PREFERENCE RELATIONS, TRANSITIVITY AND THE RECIPROCAL PROPERTY

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Abstract. Theories that attempt to represent decision makers' preferences are usually based on the axiom of transitivity. On the other hand, Saaty's Eigenvalue Method of deriving ratio scales from pairwise comparisons is based on the assumption that the numerical preferences satisfy the reciprocal property. We show that this property cannot be derived from the usual set of axioms used to obtain order preserving value functions, but that they in turn are consequences of the reciprocal property.

Keywords. Preferences; reciprocal property; cardinal consistency; transitivity.

INTRODUCTION

Until very recently, it was believed that transitivity of preferences was essential for constructing value functions that truly represented the preferences of decision makers. However, this does not appear to be so. The literature is full of examples that illustrate the intransitivity of decision makers. Why should then any theory assume transitivity of preferences when it does not hold in real life?

A cornerstone in the concept of understanding is coherence or consistency. What is meant by consistency varies from one subject to another. For example, a set of axioms is said to be consistent if "something satisfies the axioms" (Krantz et al., 1971). In the area of preference and choice consistency in the ordinal sense means that when, given three alternatives A, B, and C, if A is preferred to B and B is preferred to C, then to be consistent A must be preferred to C. This is the well-known transitivity condition. We are particularly interested in individual behavior that seems to violate this transitivity assumption. In other words, people in fact express contradictory preferences without being bothered by such inconsistency.

We observe that the process of understanding allows for new knowledge which may be inconsistent with old knowledge and may require adjustment in previous understanding in order for the old and the new to relate together consistently. Thus, inconsistency is allowed at first so that understanding may grow. The question is how should one define consistency in a way that tolerates a degree of inconsistency, thereby accommodating and measuring violations of consistency as a part of a theory which takes into consideration actual human behavior.

What is needed is a method which reflects the intensity of preference so that one can tradeoff the alternatives with respect to the criteria. This means that if A is preferred to B x times, and B is preferred to C y times, then A is preferred to C z times and z = xy. Heretofore we shall call this cardinal consistency, or simply, consistency. It implies ordinal consistency but not the converse (see Saaty, 1980, p. 191). Thus it is possible to be cardinally inconsistent but still transitive. Cardinal consistency implies that if A is preferred to B x times, then B is preferred to A 1/x times so that A would be equally preferred to itself. This is known as the reciprocal property of pairwise comparisons. Thus it is possible to assume the reciprocal property for pairwise comparisons but violate cardinal consistency. The reciprocal property, which is weaker than cardinal consistency and holds between pairs of elements, enables us to study cardinal consistency and is a more general notion than cardinal consistency (Saaty, 1980). It has been possible to derive conditions under which reciprocal paired comparisons give rise to consistency and to indicate when though as a set they are inconsistent they can still be thought of as permissible, thus accounting for the kind of inconsistency and intransitivity we spoke of earlier. That is, a degree of inconsistency is allowed to make it possible to integrate new knowledge at "odds" with old knowledge.

The question is: given a set of behavioral preference axioms between pairs of alternatives needed in the logic of traditional decision theory: reflexivity, and strong completeness, it is always possible to assign numerical intensities to the comparisons; do these intensities satisfy the reciprocal property? It will be shown that the two axioms are necessary and sufficient for the reciprocal property to hold so long as the set of positive intensities of binary comparisons X both contains unity, is closed under the inverse of multiplication in which a value $x \in X$ $(x > 0)$ is transformed to $1/x$, and the isomorphism defined on the set X of intensities satisfies Cauchy's generalized equation $f(xy) = f(x) f(y)$.

What is the importance of the fact that pairwise comparisons can be assigned numerical intensities and that the reciprocal property should hold?

One consequence of this approach is that by adopting a fundamental scale with the reciprocal property one can obtain a derived scale of measurements which belongs to a ratio scale. Another result is that this approach allows one to determine from the binary comparisons the degree of inconsistency in the numerical comparisons among all the alternatives without imposing transitivity or consistency itself.

MEASUREMENT FOR PREFERENCE RELATIONS WITH TRANSITIVITY

Let A be a finite set of alternatives and let C be a set of criteria, properties or attributes by means of which binary relations are constructed. Let $\gtrsim_C$, $C \in C$ be a binary relation on A. Given $A_i, A_j \in A$, $A_i \gtrsim_C A_j$ represents that A_i is more preferred (or dominates) or equally preferred to A_j according to $C \in C$. Let $A_i =_C A_j$ denote that A_i is equally preferred to A_j according to $C \in C$.

Axiom 1 (Strong Completeness): For $A_i, A_j \in A$, $A_i \gtrsim_C A_j$ or $A_j \gtrsim_C A_i$.

We will write $A_i =_C A_j$ if $A_i \gtrsim_C A_j$ and $A_j \gtrsim_C A_i$.

Axiom 2 (Reflexivity): $A_i \gtrsim_C A_i$ for all $A_i \in A$.

