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STATISTICAL IMAGE RESTORATION

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Abstract. Restoration of very noisy images typical of electron-beam diagnostics of microelectronics is considered. The conditions, under which the regularization method yields strongly consistent (with probability 1) estimates of image functions are investigated. The problems of the choice of stabilizing functional and numerical computer realization are discussed.

1. INTRODUCTION

Image restoration in the diagnostic problems of microelectronics has a number of distinctine features, namely, insufficient information on the statistical characteristics of the noise values in different points of the image; simultaneity of the additive and multiplicative components of the noise function; a low signal-to-noise ratio and a specific form of the functions describing the image (a discontinuous function taking a finite number of values) [2]. Moreover, the statement of the problem is specific to itself, since it amounts to localization of the domain boundaries of its constant values rather than to the best approximation of the image function. The domain of definition of image functions in microelectronics is generally divided into a number of sets, so that the function variation is small in any set, but it is large in the neighbourhood of the boundaries between the sets. In principle, such images can be defined by discontinuous functions, but in the present work we shall consider only continuous functions, because they essentially simplify proofs of the results and allow avoiding unwieldy but insignificant detailes about sets of discontinuities and the behaviour of functions at those points.

The paper suggests a method of noisy images restoration based on regularization which, under the conditions mentioned, is more efficient than the conventional methods of filtration. The statements revealing a strong consistency of the method are proved.

2. DEFINITIONS AND AUXILIARY RESULTS

Let E be a compact metric space without isolated points, $f(x)$, $x \in E$ be a real or complex function defined on E; we consider for simplicity that $f(x)$ is real. Assume that a random function $n(x)$ is realised on some subset E' of space E and appears as

$$n(x) = (1 + e(x))f(x) + k(x), \qquad x \in E' \subseteq E,$$
$$P(n(x) > 0) = 1,$$

where $f(x)$ is the function to be restored (source image), $e(x)$ and $k(x)$ are the additive and multiplicative noise components, respectively. $f(x)$, $x \in E$, needs to be estimated by observed values of $n(x)$, $x \in E'$. Denote, as usual, the metric in the space E and the set of continuous functions on E by $r_E(\cdot,\cdot)$ and $C(E)$, respectively. Choose the uniform metric on the space of functions. With respect to random error $(e(x), k(x))$ assume that the following conditions are satisfied:

(a) random functions $e(x)$ and $k(x)$ are stochastically independent;
(b) there exist $b_1, b_2 > 0$ and $0 < a_1, a_2 < 1$ such that

$$P(-b_2 \le k(x) \le b_1) = 1, P(-a_2 \le e(x) \le a_1) = 1$$

for any $x \in E'$;

(c) for any $e_1, e_2 > 0$ there exist $d_1, d_2 > 0$ such that for any $x \in E'$ and any finite subset $E'' \subset E'$, $x \notin E''$ the inequalities

$$P(k(x) \le -b_2 + e_1 \mid k(y),\, y \in E'') \ge d_1,$$
$$P(k(x) \ge b_1 - e_1 \mid k(y),\, y \in E'') \ge d_1,$$
$$P(e(x) \le -a_2 + e_2 \mid k(y),\, y \in E'') \ge d_2,$$
$$P(e(x) \ge a_1 - e_2 \mid k(y),\, y \in E'') \ge d_2,$$

are satisfied (it is assumed that the probability space on which the random functions $e(x)$ and $k(x)$ are defined is complete, that is, every set of elementary occurrences which is contained in an event of zero probability is also an event).

Lemma 1. Let the random functions $e(x)$ and $k(x)$ satisfy conditions (a)-(c), and let the set E' be infinite. Then for any sequence $x_1, x_2, \ldots \in E'$ ($x_i \neq x_j$ for $i \neq j$) the equalities

$$P(\sup_{n\ge1} k(x_n) = b_1\ ,\ \sup_{n\ge1} e(x_n) = a_1) = 1, \tag{1}$$
$$P(\inf_{n\ge1} k(x_n) = -b_2\ ,\ \inf_{n\ge1} e(x_n) = -a_2) = 1$$

are satisfied.

