Mode III eccentric crack in a functionally graded piezoelectric strip

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Abstract

This paper studies the internal crack problem located within one functionally graded piezoelectric strip. One crack is normal to the edge of the strip and the material properties vary along the direction of crack length. Three different boundary conditions and both impermeable and permeable cases are discussed. The problem can be reduced to a system of singular integral equations and solved by using the Gauss–Chebyshev formulas. The results show that the edge boundary conditions and the nonhomogeneous parameter significantly control the magnitudes of stress and electric displacement intensity factors.

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1. Introduction

Piezoelectric materials have been widely used as a smart material in electromechanical devices due to the demand of transform from mechanical to electrical loadings, and vice versa. To get rid of the possible interface failure, the functionally graded piezoelectric materials (FGPM) have been introduced to reduce the stress concentration at the interface.

Recently, some researchers have investigated the fracture behavior of a crack in FGPM. The material properties may vary along the crack line or normal to the crack surface. Wang and Node (2001) firstly studied the thermopiezoelectric fracture problem of a functionally graded piezoelectric layer bonded to a metal. They obtained the thermal flow, stress and electric displacement intensity factors and predict the direction of crack extension by using the energy density theory. Li and Weng (2002) solved the problem of a FGPM...
strip containing a finite crack normal to boundary surfaces. The elastic stiffness, piezoelectric constant and the dielectric permittivity vary continuously along the thickness. It is found that the singular behaviors of both stress and electric displacement are the same as those in a homogeneous piezoelectric material. In this crack problem, an increase in the gradient of the material properties will reduce the magnitude of the intensity factors. Jin and Zhong (2002) studied the problem of a moving mode-III permeable crack in FGPM. Wang (2003) considered the mode III crack problem in FGPM in which the material properties are assumed in a class of functional form such that an analytical solution is possible. Kwon and Lee (2003) studied the problem of a finite crack propagating at constant speed in a functionally graded piezoelectric strip bonded to a homogeneous piezoelectric strip. Jin et al. (2003) solved the propagation problem of an anti-plane moving crack in a functionally graded piezoelectric strip. Chen et al. (2003a) studied the dynamics response of a crack in a functionally graded interface of two dissimilar piezoelectric half-planes. They found that the presence of the dynamic electric field could impede or enhance the crack propagation depending on the time elapsed and the direction of the applied electric impact. All of the above researches assumed that the material properties vary in the direction normal to the crack line.

Chen et al. (2003b) assumed the material properties to vary continuously along the crack direction. They obtained the mode-I transient response of a FGPM containing a through crack under the in-plane mechanical and electrical impact. Later, Ueda (2003) solved the anti-plane problem of a crack in a functionally graded piezoelectric strip bonded to two elastic surface layers. Chen et al. (2004) obtained the transient response of an infinite functionally graded piezoelectric medium containing a through crack under the mixed-mode in-plane mechanical and electrical load. Chue and Ou (2005) studied the problem of a crack in one of two bonded functionally graded piezoelectric half-planes. Several degenerated sub-problems are also considered to discuss the effects of the nonhomogeneous parameters and crack locations on the stress and electric displacement intensity factors.

In this paper, the permeable and impermeable crack problems in a functionally graded piezoelectric strip are considered. The crack is normal to the edges of strip. The material properties are assumed in an exponential form and vary along the crack line. Integral transform techniques are used to obtain a system of singular integral equations, which are then solved by employing the Gauss–Chebyshev integration formula. The stress and electrical displacement intensity factors are computed under three different mechanical and electrical boundary conditions. Several simplified cases are also discussed.

