\right\} + \left(\int_s^T |Z_r^n|^2 dr\right)^p + \big(K_T^n\big)^p\right] \\ &\le \sum_{i=0}^{j_n} E\left[\exp\left\{p \sup_{t' \in [s,T]} \left|Y_{t'}^{s,x_i^n}\right|\right\} + \left(\int_s^T \left|Z_r^{s,x_i^n}\right|^2 dr\right)^p + \left(K_T^{s,x_i^n}\right)^p\right] < \infty. \end{aligned} \tag{7.16}$$

(2) Fix $m \in \mathbb{N}_0$. As $\mathcal{X}_{t'}^m \triangleq \mathbf{1}_{\{t' < s\}} E[X_s^m | \mathcal{F}_{t'}] + \mathbf{1}_{\{t' \ge s\}} X_{t'}^m$, $t' \in [0, T]$ is an $\mathbf{F}$-adapted continuous process, the continuity of function l and f shows that $\mathcal{L}_{t'}^m \triangleq l\big(t', \mathcal{X}_{t'}^m\big)$, $t' \in [0, T]$ is also an $\mathbf{F}$-adapted continuous process and

$$f_m(t', \omega, y, z) \triangleq f\big(t', \mathcal{X}_{t'}^m(\omega), y, z\big), \quad \forall\, (t', \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^d$$

is a $\mathscr{P} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function. Moreover, (7.3)–(7.4) show that f_m satisfies (H1)–(H3) with the same constants $\alpha, \beta, \kappa \geq 0$ and $\gamma > 0$ as f. For any $p \in (1, \infty)$, the convexity of function $y \to e^{|y|^{\varpi}}$ on $\mathbb{R}$ and Jensen's inequality imply that $\Big\{\exp\Big\{\Big(E\big[|X^n_s|\,|\mathcal{F}_{t'}\big]\Big)^{\varpi}\Big\}\Big\}_{t'\in[0,\infty)}$ is a continuous positive submartingale. Doob's Martingale Inequality then shows that

$$E\bigg[\sup_{t'\in[0,s]}\Big(\exp\Big\{\Big(E\big[|X^m_s|\,|\mathcal{F}_{t'}\big]\Big)^{\varpi}\Big\}\Big)^p\bigg] \leq \bigg(\frac{p}{p-1}\bigg)^p E\Big[\Big(\exp\big\{|X^m_s|^{\varpi}\big\}\Big)^p\Big],$$

which together with (7.15) and Lemma 7.1 leads to that

$$\begin{aligned}
E\Big[\exp\Big\{p\big(\mathcal{X}^m_*\big)^{\varpi}\Big\}\Big] &\leq E\bigg[\sup_{t'\in[0,s]}\exp\Big\{p\Big(E\big[|X^m_s|\,|\mathcal{F}_{t'}\big]\Big)^{\varpi}\Big\}\bigg] + E\bigg[\sup_{t'\in[s,T]}\exp\Big\{p|X^m_{t'}|^{\varpi}\Big\}\bigg] \\
&\leq \widetilde{c}_p E\bigg[\exp\bigg\{p\sup_{t'\in[s,T]}|X^m_{t'}|^{\varpi}\bigg\}\bigg] \\
&\leq \widetilde{c}_p + \widetilde{c}_p E\bigg[\exp\bigg\{p\,2^{3\varpi-1}e^{\kappa\varpi T}\sup_{t'\in[s,T]}|X^0_{t'}|^{\varpi}\bigg\}\bigg] \\
&\leq \widetilde{c}_p + \widetilde{c}_p \exp\Big\{p\,2^{3\varpi-1}3^{\varpi-1}e^{2\kappa\varpi T}|x|^{\varpi}\Big\}.
\end{aligned}$$

Hence it follows from (7.2) that

$$\begin{aligned}
E\Big[\exp\Big\{p\Big(|h(\mathcal{X}^m_T)| \vee \mathcal{L}^m_*\Big)\Big\}\Big] &\leq e^{p\kappa} E\Big[\exp\big\{(1\vee p\kappa)\big(\mathcal{X}^m_*\big)^{\varpi}\big\}\Big] \\
&\leq \widetilde{c}_p + \widetilde{c}_p \exp\Big\{(1\vee p\kappa)\,2^{3\varpi-1}3^{\varpi-1}e^{2\kappa\varpi T}|x|^{\varpi}\Big\}. \quad (7.17)
\end{aligned}$$

As $Y^{t,x} \in \mathbb{E}^p_{\mathbf{F}}[0,T]$, we also see from (7.16) that $E\Big[e^{p\,|Y^m_s|}\Big] < \infty$. Since $Y^m_s \geq l\big(s, X^m_s\big) = l\big(s, \mathcal{X}^m_s\big) = \mathcal{L}^m_s$, P-a.s., Theorems 3.2 and 4.1 imply that the quadratic RBSDE$(Y^m_s, f_m, \mathcal{L}^m)$ over time interval $[0,s]$ admits a unique solution $\big\{(\mathcal{Y}^m_r, \mathcal{Z}^m_r, \mathcal{K}^m_r)\big\}_{r\in[0,s]}$ in $\cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,s]$.

We extend the processes $(\mathcal{Y}^m, \mathcal{Z}^m, \mathcal{K}^m)$ to the period $(s,T]$ by setting $(\mathcal{Y}^m_{t'}, \mathcal{Z}^m_{t'}, \mathcal{K}^m_{t'}) \triangleq (Y^m_{t'}, Z^m_{t'}, \mathcal{K}^m_s + K^m_{t'} - K^m_s)$, $\forall\, t' \in (s,T]$. Then (7.11) and (7.12) imply that $\{(\mathcal{Y}^m_{t'}, \mathcal{Z}^m_{t'}, \mathcal{K}^m_{t'})\}_{t'\in[0,T]}$ solves the quadratic RBSDE$\big(h(\mathcal{X}^m_T), f_m, \mathcal{L}^m\big)$. As $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,T]$, (7.16) shows that $(\mathcal{Y}^m, \mathcal{Z}^m, \mathcal{K}^m) \in \cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,T]$. Moreover, Theorems 3.2 and 4.1 and (7.17) yield that $(\mathcal{Y}^m, \mathcal{Z}^m, \mathcal{K}^m)$ is the unique solution of the quadratic RBSDE$\big(h(\mathcal{X}^m_T), f_m, \mathcal{L}^m\big)$ in $\cap_{p\in[1,\infty)}\mathbb{S}^p_{\mathbf{F}}[0,T]$.

(3) Squaring both sides of (7.13), one can deduce from Hölder's inequality, Doob's martingale inequality, Fubini's Theorem and (7.1) that

$$\begin{aligned}
&E\bigg[\sup_{s'\in[s,t']}|X^n_{s'} - X^0_{s'}|^2\bigg] \\
&\quad\leq 3E\Big[|X^n_s - X^0_s|^2\Big] + 3\kappa^2(T+4)\int_s^{t'} E\bigg[\sup_{s'\in[s,r]}|X^n_{s'} - X^0_{s'}|^2\bigg]dr, \quad t' \in [s,T].
\end{aligned}$$

Then Gronwall's inequality and (7.14) imply that

$$E\Big[\sup_{t'\in[s,T]}|X^n_{t'}-X^0_{t'}|^2\Big] \le 3E\Big[|X^n_s-X^0_s|^2\Big]\,e^{3\kappa^2(T^2+4T)} \le \widetilde{c}_0\Big(2^{-2n}+E[\mathbf{1}_{\{|X^0_s|\ge 2^n\}}|X^0_s|^2]\Big).$$

As $E\Big[|X^{t,x}_s|^2\Big]<\infty$, letting $n\to\infty$ yields that $\lim_{n\to\infty}E\Big[\sup_{t'\in[s,T]}|X^n_{t'}-X^0_{t'}|^2\Big]=0$. By Doob's martingale inequality

$$E\Big[\sup_{t'\in[0,T]}|\mathcal{X}^n_{t'}-\mathcal{X}^0_{t'}|^2\Big] \le E\Big[\sup_{t'\in[0,s]}|E[X^n_s-X^0_s|\mathcal{F}_{t'}]\,|^2\Big]+E\Big[\sup_{t'\in[s,T]}|X^n_{t'}-X^0_{t'}|^2\Big] \le 5E\Big[\sup_{t'\in[s,T]}|X^n_{t'}-X^0_{t'}|^2\Big].$$