PROOF. Let us prove the first equality. The second one can be proved by similar arguments. Show that

$$P(\sup_{n\ge1} k(x_n) = b_1) = 1 \tag{2}$$

and

$$P(\sup_{n\ge1} e(x_n) = a_1) = 1\ , \tag{3}$$

whence, equality (1) follows from independence of the random functions $k(x)$ and $e(x)$. To prove (2) it is sufficient to show that for any $e_1 > 0$

$$P(\sup_{n\ge1} k(x_n) < b_1 - e_1) = 0\ . \tag{4}$$

It is clear that

$$\sup_{1\le n\le N} k(x_n) < \sup_{n\ge1} k(x_n)$$

for any $N = 1, 2, \cdots$; therefore, taking into consideration condition (c), we have

$$P(\sup_{n\ge1} k(x_n) \le b_1 - e_1) \le P(\sup_{1\le n\le N} k(x_n) \le b_1 - e_1) =$$
$$P(k(x_1) \le b_1 - e_1, k(x_2) \le b_1 - e_1, \cdots, k(x_N) \le b_1 - e_1) =$$
$$P(k(x_1) \le b_1 - e_1 \mid k(x_2) \le b_1 - e_1, \cdots, k(x_N) \le b_1 - e_1)\times$$
$$\times\ P(k(x_2) \le b_1 - e_1 \mid k(x_3) \le b_1 - e_1, \cdots, k(x_N) \le b_1 - e_1)\times$$
$$\times \cdots \times P(k(x_{N-1}) \le b_1 - e_1 \mid k(x_N) \le b_1 - e_1)\times$$
$$\times\ P(k(x_N) \le b_1 - e_1) < (1 - d_1)^{N-1}\ ,$$

where d_1 depends on e_1, being independent of N. Hence, (4) follows from arbitrariness of N. Thus (2) is proved. Equality (3) is proved in a similar way. The Lemma is proved.

Now we turn to the construction of a nonparametric estimate of the function $f(x)$ based on the regularization method [1]. Let A be a set of functions, which are continuous on E and uniformly bounded at some point $x_0 \in E$, comprising function f; $T[f]$ a stabilizing functional defined on A, i. e. a nonnegative functional such that the set

$$A_x = \{f \in A : T[f] \le x\}$$

is compact at any $x > 0$. Let

$$q_1(x) = (n(x) - b_1)/(1 + a_1) \quad , \ q_2(x) = (n(x) + b_2)/(1 - a_2).$$

Suppose that the function $f(x)$ is estimated by $F(x; E) \in A$ for which

1. $$q_1(x) \leq F(x; E') \leq q_2(x),$$

2. $$T[F] = \min T[g] \ ,$$

where min is taken by all $g \in A$, confined between $q_1(x)$ and $q_2(x)$ at $x \in E'$ (note, that the set of such functions is not empty). If there are several functions of A satisfying conditions 1 and 2, then any of them are taken as an estimate of F.

3. STRONG CONSISTENCY OF F ESTIMATE

THEOREM 1. *If E' is dense in E, then*

$$P(F(x; E') = f(x)) = 1. \tag{5}$$

PROOF. By the assumption of completeness of the probability space, it is sufficient to prove the Theorem for the case when E' is countable. Consider the set of functions

$$H = H(E') = \{g \in A : \sup_{x \in E'} \mid g(x) - f(x) \mid > 0\}.$$

Denote the event A_g to be

$$\bigcap_{x \in E'} \{q_1(x) \leq g(x) \leq q_2(x)\}$$

and assume, that

$$G = \bigcup_{g \in H} A_g. \tag{6}$$

It is clear that (5) follows from the equality

$$P(G) = 0. \tag{7}$$

Therefore, to prove the Theorem it is sufficient to ascertain correctness of (7). First suppose that $n_1(x)$ and $n_2(x)$, $x \in E'$, are two random functions, B is some set of functions from $C(E'')$; then the set of elementary occurrences appears as

$$\bigcup_{g \in B} \bigcap_{x \in E'} \{n_1(x) \leq g(x) \leq n_2(x)\}$$

which, generally speaking, may not be an event, if B is not countable. Therefore, it is not a priori clear whether expression (7) makes any sense. However, as will be shown below, all the sets of elementary occurrences in question are measurable relative to the σ-algebra of events (under the assumption of completeness of probability space), and the term 'event' is quite correct here.