2. Formulation of the problem

Consider a functionally graded piezoelectric strip containing an eccentric crack normal to the edge (Fig. 1). The width of the strip is \( h \) and the crack with crack length \( 2a_0 \) is centered at \( x = c \). The material properties of FGPM vary continuously along \( x \)-direction in exponential form. The FGPM is polarized normal to the \( x-y \) plane. When the strip is subjected to antiplane mechanical and inplane electrical loads, the crack problem involves the antiplane elastic field coupled \( (w, \tau_{xz}, \tau_{yz}) \) with the inplane electric field \( (\phi, D_x, D_y) \). The constitutive and equilibrium equations are as follows:

\[
\begin{align*}
\tau_{xz} &= e_0 e_{6x} \frac{\partial w}{\partial x} + e_0 e_{6x} \frac{\partial \phi}{\partial x} \\
\tau_{yz} &= e_0 e_{6x} \frac{\partial w}{\partial y} + e_0 e_{6x} \frac{\partial \phi}{\partial y} \\
D_x &= e_0 e_{6x} \frac{\partial w}{\partial x} - e_0 e_{6x} \frac{\partial \phi}{\partial x} \\
D_y &= e_0 e_{6x} \frac{\partial w}{\partial y} - e_0 e_{6x} \frac{\partial \phi}{\partial y}
\end{align*}
\]
where $c_0$, $e_0$ and $\varepsilon_0$ are the shear modulus, the piezoelectric constant and the dielectric constants defined at the boundary $x = 0$, respectively. The parameter $\beta$ is called the nonmohomogeneous parameter of the FGPM.

Substituting Eqs. (1) into Eqs. (2), the equilibrium equations can be rewritten as

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \quad (2a)$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \quad (2b)$$

The solutions of Eq. (3) are as follows:

$$W(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x, y)e^{-i\alpha x} \, dx + \frac{2}{\pi} \int_{0}^{\infty} g_1(x, \alpha) \sin(\alpha y) \, d\alpha$$

$$\phi(x, y) = \frac{e_0}{\varepsilon_0} W(x, y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(x, y)e^{-i\alpha x} \, dx + \frac{2}{\pi} \int_{0}^{\infty} g_2(x, \alpha) \sin(\alpha y) \, d\alpha \quad (4)$$

If the problem is symmetric with respect to the x-axis, we only consider the upper-half plane, i.e., $y > 0$. In addition, we assume that the displacement and electric potential vanish as $y$ approaches infinity. The unknown functions of Eq. (4) may be expressed as

$$f_1(x, y) = A_1(x)e^{my}$$
$$f_2(x, y) = B_1(x)e^{my}$$
$$g_1(x, \alpha) = C_1(x)e^{\alpha x} + C_2(x)e^{-\alpha x}$$
$$g_2(x, \alpha) = D_1(x)e^{\alpha x} + D_2(x)e^{-\alpha x} \quad (5)$$

where $m = -\sqrt{x^2 + i\beta}$, $p_1 = -\beta/2 - \alpha_1$, $p_2 = -\beta/2 + \alpha_1$, $\alpha_1 = \sqrt{x^2 + \beta^2/4}$. 

![Fig. 1. Configuration of a FGPM strip containing an eccentric crack.](image-url)
The eccentric crack can be categorized into two types: impermeable and permeable. In addition, both the mechanical and electrical loads are exerted on the crack surfaces. Two types of boundary conditions assigned at the strip edges are considered. Type A represents that the surface is clamped/electrically closed whereas Type B is traction-free/electrically opened. In the following formulations, we derive the stress and electric displacement intensity factors for impermeable crack problem and permeable crack problem when the conditions at the boundary surfaces \( x = 0 \) and \( x = h \) are A–A, B–A, and B–B respectively.

3. Impermeable crack problem

For impermeable crack problem, the conditions on the line \( y = 0 \) are

\[
\begin{align*}
  w(x,0) &= 0 \quad \text{for } 0 \leq x < a \text{ and } b < x < h \quad (6a) \\
  \phi(x,0) &= 0 \quad \text{for } 0 \leq x < a \text{ and } b < x < h \quad (6b) \\
  \tau_y(x,0) &= \tau(x)\text{ for } a \leq x \leq b \quad (6c) \\
  D_y(x,0) &= D(x)\text{ for } a \leq x \leq b \quad (6d)
\end{align*}
\]

where \( \tau(x) \) and \( D(x) \) are the shear stress and electric displacement applied on the crack surface, respectively.