It follows that $\lim_{n\to\infty}E\Big[\sup_{t'\in[0,T]}|\mathcal{X}^n_{t'}-\mathcal{X}^0_{t'}|^2\Big]=0$. Hence, we can pick up a subsequence of $\{\mathcal{X}^n\}_{n\in\mathbb{N}}$ (we still denote it by $\{\mathcal{X}^n\}_{n\in\mathbb{N}}$) such that except on a P-null set $\mathscr{N}$,

$$\lim_{n\to\infty}\Big(\sup_{t'\in[0,T]}|\mathcal{X}^n_{t'}-\mathcal{X}^0_{t'}|\Big)=0 \quad \text{and the path } t'\to\mathcal{X}^0_{t'} \text{ is continuous.} \tag{7.18}$$

To apply Theorem 6.1 to the sequence $\big\{(\mathcal{Y}^n,\mathcal{Z}^n,\mathcal{K}^n)\big\}_{n\in\mathbb{N}}$, let us check the assumptions of this theorem first. We have seen that the sequence $\{f_m\}_{m\in\mathbb{N}_0}$ satisfies (S1), and that (7.17) justifies (S3). Fix $\omega\in\mathscr{N}^c$. For any $\varepsilon>0$, the continuity of h assures that there exists a $\delta(\omega)\in(0,1)$ such that

$$|h(\tilde{x})-h(x')|\vee|l(\tilde{s},\tilde{x})-l(s',x')|<\varepsilon, \qquad \forall\,(\tilde{s},\tilde{x}),(s',x')\in[0,T]\times\mathscr{D}(\omega) \quad \text{with} \quad |\tilde{s}-s'|^2+|\tilde{x}-x'|^2<\delta^2(\omega),$$

where $\mathscr{D}(\omega)\triangleq\Big\{\tilde{x}\in\mathbb{R}^k:|\tilde{x}|\le 1+\sup_{t'\in[0,T]}|\mathcal{X}^0_{t'}(\omega)|<\infty\Big\}$. In light of (7.18), there exists an $N(\omega)\in\mathbb{N}$ such that for any $n\ge N(\omega)$, $\sup_{t'\in[0,T]}|\mathcal{X}^n_{t'}(\omega)-\mathcal{X}^0_{t'}(\omega)|<\delta(\omega)$. Then it holds for any $n\ge N(\omega)$ that $|h\big(\mathcal{X}^n_T(\omega)\big)-h\big(\mathcal{X}^0_T(\omega)\big)|<\varepsilon$ and that $|\mathcal{L}^n_{t'}(\omega)-\mathcal{L}^0_{t'}(\omega)|=|l\big(t'\mathcal{X}^n_{t'}(\omega)\big)-l\big(t'\mathcal{X}^0_{t'}(\omega)\big)|<\varepsilon$ for any $t'\in[0,T]$. Thus (S2) is satisfied. Given $(t',\omega)\in[0,T]\times\mathscr{N}^c$, the continuity of f and (7.18) imply that $\lim_{n\to\infty}f_n\big(t',\omega,\mathcal{Y}^0_{t'}(\omega),\mathcal{Z}^0_{t'}(\omega)\big)=\lim_{n\to\infty}f\big(t',\mathcal{X}^n_{t'}(\omega),\mathcal{Y}^0_{t'}(\omega),\mathcal{Z}^0_{t'}(\omega)\big)=f\big(t',\mathcal{X}^0_{t'}(\omega),\mathcal{Y}^0_{t'}(\omega),\mathcal{Z}^0_{t'}(\omega)\big)=f_0\big(t',\omega,\mathcal{Y}^0_{t'}(\omega),\mathcal{Z}^0_{t'}(\omega)\big)$.

Now, applying Theorem 6.1 yields that $\lim_{n\to\infty}E\Big[\exp\Big\{\sup_{t'\in[0,T]}|\mathcal{Y}^n_{t'}-\mathcal{Y}^0_{t'}|\Big\}\Big]=1$, thus $\{\mathcal{Y}^n\}_{n\in\mathbb{N}}$ has a subsequence (we still denote it by $\{\mathcal{Y}^n\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty}\sup_{t'\in[0,T]}|\mathcal{Y}^n_{t'}-\mathcal{Y}^0_{t'}|=0$, P-a.s. In particular,

$$\lim_{n\to\infty}Y^n_s=\lim_{n\to\infty}\mathcal{Y}^n_s=\mathcal{Y}^0_s=Y^0_s=Y^{t,x}_s, \qquad P\text{-a.s.} \tag{7.19}$$

where $Y^n_s=\sum_{i=0}^{j_n}\mathbf{1}_{\mathcal{A}^n_i}Y^{s,x^n_i}_s=\sum_{i=0}^{j_n}\mathbf{1}_{\mathcal{A}^n_i}u\big(s,x^n_i\big)=\sum_{i=0}^{j_n}\mathbf{1}_{\mathcal{A}^n_i}u\big(s,X^n_s\big)=u\big(s,X^n_s\big)$ for any $n\in\mathbb{N}$. Since $\lim_{n\to\infty}X^n_s=X^0_s=X^{t,x}_s$, P-a.s. by (7.14), Proposition 7.1 and (7.19) then imply that $Y^{t,x}_s=\lim_{n\to\infty}u\big(s,X^n_s\big)=u\big(s,X^{t,x}_s\big)$, P-a.s. Eventually, the continuity of processes $X^{t,x}$, $Y^{t,x}$ and Proposition 7.1 leads to (7.9). □

Theorem 7.1. *The function u defined in (7.8) is a viscosity solution of (7.6).*

Proof. (1) For any $x \in \mathbb{R}^k$, it is clear that $u(T,x) = \widetilde{Y}_0^{T,x} = h\big(X_T^{T,x}\big) = h(x)$. We first show that u is a viscosity subsolution of (7.6). Let $(t_0, x_0, \varphi) \in (0,T) \times \mathbb{R}^k \times C^{1,2}\big([0,T] \times \mathbb{R}^k\big)$ be such that $u(t_0,x_0) = \varphi(t_0,x_0)$ and that $u - \varphi$ attains a local maximum at (t_0,x_0). We prove by contradiction. Suppose that

$$\varepsilon \triangleq \frac{1}{2} \min\{(u-l)(t_0,x_0), -\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) \\ - f\Big(t_0, x_0, \varphi(t_0,x_0), (\sigma^T \nabla_x \varphi)(t_0,x_0)\Big)\Big\} > 0.$$

Since $\varphi \in C^{1,2}\big([0,T]\times\mathbb{R}^k\big)$, the continuity of functions u, l, b, σ and f as well as the assumption on local maximum of $u-\varphi$ assure that there is a $\delta \in \big(0, T-t_0\big]$ such that for any $t \in [t_0, t_0+\delta]$ and any $x \in \mathbb{R}^k$ with $|x - x_0| \le \delta$

$$|u(t,x) - u(t_0,x_0)| \le \frac{1}{3}\varepsilon, \quad (u-l)(t,x) \ge \varepsilon, \ (u-\varphi)(t,x) \le 0, \tag{7.20}$$

$$\text{and} \quad -\partial_t\varphi(t,x) - \mathcal{L}\varphi(t,x) - f\Big(t,x,\varphi(t,x),(\sigma^T\nabla_x\varphi)(t,x)\Big) \ge \varepsilon. \tag{7.21}$$

Since $\big\{\widetilde{X}_s^{t_0,x_0}\big\}_{s\in[0,T-t_0]}$ and $\widetilde{Y}^{t_0,x_0}$ are both $\mathbf{F}^{t_0}$-adapted continuous processes,

$$\nu \triangleq \inf\Big\{s \in [0,\delta] : |\widetilde{X}_s^{t_0,x_0} - x_0| > \delta\Big\} \wedge \inf\Big\{s \in [0,\delta] : |\widetilde{Y}_s^{t_0,x_0} - \widetilde{Y}_0^{t_0,x_0}| > \frac{1}{3}\varepsilon\Big\} \wedge \delta \tag{7.22}$$

defines an $\mathbf{F}^{t_0}$-stopping time such that $\nu > 0$, P-a.s. For any $\omega \in \Omega$ and $s \in [0, \nu(\omega)]$, (7.20) implies that

$$\widetilde{Y}_s^{t_0,x_0}(\omega) \ge \widetilde{Y}_0^{t_0,x_0} - \frac{1}{3}\varepsilon = u(t_0,x_0) - \frac{1}{3}\varepsilon \ge u\big(t_0+s, \widetilde{X}_s^{t_0,x_0}(\omega)\big) - \frac{2}{3}\varepsilon \\ \ge l\big(t_0+s, \widetilde{X}_s^{t_0,x_0}(\omega)\big) + \frac{1}{3}\varepsilon = \widetilde{L}_s^{t_0,x_0}(\omega) + \frac{1}{3}\varepsilon.$$