Now let us prove (7). Since E is compact, every function of $C(E)$, and, hence $C(E')$, is uniformly continuous. Therefore, for every function $g \in C(E')$, its modulus of continuity is defined and is finite, that is,

$$w_g(\delta) = \max_{\substack{x,y \in E' \\ r_E(x,y) \leq \delta}} \mid g(x) - g(y) \mid \quad .$$

Consider the sets of functions

$$H_n = H_n(E') = \{g \in A; \sup_{x\in E'} \mid g(x) - f(x) \mid > (2/n)\} \ ,$$
$$D_{n,m} = D_{n,m}(E') = \{g \in A; w_g(1/m) < 1/(4n)\} \ , \quad n,m = 1,2,\cdots .$$

Evidently $\bigcup_{m=1}^{\infty} D_{n,m} = A$ for any n , and $\bigcup_{n=1}^{\infty} H_n = H$. Consequently

$$\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}(D_{n,m} \cap H_n) = H$$

and, therefore

$$G = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{g\in A} A_g \ , \tag{8}$$

where $A_{n,m} = D_{n,m} \cap H_n$. Choose $\sigma > 0$ so small that for $r_E(x,y) < \sigma$ inequality

$$\mid f(x) - f(y) \mid < 1/(4n) \tag{9}$$

is satisfied.

Suppose n and m are fixed. Let $\delta_0 = \min\{\delta/2, 1/m\}$. Since the closure of E' is compact, for any $e > 0$ in E' a finite e-net exists. Let $x_1, x_2, \cdots, x_N$ be elements of E' making an e-net for $e = e_0$ (it is clear that $N = N(e) = N(m,\delta)$). Consider a set of solid spheres (in E') $O_1, O_2, \cdots, O_N$ with the same radius δ_0 and with centres in points $x_1, x_2, \cdots, x_N$ accordingly (since E has no isolated points, every solid sphere contains an infinite number of points). We have

$$\bigcup_{j=1}^{N} O_j = E'.$$

Let $L = \sup_{x\in E} f(x)$ and consider the events

$$C'_j = \{\sup_{x\in O_j} k(x) < b_1 - (1+a_1)/(4n)\} \cup \{\sup_{x\in O_j} e(x) < a_1 - (1+a_1)/(4nL)\},$$
$$C''_j = \{\inf_{x\in O_j} k(x) > -b_2 + (1+a_2)/(4n)\} \cup \{\inf_{x\in O_j} e(x) > -a_2 + (1+a_2)/(4nL)\},$$
$$C' = \bigcup_{j=1}^{N} C'_j,$$
$$C'' = \bigcup_{j=1}^{N} C''_j.$$

By virtue of Lemma 1

$$P(C' \cup C'') = 0 \ . \tag{10}$$

Let us show that for any $g \in A_{n,m}$ the inclusion

$$A_g \subseteq C' \cup C'' \tag{11}$$

is satisfied.