3.1. Case 1: A–A boundary condition

In this case, both surfaces of the strip are clamped/electrically closed. The required conditions are as follows:

\[
\begin{align*}
  w(0,y) &= 0 \quad (7a) \\
  \phi(0,y) &= 0 \quad (7b) \\
  w(h,y) &= 0 \quad (7c) \\
  \phi(h,y) &= 0 \quad (7d)
\end{align*}
\]

Using Eqs. (4) and (3) and conditions Eqs. (7a–d), we obtain the following relations after performing the inverse Fourier Transform:

\[
\begin{align*}
  C_1(x) + C_2(x) &= R_1(x) \\
  \frac{c_0}{\varepsilon_0} [C_1(x) + C_2(x)] + D_1(x) + D_2(x) &= R_2(x) \\
  C_1(x)e^{\eta h} + C_2(x)e^{\eta h} &= R_3(x) \\
  \frac{c_0}{\varepsilon_0} [C_1(x)e^{\eta h} + C_2(x)e^{\eta h}] + D_1(x)e^{\eta h} + D_2(x)e^{\eta h} &= R_4(x)
\end{align*}
\]

where \( R_i(x), (i = 1–4) \) are given in Appendix A. Define two unknown discontinuous functions as

\[
\begin{align*}
  g_1(x) &= \frac{\partial}{\partial x} w(x,0), \quad g_2(x) = \frac{\partial}{\partial x} \phi(x,0)
\end{align*}
\]

which satisfy the conditions:

\[
\int_a^b g_1(t) \, dt = \int_a^b g_2(t) \, dt = 0
\]
The functions $A_1(x)$ and $B_1(x)$ can be solved in a new form as

$$A_1(x) = \frac{i}{2} \int_a^b g_1(t) e^{i\alpha t} \, dt$$

$$B_1(x) = \frac{i}{2} \int_a^b g_2(t) e^{i\alpha t} \, dt - \frac{e_0}{\hat{v}_0} \frac{i}{\alpha} \int_a^b g_1(t) e^{i\alpha t} \, dt$$

By using the residue theorem, we obtained:

$$C_1(x) = \frac{\alpha}{4\pi \sinh(\alpha h)} \int_a^b g_1(t) e^{i\alpha t} \left( \frac{1}{p_1} e^{-\alpha_1(t-x)} - \frac{1}{p_2} e^{\alpha_2(t-x)} \right) \, dt$$

$$C_2(x) = \frac{\alpha}{4\pi \sinh(\alpha h)} \int_a^b g_1(t) e^{i\alpha t} \left( \frac{1}{p_2} e^{-\alpha_1(t-x)} - \frac{1}{p_1} e^{\alpha_2(t-x)} \right) \, dt$$

$$D_1(x) = \frac{e_0}{\hat{v}_0} \frac{\alpha}{4\pi \sinh(\alpha h)} \int_a^b g_1(t) e^{i\alpha t} \left( \frac{1}{p_1} e^{-\alpha_1(t-x)} - \frac{1}{p_2} e^{\alpha_2(t-x)} \right) \, dt$$

$$D_2(x) = \frac{e_0}{\hat{v}_0} \frac{\alpha}{4\pi \sinh(\alpha h)} \int_a^b g_1(t) e^{i\alpha t} \left( \frac{1}{p_2} e^{-\alpha_1(t-x)} - \frac{1}{p_1} e^{\alpha_2(t-x)} \right) \, dt$$