The general approach given here subsumes the special case of preference relations that are strongly complete quasi orders. A strongly complete quasi order is equivalent to the above two axioms together with transitivity. We begin by introducing this special case and prove the existence of a mapping P_C: $A \times A \to \mathbf{R}^+$ and show that it is reciprocal and order preserving. We then drop the transitivity requirement from the special case and prove the existence of P_C in general.

Definition 1: $\gtrsim_C$ is a transitive binary relation on A if and only if for all A_i, A_j and $A_k \in A$, $A_i \gtrsim_C A_j$ and $A_j \gtrsim_C A_k$ imply $A_i \gtrsim_C A_k$.

Definition 2: $\gtrsim_C$ is a quasi order on A if and only if it is reflexive (Axiom 2), and transitive.

Definition 3: A mapping P_C: $A \times A \to \mathbf{R}^+$ is said to be consistent if and only if $P_C(A_i,A_j) \cdot P_C(A_j,A_k) = P_C(A_i,A_k)$ for all A_i, A_j, $A_k \in A$.

Definition 4: A mapping P_C: $A \times A \to \mathbf{R}^+$ is said to be order preserving if and only if

$$A_i \gtrsim_C A_j \iff P_C(A_i,A_j) \geq 1.$$

Theorem 1: If $\gtrsim_C$ is a strongly complete quasi order on A, then there is a real-valued order preserving function P_C on $A \times A$ such that

$$P_C(A_i,A_j) \cdot P_C(A_j,A_k) = P_C(A_i,A_k) \text{ for all } A_i, A_j, A_k \in A.$$

Proof: To prove this theorem we use the following lemma [Debreu, 1954].

> "Let A be a completely ordered set whose quotient $A/=_C$ is countable. There exists on A a real, order preserving function, continuous in any natural typology."

In our case A is a finite set. Hence the quotient is countable. Thus, there exists a real function $g: A \times A \to \mathbf{R}^+$ such that

$$A_i \gtrsim_C A_j \text{ if and only if } g(A_i) \geq g(A_j) \text{ for all } A_i, A_j \in A.$$

Let $P_C(A_i,A_j) \equiv g(A_i)/g(A_j)$ for all $A_i, A_j \in A$. By construction P_C is consistent and the result follows.

From Theorem 1 it can be easily seen that consistency is a more restrictive property than transitivity. If P_C is consistent then the binary relation $\gtrsim_C$ must be transitive. However, the converse is not always true. For example, if $P_C(A_i,A_j) \geq 1$, $P_C(A_j,A_k) \geq 1$, and $P_C(A_i,A_k) \geq 1$, for all $A_i,A_j,A_k \in A$, then we have $A_i \gtrsim_C A_j$, $A_j \gtrsim_C A_k$ and $A_i \gtrsim_C A_k$, for all $A_i,A_j,A_k \in A$. However, one could have $P_C(A_i,A_j) \cdot P_C(A_j,A_k) \neq P_C(A_i,A_k)$, for some A_i, A_j and A_k. Then P_C would not be a consistent function and $\gtrsim_C$ is still transitive.

FUNDAMENTAL MEASUREMENT FOR PREFERENCES WITHOUT TRANSITIVITY

Let A be a countable set of alternatives.

Lemma 1: If $\gtrsim_C$ satisfies Axioms 1 and 2 on A, then there exists a real valued function P_C: $A \times A \to \mathbf{R}^+$ such that

(1) $P_C(A_i,A_i) = 1$, $A_i \in A$,

(2) If $A_i \gtrsim_C A_j$ then $P_C(A_j,A_i) \leq P_C(A_i,A_j)$, $A_i,A_j \in A$,

(3) If $P_C(A_i,A_j) \geq P_C(A_h,A_k)$ then $P_C(A_j,A_i) \leq P_C(A_k,A_h)$ $A_i,A_j,A_h,A_k \in A$.

Proof: Select all pairs $(A_i,A_j) \in A \times A$ such that $A_i \gtrsim_C A_j$. Define P_C as follows:

$$P_C(A_i,A_i) = a \in \mathbf{R}^+, \text{ for all } A_i \in A,$$

$$P_C(A_i,A_j) = \begin{cases} a_{ij} > a, & \text{if } A_i \gtrsim_C A_j \text{ and } A_i \neq_C A_j \\ a_{ij} = a, & \text{if } A_i =_C A_j \end{cases}$$

There is no loss in generality if a is made equal to unity ($a = 1$).

Let $X^+ = \{a_{ij} > 1,\ a_{ij} = P_C(A_i,A_j),\ A_i,A_j \in A\}$.

$(X^+, \geq)$ is a completely ordered set. Hence, given a_{ij} and a_{hk} in X^+, either $a_{ij} \geq a_{hk}$ or $a_{hk} \geq a_{ij}$. Assume that $a_{ij} \geq a_{hk}$.

For the remaining pairs (A_j,A_i) such that $A_i \gtrsim_C A_j$, define P_C as follows:

$$P_C(A_j,A_i) = a_{ji} \leq a_{ij},\ P_C(A_k,A_h) = a_{kh} \leq a_{hk}, \text{ and } a_{ji} \leq a_{kh} \text{ if } a_{ij} \geq a_{hk}.$$

By construction P_C satisfies (1), (2) and (3). □

Let $X^- = \{a_{ji} < 1,\ a_{ji} = P_C(A_j,A_i),\ A_i,A_j \in A\}$.