Actually, at $g \in A_{n,m}$ the event A_g means that

$$q_1(x) \le g(x) \le q_2(x), \tag{12}$$
$$\sup_{x\in E'} \mid f(x) - g(x) \mid > 2/n, \tag{13}$$
$$w_g(1/m) < 1/(4n). \tag{14}$$

From (13) $x_0 \in E'$ such that

$$\mid f(x_0) - g(x_0) \mid > 1/n. \tag{15}$$

From (9), (14) and (15) we obtain

$$| g(x) - f(x) |=| g(x) - g(x_0) + g(x_0) - f(x_0) + f(x_0) - f(x) |>$$
$$>| g(x_0) - f(x_0) | - | g(x) - g(x_0) | - | f(x_0) - f(x) |> 1/(2n)$$

for all $x \in E'$ such that $r_E(x, x_0) < 2\delta_0$, whence it follows that at these x one of the inequalities

$$g(x) > f(x) + 1/(2n), \tag{16}$$

$$f(x) - 1/(2n) > g(x) \tag{17}$$

is satisfied. Note that the set $\{x \in E' : r_E(x, x_0) < 2\delta_0\}$ contains completely at least one of the solid spheres $O_1, O_2, \cdots, O_N$. Thus, for some j_0, $1 \le j_0 \le N$, for all $x \in O_{j_0}$ either (16) or (17) is satisfied. Then taking into consideration equation (12) we obtain

$$(a_1 - e(x))f(x) + (b_1 - k(x)) > (1 + a_1)/(4n), \quad x \in O_j,$$

which is possible either at

$$e(x) < a_1 - (1 + a_1)/(4nL)$$

or at

$$k(x) < b_1 - (1 - a_1)/(4n) \ .$$

By analogy, (16) implies validity of one of the inequalities:

$$e(x) > -a_2 + (1 + a_2)/(4nL),$$
$$k(x) > -b_2 + (1 + a_2)/(4n).$$

Thus (11) is proved. Since the function g from class $A_{n,m}$ is arbitrary, we get $\bigcup_{g \in A_{n,m}} A_g \subseteq C' \cup C''$, whence, taking into account (10) and completeness of probability space we conclude that $\bigcup_{g \in A_{n,m}} A_g$ is an event, and

$$P(\bigcup_{g \in A_{n,m}} A_g) = 0. \tag{18}$$

Finally, from (8) and (18) we obtain (7).
The Theorem is proved.

THEOREM 2. *Let $E_1, E_2, \cdots$ be an increasing sequence of finite subsets of the space E, such that their union $E' = \bigcup_{n=1}^{\infty} E_n$ is dense in E. Then*

$$\sup_{x \in E} | F(x; E_n) - f(x) | \to 0$$

with probability 1 as $n \to \infty$.

PROOF. Let

$$Q_n = \{g \in A; \quad q_1(x) \le g(x) \le q_2(x), x \in E_n\},$$
$$W = \{g \in A; \quad T[g] \le T[f]\}.$$

It is clear that $Q_{n+1} \subseteq Q_n$; $F(x; E_n) \in Q_n \cap W$, $n = 1, 2, \cdots$. Then, because of Theorem 1, the set $\bigcap_{n=1}^{\infty} Q_n$ consists of one element $f(x)$, with probability 1, whence the set $W \cap (\bigcap_{n=1}^{\infty} Q_n)$ also consists of the element $f(x)$ with probability 1 (clearly, $f(x) \in W$). Since every set Q_n is closed, and the set W, by definition of the stabilizing functional, is compact, the sequence of compact sets $Q_n \cap W$, $n = 1, 2, \cdots$ has an intersection set consisting of one element $f(x)$. Whence, taking into consideration that for every n $F(x; E_n)$ belongs to the set $Q_n \cap W$, we obtain that the sequence $F(x; E_n), n = 1, 2, \cdots$, converges to $f(x)$ with probability 1. The Theorem is proved.

4. NUMERICAL REALIZATION

In this paragraph we assume that E is a compact subset of the m-dimensional Euclidean space R^m. For example, let E be a rectangle on plane:

$$E = \{x = (x^{(1)}, x^{(2)}) \in R^2 : a_i \le x^{(i)} \le b_i \quad , \ i = 1, 2\}.$$

Take as the set A the set of all the functions $C(E)$, continuous on E.