After lengthy mathematical derivations, the mechanical and electrical tractions acted on the crack surface can be written as

$$\tau_{ce}(x, 0) = \tau(x) = e_0 e^{i\phi_{ct}} \frac{1}{\pi} \int_a^b \left[ k_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_1(t) \, dt$$

$$D_3(x, 0) = D(x) = e_0 e^{i\phi_{ct}} \frac{1}{\pi} \int_a^b \left[ k_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_2(t) \, dt$$

where

$$k_1(x, t) = \frac{i}{2} \int_{-\infty}^{\infty} \frac{m}{\alpha} e^{i\alpha_1(t-x)} \, dz$$

$$k_2(x, t) = e^{i\phi_{ct}} \int_0^{\infty} \frac{-x^2}{p_2 x \sinh(\alpha h)} e^{-\alpha_2(t-x)} \, dz$$

$$k_3(x, t) = e^{i\phi_{ct}} \int_0^{\infty} \frac{-x^2}{p_2 x \sinh(\alpha h)} e^{-\alpha_2(t-x)} \, dz$$
Separating the asymptotic term of these kernels when \( z \) approaches infinity, they can be expressed as follows:

\[
k_1(x, t) = \frac{1}{t - x} + h_1(x, t) \tag{16a}
\]
\[
k_2(x, t) = e^{\beta(t-x)} \left\{ \frac{1}{2h - t - x} + \int_0^\infty \left[ \frac{-x^2 \sinh(x_1 x)}{p_1 x_1 \sinh(x_1 h)} e^{-z_1(h-t)} - e^{-x^2(2h-t-x)} \right] \, dz \right\} \tag{16b}
\]
\[
k_3(x, t) = e^{\beta(t-x)} \left\{ \frac{-1}{t + x} + \int_0^\infty \left[ \frac{-x^2 \sinh(x_1 (h-x))}{p_2 x_1 \sinh(x_1 h)} e^{-z_1 t} + e^{-x^2(t+x)} \right] \, dz \right\} \tag{16c}
\]

where

\[
h_1(x, t) = \text{Im} \left\{ \int_0^\infty (\sqrt{1 + \frac{i \beta}{z}} - 1) e^{iz(t-x)} \, dz \right\} \tag{17}
\]

which is a bounded function. Therefore, Eqs. (13) and (14) can be rewritten as

\[
\tau_z(x, 0) = \tau(x) = c_0 e^{\beta x} \frac{1}{\pi} \int_a^b \left[ \frac{1}{t - x} + h_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_1(t) \, dt 
\]
\[
+ e_0 e^{\beta x} \frac{1}{\pi} \int_a^b \left[ \frac{1}{t - x} + h_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_2(t) \, dt \tag{18a}
\]
\[
D_y(x, 0) = D(x) = e_0 e^{\beta x} \frac{1}{\pi} \int_a^b \left[ \frac{1}{t - x} + h_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_1(t) \, dt 
\]
\[
- e_0 e^{\beta x} \frac{1}{\pi} \int_a^b \left[ \frac{1}{t - x} + h_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_2(t) \, dt \tag{18b}
\]

Consider a simplified problem for a FGPM half plane contains a crack of length \( 2a_0 \) which is bonded to a rigid surface at \( x = 0 \). The solution can be obtained directly by setting \( h \to \infty \) in both Eqs. (16b) and (16c). They are

\[
k_2(x, t) = 0 \tag{19a}
\]
\[
k_3(x, t) = e^{\beta(t-x)} \int_0^\infty \frac{-x^2}{x_1(x_1 - \beta/2)} e^{-z_1(t+x)} \, dz \tag{19b}
\]

These results are exactly the same as those of one degenerated case in Chue and Ou (2005).