Because $\big(\widetilde{Y}^{t_0,x_0}, \widetilde{Z}^{t_0,x_0}, \widetilde{K}^{t_0,x_0}\big) \in \cap_{p\in[1,\infty)} \mathbb{S}^p_{\mathbf{F}^{t_0}}[0, T-t_0]$ solves the quadratic RBSDE$\Big(h\big(X_T^{t_0,x_0}\big)$ $, \tilde{f}^{t_0,x_0}, \widetilde{L}^{t_0,x_0}\Big)$ with respect to B^{t_0} over the period $[0, T-t_0]$, its flat-off condition shows that P-a.s., $\widetilde{K}_s^{t_0,x_0} = 0$ for any $s \in [0,\nu]$. Hence, it holds P-a.s. that

$$\widetilde{Y}_{\nu\wedge s}^{t_0,x_0} = \widetilde{Y}_\nu^{t_0,x_0} + \int_{\nu\wedge s}^{\nu} \tilde{f}^{t_0,x_0}\Big(r, \widetilde{Y}_r^{t_0,x_0}, \widetilde{Z}_r^{t_0,x_0}\Big)dr - \int_{\nu\wedge s}^{\nu} \widetilde{Z}_r^{t_0,x_0} dB_r^{t_0}, \quad s \in [0,\delta].$$

To wit, $\big(\mathcal{Y}, \mathcal{Z}\big) \triangleq \Big\{\Big(\widetilde{Y}_{\nu\wedge s}^{t_0,x_0}, \mathbf{1}_{\{s<\nu\}}\widetilde{Z}_s^{t_0,x_0}\Big)\Big\}_{s\in[0,\delta]} \in \mathbb{C}^\infty_{\mathbf{F}^{t_0}}[0,\delta] \times \cap_{p\in[1,\infty)} \mathbb{H}^{2,2p}_{\mathbf{F}^{t_0}}([0,\delta];\mathbb{R}^d)$ solves the BSDE:

$$\mathcal{Y}_s = \widetilde{Y}_\nu^{t_0,x_0} + \int_s^\delta \mathfrak{f}(r, \mathcal{Y}_r, \mathcal{Z}_r)\,dr - \int_s^\delta \mathcal{Z}_r dB_r^{t_0}, \quad s \in [0,\delta], \\ \text{with } \mathfrak{f}(s,\omega,y,z) \triangleq \mathbf{1}_{\{s<\nu(\omega)\}} \tilde{f}^{t_0,x_0}(s,\omega,y,z), \\ \forall\,(s,\omega,y,z) \in [0,\delta] \times \Omega \times \mathbb{R} \times \mathbb{R}^d. \tag{7.23}$$

Like $\tilde{f}^{t_0,x_0}$, $\mathfrak{f}$ is a generator with respect to $\mathbf{F}^{t_0}$ over the period $[0,\delta]$ that satisfies (H1)–(H3).

On the other hand, since

$$\widetilde{X}_s^{t_0,x_0} = x + \int_0^s b(r+t_0, \widetilde{X}_r^{t_0,x_0})dr + \int_0^s \sigma(r+t_0, \widetilde{X}_r^{t_0,x_0})dB_r^{t_0}, \quad s \in [0, T-t_0],$$

applying Itô's formula to the process $\varphi(t_0+\cdot, \widetilde{X}_\cdot^{t_0,x_0})$ yields that

$$\varphi\big(t_0+\nu\wedge s, \widetilde{X}_{\nu\wedge s}^{t_0,x_0}\big) = \varphi\big(t_0+\nu, \widetilde{X}_\nu^{t_0,x_0}\big) - \int_{\nu\wedge s}^{\nu} \big(\partial_t\varphi + \mathcal{L}\varphi\big)\big(t_0+r, \widetilde{X}_r^{t_0,x_0}\big)dr$$
$$- \int_{\nu\wedge s}^{\nu} (\sigma^T\nabla_x\varphi)\big(t_0+r, \widetilde{X}_r^{t_0,x_0}\big)dB_r^{t_0}, \quad s\in[0,\delta].$$

Namely, $(\mathcal{Y}', \mathcal{Z}') \triangleq \big\{\big(\varphi\big(t_0+\nu\wedge s, \widetilde{X}_{\nu\wedge s}^{t_0,x_0}\big), \mathbf{1}_{\{s<\nu\}}(\sigma^T\nabla_x\varphi)\big(t_0+s, \widetilde{X}_s^{t_0,x_0}\big)\big)\big\}_{s\in[0,\delta]}$ solves the BSDE

$$\mathcal{Y}'_s = \varphi\big(t_0+\nu, \widetilde{X}_\nu^{t_0,x_0}\big) + \int_s^\delta \mathfrak{f}'_r dr - \int_s^\delta \mathcal{Z}'_r dB_r^{t_0}, \quad s\in[0,\delta],$$

where $\mathfrak{f}'_s \triangleq -\mathbf{1}_{\{s<\nu\}}\big(\partial_t\varphi + \mathcal{L}\varphi\big)\big(t_0+s, \widetilde{X}_s^{t_0,x_0}\big)$, $\forall s\in[0,\delta]$. Since $\widetilde{X}^{t_0,x_0}$ is an $\mathbf{F}^{t_0}$-adapted continuous process, and since $\varphi \in C^{1,2}\big([0,T]\times\mathbb{R}^k\big)$, the continuity of function σ implies that $\mathcal{Y}'$ is an $\mathbf{F}^{t_0}$-adapted continuous process as well as that $\mathcal{Z}'$ and $\mathfrak{f}'$ are both $\mathbf{F}^{t_0}$-progressively measurable processes. Moreover, since $|\widetilde{X}_s^{t_0,x_0} - x_0| \le \delta$ holds for P-a.s. $\omega\in\Omega$ and $s\in\big[0,\nu(\omega)\big]$, and since $\varphi\in C^{1,2}\big([0,T]\times\mathbb{R}^k\big)$, we further see from the continuity of function b and the boundedness of function σ that $\mathcal{Y}'$, $\mathcal{Z}'$ and $\mathfrak{f}'$ are all bounded processes.

Proposition 7.2 and (7.20)–(7.22) imply that $\widetilde{Y}_\nu^{t_0,x_0} = u\big(t_0+\nu, \widetilde{X}_\nu^{t_0,x_0}\big) \le \varphi\big(t_0+\nu, \widetilde{X}_\nu^{t_0,x_0}\big)$, P-a.s., and that on Ω

$$\begin{aligned}\mathfrak{f}'_s - \mathfrak{f}\big(s, \mathcal{Y}'_s, \mathcal{Z}'_s\big) &= -\mathbf{1}_{\{s<\nu\}}\big(\partial_t\varphi+\mathcal{L}\varphi\big)\big(t_0+s, \widetilde{X}_s^{t_0,x_0}\big)\\ &\quad - \mathbf{1}_{\{s<\nu\}} f\Big(t_0+s, \widetilde{X}_s^{t_0,x_0}, \varphi\big(t_0+s, \widetilde{X}_s^{t_0,x_0}\big),\\ &\quad (\sigma^T\nabla_x\varphi)\big(t_0+s, \widetilde{X}_s^{t_0,x_0}\big)\Big) \ge \varepsilon\mathbf{1}_{\{s<\nu\}}, \quad \forall s\in[0,\delta]. \end{aligned} \tag{7.24}$$

The first part of Proposition 5.1 gives that P-a.s., $\mathcal{Y}'_s \ge \mathcal{Y}_s$ for any $s\in[0,\delta]$. Since $\mathcal{Y}'_0 = \varphi(t_0,x_0) = u(t_0,x_0) = \widetilde{Y}_0^{t_0,x_0} = \mathcal{Y}_0$, the second part of Proposition 5.1 further shows that $P\Big(\int_0^\delta \big(\mathfrak{f}'_s - \mathfrak{f}\big(s,\mathcal{Y}'_s,\mathcal{Z}'_s\big)\big)ds = 0\Big) > 0$. However, (7.24) and (7.22) show that P-a.s., $\int_0^\delta \Big(\mathfrak{f}'_s - \mathfrak{f}\big(s,\mathcal{Y}'_s,\mathcal{Z}'_s\big)\Big)ds \ge \varepsilon\nu > 0$, which leads to a contradiction.