Consider the sequence of rectangular lattices $E_1, E_2, \cdots$ getting denser (without lack of generality, assume that in the lattice E_n the number of rows and the number of columns are equal to n) and introduce on A some stabilizing functional T. Let $f(x) = f(x^{(1)}, x^{(2)})$ be an image function. Choose, as a restored image on every lattice E_n, a matrix of values $F(x_{ij}; E_n)$, where $x_{ij}, \quad i, j = 1, 2, \cdots, n$ are elements of the lattice E_n. Therefore, the problem of image restoration reduces to minimization of some function in a n^2-dimensional cube. At moderate values of n ($n \sim 10^1 - 10^2$) the solution of the problem does not present any difficultes, but computation time tends to increase with increasing n. Therefore, when images are described by complicated functions that can be satisfactorily defined by grids with a large number of nodes, it is reasonable to divide the set E into some smaller (intersecting or not) sets to restore images by parts.

The choice of the stabilizing functional is important here for the effectiveness of numerical algorithm for image restoration. For example, numerical realization of the methods shows that application of stabilizing functional such as

$$T_p[f] = \iint_E (\mid \partial f / \partial x^{(1)} \mid^p + \mid \partial f / \partial x^{(2)} \mid^p) dx^{(1)} dx^{(2)}, \quad p \ge 1$$

for different values of p leads to considerably different results; for instance, from the point of view of the boundary localization problem mentioned above. Fig. 1 shows the results of restoration the boundary between two areas with function $f(x)$ equal to 1 in one area and 0 in the other. Original and noisy images are shown in Figs. 1a,b, and those restored with the use of minimization of T_2 and T_1 are presented in Figs. 1c,d, respectively.

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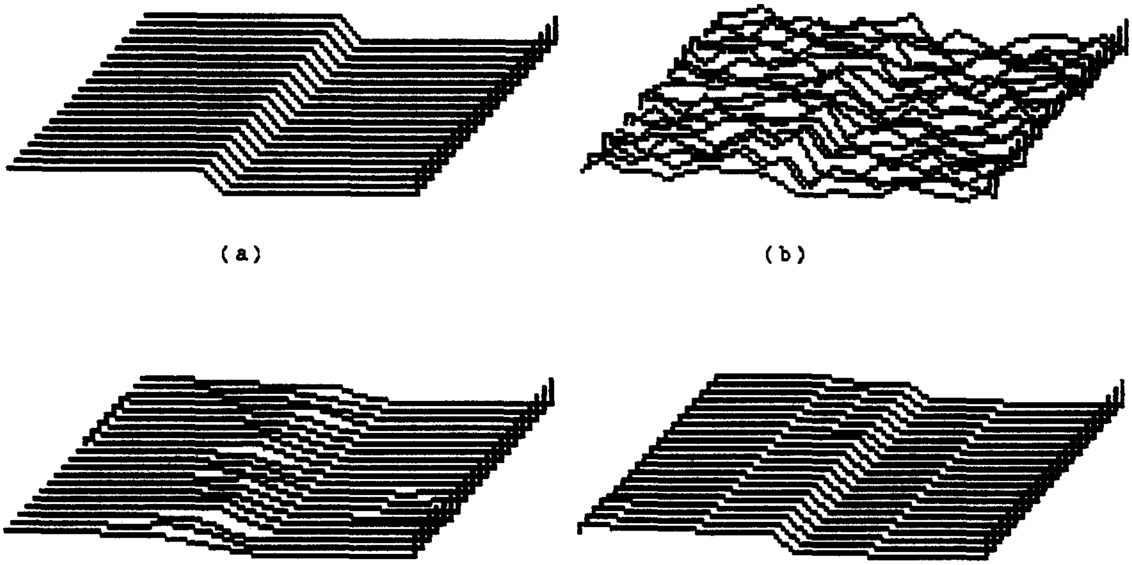
(See overleaf for Fig. 1.)



(a) (b) (c) (d)

Fig. 1. Restoration of noisy image. (a) Original image. (b) Noisy image. (c,d) Restored image.