3.2. Case 2: B–A boundary condition

In this case, we assume that the two surfaces at \( x = 0 \) and \( x = h \) are traction-free/electrically opened and clamped/electrically closed, respectively. The required conditions are as follows:

\[
\tau_z(0, y) = 0 \tag{20a}
\]
\[
D_y(0, y) = 0 \tag{20b}
\]
\[
w(h, y) = 0 \tag{20c}
\]
\[
\phi(h, y) = 0 \tag{20d}
\]
Similarly, use Eqs. (4) and (3) and conditions Eqs. (20a)–(20d), the following relations can also obtained.

\[
\left( c_0 + \varepsilon_0^2 \right) [p_1 C_1(x) + p_2 C_2(x)] + \varepsilon_0 [p_1 D_1(x) + p_2 D_2(x)] = R_3(x)
\]

\[- \varepsilon_0 [p_1 D_1(x) + p_2 D_2(x)] = R_6(x)
\]

\[C_1(x) \varepsilon_1^h + C_2(x) \varepsilon_2^h = R_7(x)
\]

\[
\frac{\varepsilon_0}{\varepsilon_0} \left[ C_1(x) \varepsilon_1^h + C_2(x) \varepsilon_2^h \right] + D_1(x) \varepsilon_1^h + D_2(x) \varepsilon_2^h = R_8(x)
\]

where \( R_i(x), (i = 5–8) \) are given in Appendix A. The four unknowns can also be solved by using the defined functions (11a) and (11b). They are

\[C_1(x) = \frac{\alpha}{2 p_1 z_1} \frac{1}{p_2 \varepsilon_1^h - p_1 \varepsilon_1^h} \int_a^b g_1(t) e^{-\frac{\varepsilon_1^h(t)}{2}} (p_1 e^{z_1(h-t)} - p_2 e^{-z_1(h-t)}) \, dt
\]

\[C_2(x) = \frac{\alpha}{z_1} \left( \frac{e_1^h}{p_2 \varepsilon_1^h - p_1 \varepsilon_1^h} \right) \int_a^b g_1(t) e^{\frac{\varepsilon_1^h(t)}{2}} \sinh(z_1 t) \, dt
\]

\[D_1(x) = \frac{\varepsilon_0}{\varepsilon_0} \frac{\alpha}{2 p_1 z_1} \frac{1}{p_2 \varepsilon_1^h - p_1 \varepsilon_1^h} \int_a^b g_1(t) e^{-\frac{\varepsilon_1^h(t)}{2}} (p_1 e^{z_1(h-t)} - p_2 e^{-z_1(h-t)}) \, dt
\]

\[D_2(x) = \frac{\varepsilon_0}{\varepsilon_0} \frac{\alpha}{z_1} \left( \frac{e_1^h}{p_2 \varepsilon_1^h - p_1 \varepsilon_1^h} \right) \int_a^b g_1(t) e^{\frac{\varepsilon_1^h(t)}{2}} \sinh(z_1 t) \, dt
\]

The mechanical and electrical tractions acted on the crack surface can also be expressed as in Eqs. (13) and (14), and the kernel \( k_2(x,t) \) is the same as that of Case 1. The kernel \( k_2(x,t) \) and \( k_3(x,t) \) are

\[k_2(x,t) = e^{\frac{t-x}{2}} \int_0^\infty \frac{x^2}{z_1} \frac{1}{p_2 \varepsilon_1^h - p_1 \varepsilon_1^h} e^{-\frac{\varepsilon_1^h}{2}(e^{z_1(h-t-x)} + e^{-z_1(h-t-x)})} \, dx
\]

\[k_3(x,t) = e^{\frac{t-x}{2}} \int_0^\infty \frac{-x^2}{z_1} \frac{e_1^h}{p_2 \varepsilon_1^h - p_1 \varepsilon_1^h} (p_2 e^{z_1(t-x)} + p_1 e^{-z_1(t-x)}) \, dx
\]