(2) Next, we show that u is a viscosity supersolution of (7.6). Let $(t_0,x_0,\varphi)\in(0,T)\times\mathbb{R}^k\times C^{1,2}\big([0,T]\times\mathbb{R}^k\big)$ be such that $u(t_0,x_0)=\varphi(t_0,x_0)$ and that $u-\varphi$ attains a local minimum at (t_0,x_0). Since $u(t_0,x_0) = Y_{t_0}^{t_0,x_0} \ge L_{t_0}^{t_0,x_0} = l\big(t_0, X_{t_0}^{t_0,x_0}\big) = l(t_0,x_0)$, it suffices to show that

$$-\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) - f\Big(t_0,x_0,\varphi(t_0,x_0),(\sigma^T\nabla_x\varphi)(t_0,x_0)\Big) \ge 0.$$

To make a contradiction, we assume that

$$\varepsilon \triangleq \frac{1}{2}\Big(\partial_t\varphi(t_0,x_0) + \mathcal{L}\varphi(t_0,x_0) + f\Big(t_0,x_0,\varphi(t_0,x_0),(\sigma^T\nabla_x\varphi)(t_0,x_0)\Big)\Big) > 0.$$

Since $\varphi\in C^{1,2}\big([0,T]\times\mathbb{R}^k\big)$, the continuity of functions b,σ and f as well as the assumption on local minimum of $u-\varphi$ assures that there is a $\delta\in\big(0,T-t_0\big]$ such that for any $t\in[t_0,t_0+\delta]$

and any $x \in \mathbb{R}^k$ with $|x - x_0| \le \delta$

$$\partial_t \varphi(t,x) + \mathcal{L}\varphi(t,x) + f\Big(t,x,\varphi(t,x),(\sigma^T\nabla_x\varphi)(t,x)\Big) \ge \varepsilon \quad \text{and} \quad (u-\varphi)(t,x) \ge 0. \tag{7.25}$$

We still define the $\mathbf{F}^{t_0}$-stopping time ν as in (7.22). It is easy to see that the processes

$$\big(\mathcal{Y},\mathcal{Z},\mathcal{V}\big) \triangleq \Big\{\Big(\widetilde{Y}^{t_0,x_0}_{\nu\wedge s}, \mathbf{1}_{\{s<\nu\}}\widetilde{Z}^{t_0,x_0}_s, \widetilde{K}^{t_0,x_0}_{\nu\wedge s}\Big)\Big\}_{s\in[0,\delta]} \in \mathbb{C}^{\infty}_{\mathbf{F}^{t_0}}[0,\delta] \\ \times \bigcap_{p\in[1,\infty)} \mathbb{H}^{2,2p}_{\mathbf{F}^{t_0}}([0,\delta];\mathbb{R}^d) \times \bigcap_{p\in[1,\infty)} \mathbb{K}^{p}_{\mathbf{F}^{t_0}}[0,\delta]$$

solves the BSDE (5.5) with generator $\mathfrak{f}$ defined in (7.23) over the period $[0,\delta]$. Let $(\mathcal{Y}',\mathcal{Z}')$ be the pair of processes considered in part 1. Proposition 7.2, (7.25) and the definition of ν imply that $\widetilde{Y}^{t_0,x_0}_{\nu} = u\big(t_0+\nu, \widetilde{X}^{t_0,x_0}_{\nu}\big) \ge \varphi\big(t_0+\nu, \widetilde{X}^{t_0,x_0}_{\nu}\big)$, P-a.s., and that on Ω

$$\mathfrak{f}\big(s,\mathcal{Y}',\mathcal{Z}'\big) - \mathfrak{f}'_s = \mathbf{1}_{\{s<\nu\}} f\Big(t_0+s, \widetilde{X}^{t_0,x_0}_s, \varphi\big(t_0+s,\widetilde{X}^{t_0,x_0}_s\big), (\sigma^T\nabla_x\varphi)\big(t_0+s,\widetilde{X}^{t_0,x_0}_s\big)\Big) \\ +\mathbf{1}_{\{s<\nu\}}\big(\partial_t\varphi+\mathcal{L}\varphi\big)\big(t_0+s,\widetilde{X}^{t_0,x_0}_s\big) \ge \varepsilon\mathbf{1}_{\{s<\nu\}}, \quad \forall\, s\in[0,\delta].$$

Using similar arguments to those that follow (7.24), we reach a contradiction. □

For the uniqueness of the viscosity solution of (7.6), we establish a comparison principle between its viscosity subsolution and viscosity supersolution, whose proof is inspired by the techniques used in Theorem 3.1 of [8].

Theorem 7.2. *Suppose that there exists an increasing function $\mathfrak{M} : (0,\infty) \to (0,\infty)$ such that for any $R > 0$,*

$$|f(t,x,y,z) - f(t,x',y,z)| \le \mathfrak{M}(R)\big(1+|z|\big)|x-x'| \tag{7.26}$$

holds for any $(t,x,x',y,z) \in [0,T]\times\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$ with $|x|\vee|x'|\vee|y| \le R$. Let $u \in C\big([0,T]\times\mathbb{R}^k\big)$ (resp. $v \in C\big([0,T]\times\mathbb{R}^k\big)$) be a viscosity subsolution (resp. viscosity supersolution) of (7.6) such that for some $\widetilde{\kappa} > 0$,

$$|u(t,x)|\vee|v(t,x)| \le \widetilde{\kappa}(1+|x|^{\varpi}), \quad \forall\,(t,x)\in[0,T]\times\mathbb{R}^k. \tag{7.27}$$

Then $u(t,x) \le v(t,x)$ for all $(t,x)\in[0,T]\times\mathbb{R}^k$.

Proof. For any $\theta \in (0,1]$, we define

$$\tilde{u}_\theta(t,x) \triangleq \theta e^{\kappa t}u(t,x) \quad \text{and} \quad \tilde{v}_\theta(t,x) \triangleq \theta e^{\kappa t}v(t,x), \quad \forall\,(t,x)\in[0,T]\times\mathbb{R}^k.$$

One can show that $\tilde{u}_\theta$ and $\tilde{v}_\theta$ are respectively a viscosity subsolution and a viscosity supersolution of

$$\begin{cases} \min\Big\{\tilde{u}(t,x)-\theta e^{\kappa t}l(t,x), -\partial_t\tilde{u}(t,x)-\mathcal{L}\tilde{u}(t,x)-\tilde{f}_\theta\Big(t,x,\tilde{u}(t,x),\nabla_x\tilde{u}(t,x)\Big)\Big\} = 0, \\ \quad \forall\,(t,x)\in(0,T)\times\mathbb{R}^k, \\ \tilde{u}(T,x) = \theta e^{\kappa T}h(x), \quad \forall\, x\in\mathbb{R}^k, \end{cases} \tag{7.28}$$

with $\tilde{f}_\theta(t,x,y,z) \triangleq -\kappa y + \theta e^{\kappa t} f\Big(t,x,\frac{1}{\theta}e^{-\kappa t}y, \frac{1}{\theta}e^{-\kappa t}\sigma^T(t,x)\cdot z\Big)$, $\forall\,(y,z)\in\mathbb{R}\times\mathbb{R}^k$ (see e.g., Lemma 9.3 of [2]).

Let $\lambda \triangleq 8(b_0+\kappa)+4(1+4\gamma e)\sigma_*^2+2(\alpha+4\kappa\widetilde{\kappa})e^{\kappa T}$. Suppose that we have proven the following statement:

$$\begin{aligned}&\text{For any } [T_1,T_2]\subset[0,T] \quad \text{with } T_2-T_1\le\frac{1}{\lambda}, \quad \text{if } u(T_2,x)\le v(T_2,x),\\ &\quad \forall\, x\in\mathbb{R}^k,\ \text{then} u(t,x)\le v(t,x),\ \forall\,(t,x)\in[T_1,T_2]\times\mathbb{R}^k.\end{aligned} \tag{7.29}$$

Set $N\triangleq\lceil\lambda T\rceil$ and $t_i\triangleq i\frac{T}{N}$, for $i=0,1,\ldots,N$. Starting from $u(T,x)\le h(x)\le v(T,x)$, $\forall\,x\in\mathbb{R}^k$, one can use (7.29) to iteratively shows that $u(t,x)\le v(t,x)$ over $[t_{i-1},t_i]\times\mathbb{R}^k$ for $i=N,\ldots,1$. Hence, it suffices to show (7.29).