Let $X = \{a_{ij} = P_C(A_i,A_j),\ A_i,A_j \in A\}$. We have $X = X^+ \cup X^- \cup \{1\}$.

Theorem 2: If $\gtrsim_C$ satisfies Axioms 1 and 2 on A, there exists a real valued function f: $X \to X$ such that $f[f(a_{ij})] = a_{ij}$, $a_{ij} \in X$.

Proof: From lemma 1 if $a_{ij} \geq a_{hk}$ then $a_{ji} \leq a_{kh}$. Also, the cardinality of X^+ and X^- is the same. Hence, there exists a bijective mapping f: $X \to X$ such that

(1) $x \geq x'$ if and only if $f(x) \leq f(x')$,
$x, x' \in X$,
(x) $f(1) = 1$.

We now prove by induction that there does not exist a $x \in X$ for which $f[f(x)] \neq x$.

Let n be the cardinality of A.

For n=2, we have $X = \{1, a_{12}, a_{21}\}$. By Axiom 1 either $A_1 \gtrsim_C A_2$ or $A_2 \gtrsim_C A_1$. Assume that $A_1 \gtrsim_C A_2$ and $A_1 \neq_C A_2$. By definition, $P_C(A_1, A_2) = a_{12} > 1$. It is clear that $a_{21} < 1$. Thus, we have $a_{12} \geq a_{21}$.

If $f(a_{12}) = a_{21}$, then $f(a_{21}) = a_{12}$. To see this note that if $f(a_{21}) \neq a_{12}$, either $f(a_{21}) = a_{21}$ or $f(a_{21}) = 1$. Since by definition $a_{21} < 1$ we have $f(a_{21}) = a_{21}$. But $f(x) = x$ if and only if $x = 1$, hence we have $a_{21} = 1$, which contradicts the assumption that $a_{21} < 1$. Thus, for n = 2

$$f[f(x)] = x, \quad \text{for all } x \in X \tag{1}$$

Assume that (1) holds n = k. We show that it also holds for n = k + 1. Without loss of generality let $a_{1,k+1} \geq a_{2,k+1} \geq \cdots \geq a_{k,k+1}$ and let $f(a_{i,k+1}) = a_{k+1,i}$, i = 1, 2, ..., k. By definition we have

$$a_{k+1,1} \leq a_{k+1,2} \leq \cdots \leq a_{k+1,k} \leq 1.$$

Assume that (1) is violated for some i. Thus we may write

$$f(a_{k+1,i}) \neq a_{i,k+1}$$

but

$$f(a_{k+1,h}) = a_{h,k+1}, \quad \text{for } h \neq 1.$$

Let j be such that $f(a_{k+1,i}) = a_{j,k+1}$. Let $i < j$ (the same argument may be applied for $i > j$). We have $f(a_{k+1,j}) = a_{i,k+1}$. Since $a_{i,k+1} \geq a_{j,k+1}$ for $i < j$, then $f(a_{i,k+1}) \leq f(a_{j,k+1})$ and $a_{k+1,i} \geq a_{k+1,j}$, for $i < j$. This contradicts the assumption that $a_{k+1,i} \leq a_{k+1,j}$ for all $i < j$, and the result follows.

□

Let F be the family of bijective mappings from $X \subseteq \mathbb{R}^+$ to $X \subseteq \mathbb{R}^+$ such that

(1) $x \geq x'$ if and only if $f(x) \leq f(x')$,
$x, x' \in X$
(2) $f(1) = 1$, and
(3) $f[f(x)] = x$, for all $x \in X \subseteq \mathbb{R}^+$.

Theorem 3: A necessary and sufficient condition for $f \in F$ to be the reciprocal property is that f be a solution to Cauchy's generalized equation

$$f(xy) = f(x)\, f(y).$$

Proof: Let $f \in F$ and $f(xy) = f(x)\, f(y)$ for all $x, y \in X$. The solution of this equation is given by $f(x) = x^\alpha$. Since $f \in F$, we have $f[f(x)] = x^{\alpha^2}$ and hence $\alpha^2 = 1$. $f \in F$ also implies that $\alpha = -1$, and the result follows.

CONCLUSIONS

To obtain the reciprocal property from the set of axioms imposed, one must also assume that there exists an isomorphism defined in the set of numerical intensities of the pairwise comparisons and that is a solution of Cauchy's generalized equation

$$f(xy) = f(x)\, f(y).$$

To justify this equation is even more difficult than to assume that the intensity of preferences satisfy the reciprocal property. In fact, Axioms 1 and 2 are consequences of the reciprocal property if it is imposed as an axiom. Thus, the reciprocal property should be the first axiom of the theory, and not the axioms of strong completeness and reflexivity as, for example, utility theory assumes. The reciprocal property is weaker than cardinal consistency which implies ordinal consistency (or transitivity). Hence, a theory which has the reciprocal property as an axiom should be more general than a theory that assumes transitivity.

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