Separating the asymptotic term of Eqs. (23a) and (23b) when \( z \) approaches infinity, they can be expressed as follows:

\[k_2(x,t) = e^{\frac{t-x}{2}} \left\{ \frac{1}{t+x} + \frac{1}{2h-t-x} + \int_0^\infty \left( \frac{x^2}{z_1} \frac{e^{\frac{\varepsilon_1^h}{2}}}{p_1 \varepsilon_1^h - p_1 \varepsilon_1^h} (e^{z_1(h-t-x)} + e^{-z_1(h-t-x)}) - e^{-z_1(t-x)} - e^{-z_1(2h-t-x)}) \right) \, dx \right\}
\]

\[k_3(x,t) = e^{\frac{t-x}{2}} \left\{ \frac{1}{2h-t-x} + \frac{1}{2h+t-x} + \int_0^\infty \left( \frac{-x^2}{z_1} \frac{e^{\varepsilon_1^h}}{p_1 \varepsilon_1^h - p_1 \varepsilon_1^h} (p_2 e^{z_1(t-x)} + p_1 e^{-z_1(t-x)}) - e^{-z_1(2h-t-x)} + e^{-z_1(2h+t-x)}) \right) \, dx \right\}
\]

Thus we can also obtain the same form as Eqs. (18a) and (18b) with kernels (24a) and (24b). Similarly, consider a simplified problem for a FGPM half plane contains a crack of length 2a, normal to a free surface at \( x = 0 \). The solution can also be obtained by setting \( h \to \infty \) in Eqs. (23a) and (23b). The results are
where

\[ R_i = \frac{\alpha}{2} \int_0^\infty \frac{x^2}{x_1(x_1 + \beta/2)} e^{-x_1(t+x)} \, dx \quad (25a) \]

\[ k_3(x, t) = 0 \quad (25b) \]

These results are also completely the same as those of one degenerated case in Chue and Ou (2005).

### 3.3. Case 3: B–B Boundary condition

In this case, the two surfaces at \( x = 0 \) and \( x = h \) are assumed to be traction-free/electrically opened. The required conditions here are as follows:

\[ \tau_{xz}(0, y) = 0 \quad (26a) \]
\[ D_x(0, y) = 0 \quad (26b) \]
\[ \tau_{xz}(h, y) = 0 \quad (26c) \]
\[ D_x(h, y) = 0 \quad (26d) \]

Following the same procedure as stated above, the following relations can also obtained

\[
\begin{align*}
(c_0 + \frac{c_0^2}{c_0}) [p_1 C_1(x) + p_2 C_2(x)] + e_0 [p_1 D_1(x) + p_2 D_2(x)] &= R_9(x) \\
- e_0 [p_1 D_1(x) + p_2 D_2(x)] &= R_{10}(x) \\
(c_0 + \frac{c_0^2}{c_0}) [p_1 C_1(x)e^\theta h + p_2 C_2(x)e^\theta h] + e_0 [p_1 D_1(x)e^\theta h + p_2 D_2(x)e^\theta h] &= R_{11}(x) \\
- e_0 [p_1 D_1(x)e^\theta h + p_2 D_2(x)e^\theta h] &= R_{12}(x)
\end{align*}
\]

where \( R_i(x), (i = 9–12) \) are given in Appendix A. Similar to previous cases, we get:

\[ C_1(x) = \frac{-x}{2p_1x_1 \sinh(x_1h)} \int_a^b g_1(t) e^\theta \sinh(x_1(h - t)) \, dt \quad (28a) \]

\[ C_2(x) = \frac{-xe^{-x_1h}}{2p_2x_1 \sinh(x_1h)} \int_a^b g_1(t) e^\theta \sinh(x_1t) \, dt \quad (28b) \]

\[ D_1(x) = \frac{e_0}{e_0} \frac{x}{2p_1x_1 \sinh(x_1h)} \int_a^b g_1(t) e^\theta \sinh(x_1(h - t)) \, dt - \frac{x}{2p_1x_1 \sinh(x_1h)} \int_a^b g_2(t) e^\theta \sinh(x_1(h - t)) \, dt \quad (28c) \]

\[ D_2(x) = \frac{e_0}{e_0} \frac{xe^{-x_1h}}{2p_2x_1 \sinh(x_1h)} \int_a^b g_1(t) e^\theta \sinh(x_1t) \, dt - \frac{xe^{-x_1h}}{2p_2x_1 \sinh(x_1h)} \int_a^b g_2(t) e^\theta \sinh(x_1t) \, dt \quad (28d) \]