Assume that (7.29) does not hold, i.e., there exists a time interval $[T_1,T_2]\subset[0,T]$ with $T_2-T_1\le\frac{1}{\lambda}$ such that $u(T_2,x)\le v(T_2,x)$, $\forall\,x\in\mathbb{R}^k$ and that $u(\hat{t},\hat{x})-v(\hat{t},\hat{x})>\delta$ for some $(\hat{t},\hat{x})\in[T_1,T_2)\times\mathbb{R}^k$ and some $\delta>0$. By the continuity of u and v, we may assume that $\hat{t}>T_1$. We fix a $\theta\in(0,1)$ such that

$$|e^{\kappa\hat{t}}u(\hat{t},\hat{x})|\vee|e^{\kappa\hat{t}}v(\hat{t},\hat{x})|\vee e^{\lambda(T_2-\hat{t})}(1+2|\hat{x}|^2)<\frac{\delta}{4(1-\theta)}, \tag{7.30}$$

and fix a $\varrho\in\big(0,\frac{\delta}{4}(\hat{t}-T_1)\big)$. For any $\varepsilon>0$, we define

$$\begin{aligned}&\Phi_\varepsilon(t,x,x')\triangleq\frac{\varrho}{t-T_1}+e^{\lambda(T_2-t)}\left(\frac{|x-x'|^2}{\varepsilon}+(1-\theta)(1+|x|^2+|x'|^2)\right)\\ &\quad\forall\,t\in(T_1,T_2],\ \forall\,x,x'\in\mathbb{R}^k,\\ &\text{and}\quad M_\varepsilon\triangleq\sup_{(t,x,x')\in(T_1,T_2]\times\mathbb{R}^k\times\mathbb{R}^k}\big\{\tilde{u}_\theta(t,x)-\tilde{v}_1(t,x')-\Phi_\varepsilon(t,x,x')\big\}.\end{aligned}$$

Since $r^2\ge\frac{4\widetilde{\kappa}e^{\kappa T}}{1-\theta}(1+r^{\varpi})$ holds for any $r\ge R_\theta\triangleq 1\vee\left(\frac{8\widetilde{\kappa}e^{\kappa T}}{1-\theta}\right)^{\frac{1}{2-\varpi}}$, (7.27) shows that for any $(t,x,x')\in[T_1,T_2]\times\mathbb{R}^k\times\mathbb{R}^k$ with $|x|\vee|x'|\ge R_\theta$

$$\begin{aligned}\tilde{u}_\theta(t,x)-\tilde{v}_1(t,x')&\le e^{\kappa T}\big(|u(t,x)|+|v(t,x')|\big)\le 2\widetilde{\kappa}e^{\kappa T}\big(1+(|x|\vee|x'|)^{\varpi}\big)\\ &\le\frac{1}{2}e^{\lambda(T_2-t)}(1-\theta)\big(1+|x|^2+|x'|^2\big),\end{aligned} \tag{7.31}$$

which implies that $\lim_{\frac{1}{t-T_1}\vee|x|\vee|x'|\to\infty}\big(\tilde{u}_\theta(t,x)-\tilde{v}_1(t,x')-\Phi_\varepsilon(t,x,x')\big)=-\infty$. Hence, M_ε is finite and attainable at some $(t_\varepsilon,x_\varepsilon,x'_\varepsilon)\in(T_1,T_2]\times\mathbb{R}^k\times\mathbb{R}^k$. Then it follows from (7.30) that

$$\begin{aligned}&\tilde{u}_\theta(t_\varepsilon,x_\varepsilon)-\tilde{v}_1(t_\varepsilon,x'_\varepsilon)-\Phi_\varepsilon(t_\varepsilon,x_\varepsilon,x'_\varepsilon)\\ &\quad=M_\varepsilon\ge\tilde{u}_\theta(\hat{t},\hat{x})-\tilde{v}_1(\hat{t},\hat{x})-\frac{\varrho}{\hat{t}-T_1}-e^{\lambda(T_2-\hat{t})}(1-\theta)(1+2|\hat{x}|^2)\\ &\quad\ge u(\hat{t},\hat{x})-v(\hat{t},\hat{x})+(\theta-1)e^{\kappa\hat{t}}u(\hat{t},\hat{x})-\frac{\varrho}{\hat{t}-T_1}-e^{\lambda(T_2-\hat{t})}(1-\theta)(1+2|\hat{x}|^2)\\ &\quad>\frac{\delta}{4},\end{aligned} \tag{7.32}$$

which implies that

$$\frac{\delta}{4}+e^{\lambda(T_2-t)}\left(\frac{|x_\varepsilon-x'_\varepsilon|^2}{\varepsilon}+(1-\theta)\big(1+|x_\varepsilon|^2+|x'_\varepsilon|^2\big)\right)<\tilde{u}_\theta(t_\varepsilon,x_\varepsilon)-\tilde{v}_1(t_\varepsilon,x'_\varepsilon). \tag{7.33}$$

Hence, we see from (7.31) that

$$|x_\varepsilon| \vee |x'_\varepsilon| < R_\theta. \tag{7.34}$$

As $\Big\{(t_\varepsilon, x_\varepsilon, x'_\varepsilon) \;:\; \varepsilon > 0\Big\} \subset (T_1, T_2] \times B_{R_\theta}(0) \times B_{R_\theta}(0)$, we can pick up a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ with $\lim_{n\to\infty} \downarrow \varepsilon_n = 0$ such that the sequence $\big\{(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n})\big\}_{n\in\mathbb{N}}$ converges to some $(t_*, x_*, x'_*) \in [T_1, T_2] \times \overline{B}_{R_\theta}(0) \times \overline{B}_{R_\theta}(0)$. Then (7.32) and the continuity of function u and v imply that

$$\lim_{n\to\infty} \frac{\varrho}{t_{\varepsilon_n} - T_1} \le \overline{\lim_{n\to\infty}}\, \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}) \le \tilde{u}_\theta(t_*, x_*) - \tilde{v}_1(t_*, x'_*) - \frac{\delta}{4} < \infty,$$

which implies that $t_* = \lim_{n\to\infty} t_{\varepsilon_n} > T_1$, i.e., $t_* \in (T_1, T_2]$. One can also deduce from (7.33) that $\overline{\lim}_{n\to\infty} \frac{|x_{\varepsilon_n} - x'_{\varepsilon_n}|^2}{\varepsilon_n} \le \tilde{u}_\theta(t_*, x_*) - \tilde{v}_1(t_*, x'_*) < \infty$, which leads to that $\lim_{n\to\infty} |x_{\varepsilon_n} - x'_{\varepsilon_n}| = 0$, namely, $x_* = x'_*$. For any $n \in \mathbb{N}$,

$$\begin{aligned}
&\tilde{u}_\theta(t_{\varepsilon_n}, x_{\varepsilon_n}) - \tilde{v}_1(t_{\varepsilon_n}, x'_{\varepsilon_n}) - \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}) \\
&\quad = M_{\varepsilon_n} \ge \tilde{u}_\theta(t_*, x_*) - \tilde{v}_1(t_*, x_*) - \frac{\varrho}{t_* - T_1} - e^{\lambda(T_2 - t_*)}(1-\theta)(1 + 2|x_*|^2).
\end{aligned}$$

As $n \to \infty$, the continuity of functions u and v implies that

$$\lim_{n\to\infty} \frac{|x_{\varepsilon_n} - x'_{\varepsilon_n}|^2}{\varepsilon_n} = 0. \tag{7.35}$$

Now we claim that

$$\begin{aligned}
&\{\varepsilon_n\}_{n\in\mathbb{N}} \text{ has a subsequence } \{\widetilde{\varepsilon}_n\}_{n\in\mathbb{N}} \quad \text{such that for any } n \in \mathbb{N}, \text{ either } t_{\widetilde{\varepsilon}_n} = T_2 \quad \text{or} \\
&\quad u(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) \le l(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}).
\end{aligned} \tag{7.36}$$

Assume not. Then there exists an $n^o \in \mathbb{N}$ such that for any $n \ge n^o$, $t_{\varepsilon_n} \in (T_1, T_2)$ and $u(t_{\varepsilon_n}, x_{\varepsilon_n}) > l(t_{\varepsilon_n}, x_{\varepsilon_n})$.

Fix $n \ge n^o$. The continuity of u and l shows that $(t_{\varepsilon_n}, x_{\varepsilon_n})$ has an open neighborhood $\mathcal{O}_n \triangleq (t_{\varepsilon_n} - r_n, t_{\varepsilon_n} + r_n) \times B_{r_n}(x_{\varepsilon_n}) \subset (T_1, T_2) \times \mathbb{R}^k$ for some $r_n > 0$ such that $u(t, x) > l(t, x)$ for any $(t, x) \in \mathcal{O}_n$. Then $\tilde{u}_\theta$ becomes a viscosity subsolution of (7.28) without obstacle and terminal condition over $\mathcal{O}_n$, i.e.,

$$\begin{aligned}
&-\partial_t \tilde{u}(t, x) - \mathcal{L}\tilde{u}(t, x) + \kappa \tilde{u}(t, x) \\
&\quad - \theta e^{\kappa t} f\left(t, x, \frac{1}{\theta} e^{-\kappa t} \tilde{u}(t, x), \frac{1}{\theta} e^{-\kappa t} (\sigma^T \cdot \nabla_x \tilde{u})(t, x)\right) = 0, \quad \forall\, (t, x) \in \mathcal{O}_n.
\end{aligned} \tag{7.37}$$

As $\tilde{v}_1$ is a viscosity supersolution of (7.28), it is clearly a viscosity supersolution of (7.28) without obstacle and terminal condition over $(0, T) \times \mathbb{R}^k$ (thus over $\mathcal{O}'_n \triangleq (t_{\varepsilon_n} - r_n, t_{\varepsilon_n} + r_n) \times B_{r_n}(x'_{\varepsilon_n})$), i.e.,

$$\begin{aligned}
&-\partial_t \tilde{v}(t, x) - \mathcal{L}\tilde{v}(t, x) + \kappa \tilde{v}(t, x) \\
&\quad - e^{\kappa t} f\Big(t, x, e^{-\kappa t} \tilde{v}(t, x), e^{-\kappa t} (\sigma^T \cdot \nabla_x \tilde{v})(t, x)\Big) = 0, \quad \forall\, (t, x) \in \mathcal{O}'_n.
\end{aligned} \tag{7.38}$$

Since the mapping $(t, x, x') \to \tilde{u}_\theta(t, x) - \tilde{v}_1(t, x') - \Phi_{\varepsilon_n}(t, x, x')$ is maximized at $(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n})$ over $(T_1, T_2] \times \mathbb{R}^k \times \mathbb{R}^k$ (thus over $(t_{\varepsilon_n} - r_n, t_{\varepsilon_n} + r_n) \times B_{r_n}(x_{\varepsilon_n}) \times B_{r_n}(x'_{\varepsilon_n})$), Theorem 8.3 of [7] shows that there exist $p_n, p'_n \in \mathbb{R}$ and $W_n, W'_n \in \mathbb{S}^k$ such that

$$\big(p_n, \nabla_x \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}), W_n\big) \in \overline{\mathcal{P}}^{2,+}\tilde{u}_\theta(t_{\varepsilon_n}, x_{\varepsilon_n}), \tag{7.39}$$

$$\big(p'_n, -\nabla_{x'} \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}), W'_n\big) \in \overline{\mathcal{P}}^{2,-}\tilde{v}_1(t_{\varepsilon_n}, x'_{\varepsilon_n}), \tag{7.40}$$

$$p_n - p'_n = \partial_t \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}) = -\frac{\varrho}{(t_{\varepsilon_n} - T_1)^2} - \lambda \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}), \tag{7.41}$$

$$\text{and} \quad \begin{pmatrix} W_n & 0 \\ 0 & -W'_n \end{pmatrix} \le D^2_{x,x'}\Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}) + \varepsilon_n^3\Big(D^2_{x,x'}\Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n})\Big)^2, \tag{7.42}$$

where $\overline{\mathcal{P}}^{2,+}\tilde{u}_\theta$ is the second-order superjets of $\tilde{u}_\theta$ and $\overline{\mathcal{P}}^{2,-}\tilde{v}_1$ is the second-order subjets of $\tilde{v}_1$ (see [7]).

As $\tilde{u}_\theta$ is a viscosity subsolution of (7.37), one can deduce from (7.39) that

$$\begin{aligned} &- p_n - \frac{1}{2}\mathrm{trace}\big(W_n \cdot (\sigma\sigma^T)(t_{\varepsilon_n}, x_{\varepsilon_n})\big) \\ &\quad - 2e^{\lambda(T_2 - t_{\varepsilon_n})}\left\langle b(t_{\varepsilon_n}, x_{\varepsilon_n}), \frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n} + (1-\theta)x_{\varepsilon_n}\right\rangle \\ &\quad + \theta\kappa e^{\kappa t_{\varepsilon_n}} u(t_{\varepsilon_n}, x_{\varepsilon_n}) - \theta e^{\kappa t_{\varepsilon_n}} f\left(t_{\varepsilon_n}, x_{\varepsilon_n}, u(t_{\varepsilon_n}, x_{\varepsilon_n}), \frac{2}{\theta}e^{-\kappa t_{\varepsilon_n} + \lambda(T_2 - t_{\varepsilon_n})}\sigma^T(t_{\varepsilon_n}, x_{\varepsilon_n})\right. \\ &\quad \left. \times \left(\frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n} + (1-\theta)x_{\varepsilon_n}\right)\right) \le 0. \end{aligned} \tag{7.43}$$

Since $\tilde{v}_1$ is a viscosity supersolution of (7.38), it follows from (7.40) that

$$\begin{aligned} &- p'_n - \frac{1}{2}\mathrm{trace}\big(W'_n \cdot (\sigma\sigma^T)(t_{\varepsilon_n}, x'_{\varepsilon_n})\big) \\ &\quad - 2e^{\lambda(T_2 - t_{\varepsilon_n})}\left\langle b(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n} - (1-\theta)x'_{\varepsilon_n}\right\rangle \\ &\quad + \kappa e^{\kappa t_{\varepsilon_n}} v(t_{\varepsilon_n}, x'_{\varepsilon_n}) - e^{\kappa t_{\varepsilon_n}} f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), 2e^{-\kappa t_{\varepsilon_n} + \lambda(T_2 - t_{\varepsilon_n})}\sigma^T(t_{\varepsilon_n}, x'_{\varepsilon_n})\right. \\ &\quad \left. \times \left(\frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n} - (1-\theta)x'_{\varepsilon_n}\right)\right) \ge 0. \end{aligned} \tag{7.44}$$

Subtracting (7.44) from (7.43), we see from (7.41) that

$$\frac{\varrho}{(t_{\varepsilon_n} - T_1)^2} + \lambda \Phi_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}) \le I_n^1 + 2e^{\lambda(T_2 - t_{\varepsilon_n})} I_n^2 + e^{\kappa t_{\varepsilon_n}} \sum_{j=3}^{6} I_n^j, \tag{7.45}$$

where

$$\begin{aligned} I_n^1 &\triangleq \frac{1}{2}\mathrm{trace}\big(W_n \cdot (\sigma\sigma^T)(t_{\varepsilon_n}, x_{\varepsilon_n})\big) - \frac{1}{2}\mathrm{trace}\big(W'_n \cdot (\sigma\sigma^T)(t_{\varepsilon_n}, x'_{\varepsilon_n})\big), \\ I_n^2 &\triangleq \left\langle b(t_{\varepsilon_n}, x_{\varepsilon_n}) - b(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n}\right\rangle \\ &\qquad + (1-\theta)\Big(\langle b(t_{\varepsilon_n}, x_{\varepsilon_n}), x_{\varepsilon_n}\rangle + \langle b(t_{\varepsilon_n}, x'_{\varepsilon_n}), x'_{\varepsilon_n}\rangle\Big), \\ I_n^3 &\triangleq -\theta\kappa u(t_{\varepsilon_n}, x_{\varepsilon_n}) + \kappa v(t_{\varepsilon_n}, x'_{\varepsilon_n}), \end{aligned}$$

$$I_n^4 \triangleq \left[\theta f\left(t_{\varepsilon_n}, x_{\varepsilon_n}, u(t_{\varepsilon_n}, x_{\varepsilon_n}), \frac{1}{\theta}J_n\right) - \theta f\left(t_{\varepsilon_n}, x_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{1}{\theta}J_n\right)\right], \quad \text{with}$$

$$J_n \triangleq 2e^{-\kappa t_{\varepsilon_n}+\lambda(T_2-t_{\varepsilon_n})}\sigma^T(t_{\varepsilon_n}, x_{\varepsilon_n}) \cdot \left(\frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n} + (1-\theta)x_{\varepsilon_n}\right),$$

$$I_n^5 \triangleq \left[\theta f\left(t_{\varepsilon_n}, x_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{1}{\theta}J_n\right) - \theta f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{1}{\theta}J_n\right)\right],$$

$$I_n^6 \triangleq \theta f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{1}{\theta}J_n\right) - f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), J'_n\right), \quad \text{with}$$

$$J'_n \triangleq 2e^{-\kappa t_{\varepsilon_n}+\lambda(T_2-t_{\varepsilon_n})}\sigma^T(t_{\varepsilon_n}, x'_{\varepsilon_n}) \cdot \left(\frac{x_{\varepsilon_n} - x'_{\varepsilon_n}}{\varepsilon_n} - (1-\theta)x'_{\varepsilon_n}\right).$$