Similarly, the mechanical and electrical tractions acted on the crack surface can also be expressed as in Eqs. (13) and (14) with the same kernel \( k_1(x, t) \) as Case 1. However, the kernel \( k_2(x, t) \) and \( k_3(x, t) \) here are

\[ k_2(x, t) = \frac{e^{\theta(x-x)}}{\int_0^\infty \frac{-x^2}{p_1x_1} \sinh(x_1(h - t)) e^{-x_1x} \, dx} \quad (29a) \]

\[ k_3(x, t) = \frac{e^{\theta(x-x)}}{\int_0^\infty \frac{-x^2}{p_2x_1} \sinh(x_1t) e^{-x_1(h-x)} \, dx} \quad (29b) \]
And they can be rewritten as follows after separating the asymptotic term of Eqs. (29a) and (29b) when \( z \) approaches infinity.

\[
k_2(x, t) = e^{\frac{\pi}{2}(x-t)} \left\{ \frac{1}{t+x} + \int_0^\infty \left[ \frac{-x^2}{p_1 \xi_1} \frac{\sinh(\xi_1(h - t))}{\sinh(\xi_1 h)} e^{-\xi_1 x} - e^{-u(x+t)} \right] \, dx \right\}
\]

(30a)

\[
k_3(x, t) = e^{\frac{\pi}{2}(x-t)} \left\{ \frac{-1}{2h - t - x} + \int_0^\infty \left[ \frac{-x^2}{p_2 \xi_1} \frac{\sinh(\xi_1 t)}{\sinh(\xi_1 h)} e^{-\xi_1 x} + e^{-u(2h-t-x)} \right] \, dx \right\}
\]

(30b)

The same form as Eqs. (18a) and (18b) with kernels (30a) and (30b) can also be obtained. Ueda (2003) had already solved the problem of a crack in functionally graded piezoelectric strip bonded to two elastic layers. When we let the parameters \( h_1, h_2, \mu_1 \) and \( \mu_2 \) within Ueda (2003) equal to zero and through some lengthy manipulations, the integral Eqs. (31) and (32) in that paper can be reduced to the same as Eqs. (18a) and (18b) together with kernels (17), (30a) and (30b) here.

Also, if the width \( h \) of the strip approaches infinity, we obtain same expressions of Eqs. (25a) and (25b) for \( k_2(x, t) \) and \( k_3(x, t) \).

4. A degenerated example: a crack in a functionally graded piezoelectric medium

Cases 1–3 can be degenerated to a problem of a crack existed in a FGPM medium. Take Case 1 as an example. For convenience, the \( x-y \) coordinate system is shifted to a new \( x'-y' \) coordinate system shown in Fig. 1. The boundary conditions in new coordinate system become

\[
w(-c, y') = 0 \]

\[
\phi(-c, y') = 0 \quad (31a)
\]

\[
w(d, y') = 0 \]

\[
\phi(d, y') = 0 \quad (31c)
\]

Following same derivation procedures, we have

\[
C_1(x) = \frac{x}{2p_1 (\phi_{x_1}^{(c+d)} - \phi_{x_1}^{(c+d)})} \left[ \frac{\phi_{x_1}^{(c+d)}}{p_2} \int_a^b g_1(t) e^{-p_1 t} \, dt - \frac{\phi_{x_1}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt \right]
\]

(32a)

\[
C_2(x) = \frac{x}{2p_2 (\phi_{x_1}^{(c+d)} - \phi_{x_1}^{(c+d)})} \left[ \frac{\phi_{x_1}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt - \frac{\phi_{x_1}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt \right]
\]