• One can deduce from (7.42) and (7.1) that

$$\begin{aligned}
I_n^1 &= \frac{1}{2}\begin{pmatrix}\sigma(t_{\varepsilon_n}, x_{\varepsilon_n})\\ \sigma(t_{\varepsilon_n}, x'_{\varepsilon_n})\end{pmatrix}^T \begin{pmatrix}W_n & 0\\ 0 & -W'_n\end{pmatrix}\begin{pmatrix}\sigma(t_{\varepsilon_n}, x_{\varepsilon_n})\\ \sigma(t_{\varepsilon_n}, x'_{\varepsilon_n})\end{pmatrix}\\
&\le \left(\frac{1}{\varepsilon_n}e^{\lambda(T_2-t_{\varepsilon_n})} + 4\varepsilon_n e^{2\lambda(T_2-t_{\varepsilon_n})} + 4\varepsilon_n^2(1-\theta)e^{2\lambda(T_2-t_{\varepsilon_n})}\right)\\
&\quad \times |\sigma(t_{\varepsilon_n}, x_{\varepsilon_n}) - \sigma(t_{\varepsilon_n}, x'_{\varepsilon_n})|^2 + \left((1-\theta)e^{\lambda(T_2-t_{\varepsilon_n})} + 2\varepsilon_n^3(1-\theta)^2 e^{2\lambda(T_2-t_{\varepsilon_n})}\right)\\
&\quad \times \left(|\sigma(t_{\varepsilon_n}, x_{\varepsilon_n})|^2 + |\sigma(t_{\varepsilon_n}, x'_{\varepsilon_n})|^2\right)\\
&\le e\kappa^2\frac{|x_{\varepsilon_n} - x'_{\varepsilon_n}|^2}{\varepsilon_n} + 2(1-\theta)e^{\lambda(T_2-t_{\varepsilon_n})}\sigma_*^2 + c_{\sigma_*}\left(\varepsilon_n + \varepsilon_n^2 + \varepsilon_n^3\right). \qquad (7.46)
\end{aligned}$$

• By (7.1), $I_n^2 \le \kappa\frac{|x_{\varepsilon_n} - x'_{\varepsilon_n}|^2}{\varepsilon_n} + (1-\theta)\left(b_0|x_{\varepsilon_n}| + b_0|x'_{\varepsilon_n}| + \kappa|x_{\varepsilon_n}|^2 + \kappa|x'_{\varepsilon_n}|^2\right)$. (7.47)

• We see from (7.33) that $\theta u(t_{\varepsilon_n}, x_{\varepsilon_n}) - v(t_{\varepsilon_n}, x'_{\varepsilon_n}) > 0$. Then (7.3) shows that $I_n^4 \le \kappa|\theta u(t_{\varepsilon_n}, x_{\varepsilon_n}) - \theta v(t_{\varepsilon_n}, x'_{\varepsilon_n})| \le \kappa\left(\theta u(t_{\varepsilon_n}, x_{\varepsilon_n}) - v(t_{\varepsilon_n}, x'_{\varepsilon_n})\right) + \kappa(1-\theta)|v(t_{\varepsilon_n}, x'_{\varepsilon_n})|$. Thus,

$$I_n^3 + I_n^4 \le \kappa(1-\theta)|v(t_{\varepsilon_n}, x'_{\varepsilon_n})|. \qquad (7.48)$$

• (7.34) and (7.27) imply that $\sup_{i\in\mathbb{N}}\left\{|x_{\varepsilon_i}| \vee |x'_{\varepsilon_i}| \vee |v(t_{\varepsilon_i}, x'_{\varepsilon_i})|\right\} \le \widetilde{R}_\theta \triangleq \left(1 \vee \widetilde{\kappa}\right)\left(1 + R_\theta^{\varpi}\right)$. Then (7.26) shows that

$$\begin{aligned}
I_n^5 &\le \mathfrak{M}(\widetilde{R}_\theta)\left(1 + \frac{1}{\theta}|J_n|\right)|x_{\varepsilon_n} - x'_{\varepsilon_n}|\\
&\le \mathfrak{M}(\widetilde{R}_\theta)\left(1 + \frac{2e\sigma_*}{\theta}\left(\frac{|x_{\varepsilon_n} - x'_{\varepsilon_n}|}{\varepsilon_n} + R_\theta\right)\right)|x_{\varepsilon_n} - x'_{\varepsilon_n}|. \qquad (7.49)
\end{aligned}$$

• The concavity of the mapping $z \to f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), z\right)$, (7.3) implies that

$$\begin{aligned}
I_n^6 &\le -(1-\theta)f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, v(t_{\varepsilon_n}, x'_{\varepsilon_n}), \frac{J'_n - J_n}{1-\theta}\right)\\
&\le -(1-\theta)f\left(t_{\varepsilon_n}, x'_{\varepsilon_n}, 0, \frac{J'_n - J_n}{1-\theta}\right) + \kappa(1-\theta)|v(t_{\varepsilon_n}, x'_{\varepsilon_n})|,
\end{aligned}$$

where $\frac{J_n'-J_n}{1-\theta} = \frac{2e^{-\kappa t_{\varepsilon_n}+\lambda(T_2-t_{\varepsilon_n})}}{1-\theta}\Big(\Big\langle \sigma(t_{\varepsilon_n}, x'_{\varepsilon_n}) - \sigma(t_{\varepsilon_n}, x_{\varepsilon_n}), \frac{x_{\varepsilon_n}-x'_{\varepsilon_n}}{\varepsilon_n}\Big\rangle + (1-\theta)\Big(\sigma^T(t_{\varepsilon_n}, x_{\varepsilon_n})\cdot x_{\varepsilon_n} + \sigma^T(t_{\varepsilon_n}, x'_{\varepsilon_n})\cdot x'_{\varepsilon_n}\Big)\Big)$. Since $\Big|\Big\langle \sigma(t_{\varepsilon_n}, x'_{\varepsilon_n}) - \sigma(t_{\varepsilon_n}, x_{\varepsilon_n}), \frac{x_{\varepsilon_n}-x'_{\varepsilon_n}}{\varepsilon_n}\Big\rangle\Big| \le \kappa\frac{|x'_{\varepsilon_n}-x_{\varepsilon_n}|^2}{\varepsilon_n}$, (7.35) and the continuity of function σ that

$$\lim_{n\to\infty}\frac{J_n'-J_n}{1-\theta} = 4e^{-\kappa t_*+\lambda(T_2-t_*)}\sigma^T(t_*, x_*)\cdot x_*. \tag{7.50}$$

Letting $n\to\infty$ in (7.45) and using the continuity of all functions involved, we can deduce from (7.35), (7.46) through (7.50) that

$$\begin{aligned}\lambda(1-\theta)e^{\lambda(T_2-t_*)}(1+2|x_*|^2) \le\ & 2(1-\theta)e^{\lambda(T_2-t_*)}\sigma_*^2 + 4(1-\theta)(b_0+\kappa)e^{\lambda(T_2-t_*)}\\ &\times\left(1+|x_*|^2\right) + 2\kappa e^{\kappa t_*}(1-\theta)|v(t_*, x_*)| - e^{\kappa t_*}(1-\theta)\\ &\times f\Big(t_*, x_*, 0, 4e^{-\kappa t_*+\lambda(T_2-t_*)}\sigma^T(t_*, x_*)\cdot x_*\Big).\end{aligned} \tag{7.51}$$

Conditions (7.27) and (7.3) imply that

$$\begin{aligned}&2\kappa e^{\kappa t_*}|v(t_*, x_*)| - e^{\kappa t_*} f\Big(t_*, x_*, 0, 4e^{-\kappa t_*+\lambda(T_2-t_*)}\sigma^T(t_*, x_*)\cdot x_*\Big)\\ &\quad\le 2\kappa\widetilde{\kappa}e^{\kappa t_*}(1+|x_*|^{\varpi}) + e^{\kappa t_*}\left(\alpha + 8\gamma e^{-2\kappa t_*+2\lambda(T_2-t_*)}\sigma_*^2|x_*|^2\right).\end{aligned}$$

Plugging it back into (7.51) yields that

$$\begin{aligned}&\lambda(1-\theta)e^{\lambda(T_2-t_*)}(1+2|x_*|^2)\\ &\quad\le (1-\theta)e^{\lambda(T_2-t_*)}(1+|x_*|^2)\Big(4(b_0+\kappa) + 2(1+4\gamma e)\sigma_*^2 + (\alpha+4\kappa\widetilde{\kappa})e^{\kappa T}\Big)\\ &\quad= \frac{1}{2}\lambda(1-\theta)e^{\lambda(T_2-t_*)}(1+2|x_*|^2),\end{aligned}$$

which results in a contradiction. Thus we proved claim (7.36). Let $\{\widetilde{\varepsilon}_n\}_{n\in\mathbb{N}}$ be the subsequence of $\{\varepsilon_n\}_{n\in\mathbb{N}}$ as described in (7.36). For any $n\in\mathbb{N}$, since the maximum is attained at $(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}, x'_{\widetilde{\varepsilon}_n})$,

$$\begin{aligned}&\theta\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}) - \frac{\varrho}{\hat{t}-T_1} - e^{\lambda(T_2-\hat{t})}(1-\theta)(1+2|\hat{x}|^2)\\ &\quad\le M_{\widetilde{\varepsilon}_n} \le \theta\tilde{u}(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) - \tilde{v}(t_{\widetilde{\varepsilon}_n}, x'_{\widetilde{\varepsilon}_n}).\end{aligned} \tag{7.52}$$

If $t_{\widetilde{\varepsilon}_n} = T_2$, $u(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) = u(T_2, x_{\widetilde{\varepsilon}_n}) \le v(T_2, x_{\widetilde{\varepsilon}_n}) = v(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n})$ by our condition. Otherwise, $t_{\widetilde{\varepsilon}_n}\in(T_1, T_2)$ and $u(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) \le l(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n})$. As v is a viscosity supersolution of (7.6), we always have $v(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) - l(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) \ge 0$. Thus we still have $u(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) \le v(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n})$. Then (7.52) reduces to

$$\theta\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}) - \frac{\varrho}{\hat{t}-T_1} - e^{\lambda(T_2-\hat{t})}(1-\theta)(1+2|\hat{x}|^2) \le \theta\tilde{v}(t_{\widetilde{\varepsilon}_n}, x_{\widetilde{\varepsilon}_n}) - \tilde{v}(t_{\widetilde{\varepsilon}_n}, x'_{\widetilde{\varepsilon}_n}).$$

As $n\to\infty$, we obtain $\theta\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}) - \frac{\varrho}{\hat{t}-T_1} - e^{\lambda(T_2-\hat{t})}(1-\theta)(1+2|\hat{x}|^2) \le (\theta-1)\tilde{v}(t_*, x_*)$. Letting $\varrho\to 0$ and letting $\theta\to 1$ yield that $\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}) \le 0$. Thus $u(\hat{t},\hat{x}) \le v(\hat{t},\hat{x})$, which contradicts with our initial assumption. Therefore, (7.29) holds, proving the theorem. □

Thanks to Theorems 7.1 and 7.2, (7.6) has a unique viscosity solution.

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Appendix

A.1. Proof of (3.12)

Fix $n \in \mathbb{N}$. We have seen from (3.9) that $\left\{\sqrt{|\phi'(Y^{m,n})|}Z^{m,n}\right\}_{m\ge n} \subset \mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Similar to (3.5),

$$E\int_0^T |\phi'(Y_s - Y_s^n)|\,|Z_s - Z_s^n|^2 ds \le c_{\lambda,\lambda'}\,\Xi + c_{\lambda,\lambda'}\,E\left[\left(\int_0^T |Z_s|^2 ds\right)^{p_o}\right] < \infty.$$

Thus $\sqrt{|\phi'(Y-Y^n)|}\big(Z - Z^n\big) \in \mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. (Note that since $Y^n, n \in \mathbb{N}$ are $\mathbf{F}$-adapted continuous processes, $Y = \lim_{n\to\infty} Y^n$ is at least an $\mathbf{F}$-predictable process.)

For any $X \in \mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$, since $\frac{1}{p_1}+\frac{1}{p_2}+1 = 2-\frac{1}{p_o}$ by (3.4), applying Young's inequality with $q_1 = p_1\big(2-\frac{1}{p_o}\big)$, $q_2 = p_2\big(2-\frac{1}{p_o}\big)$ and $q_3 = 2-\frac{1}{p_o}$, we can deduce from (3.1) and (3.2) that

$$\begin{aligned}
&E\left[\left(\int_0^T |\phi'(Y_s - Y_s^n)||X_s|^2 ds\right)^{\frac{p_o}{2p_o-1}}\right]\\
&\quad\le c_{\lambda,\lambda'} E\left[e^{\frac{\lambda_o p_o q_1\gamma}{2p_o-1}\mathscr{L}_*} + e^{\frac{\lambda_o p_o q_2\gamma}{2p_o-1}\mathscr{Y}_*} + \int_0^T |X_s|^2 ds\right]\\
&\quad\le c_{\lambda,\lambda'}\left(\Xi + E\int_0^T |X_s|^2 ds\right) < \infty.
\end{aligned}$$

Thus $X\sqrt{|\phi'(Y-Y^n)|} \in \mathbb{H}_{\mathbf{F}}^{2,\frac{2p_o}{2p_o-1}}([0,T];\mathbb{R}^d)$. As $\{Z^m\}_{m\ge n}$ weakly converges to Z in $\mathbb{H}^{2,2p_o}_{\mathbf{F}}([0,T];\mathbb{R}^d)$,

$$\lim_{m\to\infty} E\int_0^T X_s\sqrt{|\phi'(Y_s - Y_s^n)|}\left(Z_s - Z_s^m\right) ds = 0. \tag{A.1}$$

On the other hand, for any $m \ge n$ Hölder's inequality and (3.2) imply that

$$\begin{aligned}
&\left|E\int_0^T X_s\Big(\sqrt{|\phi'(Y_s - Y_s^n)|} - \sqrt{|\phi'(Y_s^{m,n})|}\Big)Z_s^{m,n} ds\right|\\
&\quad\le c_{\lambda,\lambda'}\,\Xi^{\frac{1}{2p_o}}\left\|\,|X_s|\Big(\sqrt{|\phi'(Y_s - Y_s^n)|} - \sqrt{|\phi'(Y_s^{m,n})|}\Big)\right\|_{\mathbb{H}_{\mathbf{F}}^{2,\frac{2p_o}{2p_o-1}}([0,T];\mathbb{R})}.
\end{aligned}\tag{A.2}$$

It follows from (3.7) that P-a.s.

$$\begin{aligned}
&0 \le |X_t|\Big(\sqrt{|\phi'(Y_t - Y_t^n)|} - \sqrt{\phi'|(Y_t^{m,n})|}\Big) \le |X_t|\sqrt{|\phi'(Y_t - Y_t^n)|},\\
&\quad\forall\, t \in [0,T],\ \forall\, m \ge n.
\end{aligned}$$

Since $|X|\sqrt{|\phi'(Y-Y^n)|} \in \mathbb{H}_{\mathbf{F}}^{2,\frac{2p_o}{2p_o-1}}([0,T];\mathbb{R})$, the continuity of ϕ' and Dominated Convergence Theorem imply $\lim_{m\to\infty} \| \, |X_s|\Big(\sqrt{|\phi'(Y_s-Y_s^n)|}-\sqrt{|\phi'(Y_s^{m,n})|}\Big)\|_{\mathbb{H}_{\mathbf{F}}^{2,\frac{2p_o}{2p_o-1}}([0,T];\mathbb{R})} = 0$, which together with (A.2) and (A.1) gives

$$\lim_{m\to\infty} E\int_0^T X_s\Big(\sqrt{|\phi'(Y_s-Y_s^n)|}\big(Z_s-Z_s^n\big)-\sqrt{|\phi'(Y_s^{m,n})|}\big(Z_s^{m,n}\big)\Big)ds=0,$$

proving (3.12).

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