(32b)

\[
D_1(x) = \frac{x}{2p_2 \phi_{x_1}^{(c+d)} - \phi_{x_1}^{(c+d)}} \left[ - \frac{\phi_{x_2}^{(c+d)}}{p_2} \int_a^b g_1(t) e^{-p_1 t} \, dt + \frac{\phi_{x_2}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt - \frac{\phi_{x_2}^{(c+d)}}{p_1} \int_a^b g_2(t) e^{-p_1 t} \, dt \right]
\]

\[
+ \frac{x}{2p_1 \phi_{x_1}^{(c+d)} - \phi_{x_1}^{(c+d)}} \left[ \frac{\phi_{x_1}^{(c+d)}}{p_2} \int_a^b g_1(t) e^{-p_1 t} \, dt - \frac{\phi_{x_1}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt + \frac{\phi_{x_1}^{(c+d)}}{p_1} \int_a^b g_2(t) e^{-p_1 t} \, dt \right]
\]

(32c)

\[
D_2(x) = \frac{x}{2p_2 \phi_{x_1}^{(c+d)} - \phi_{x_1}^{(c+d)}} \left[ \frac{\phi_{x_2}^{(c+d)}}{p_2} \int_a^b g_1(t) e^{-p_1 t} \, dt - \frac{\phi_{x_2}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt \right]
\]

\[
- \frac{x}{2p_1 \phi_{x_1}^{(c+d)} - \phi_{x_1}^{(c+d)}} \left[ \frac{\phi_{x_2}^{(c+d)}}{p_1} \int_a^b g_1(t) e^{-p_1 t} \, dt - \frac{\phi_{x_2}^{(c+d)}}{p_1} \int_a^b g_2(t) e^{-p_1 t} \, dt \right]
\]

(32d)

If the two boundary are moved to infinity, that is, \( c \to \infty \) and \( d \to \infty \), the four functions \( C_1(x), C_2(x), D_1(x) \) and \( D_2(x) \) become zero. Therefore the kernels \( k_2(x, t) \) and \( k_3(x, t) \) in Eqs. (18a) and (18b) also disappear. It is clear that the existence of \( k_2(x, t) \) and \( k_3(x, t) \) come from the effects of the boundary surfaces on the crack. Similar conclusion can be made for Cases 2 and 3.
5. Solutions of the singular integral equations

The solutions of the singular integral equations Eqs. (18a) and (18b) with the Cauchy type kernel have the standard form

\[ g_i(t) = \frac{G_i(t)}{\sqrt{(t-a)(b-t)}}, \quad i = 1, 2 \] (33)

where \( G_i(t) \) are bounded functions. By using the known formula of singular integral equation (Muskhelishvili, 1953) as

\[ \frac{1}{\pi} \int_a^b \frac{g_i(t)}{t-x} \, dt = \frac{G_i(a)\exp(\eta)}{\sqrt{b-a}\sqrt{x-a}} - \frac{G_i(b)}{\sqrt{b-a}\sqrt{x-b}} + \text{other terms}, \quad i = 1, 2 \] (34)

Thus the stress and electric displacement intensity factors at crack tips can be obtained as follows:

\[ k_3(b) = \lim_{t \to b^+} \sqrt{2(x-b)}\tau_{xc}(x,0) = -c(b) \frac{G_1(b)}{(b-a)/2} - e(b) \frac{G_2(b)}{(b-a)/2} \] (35a)

\[ k_3(a) = \lim_{t \to a^-} \sqrt{2(a-x)}\tau_{xc}(x,0) = c(a) \frac{G_1(a)}{(b-a)/2} + e(a) \frac{G_2(a)}{(b-a)/2} \] (35b)

\[ k_3^D(b) = \lim_{t \to b^+} \sqrt{2(x-b)}D_t(x,0) = -e(b) \frac{G_1(b)}{(b-a)/2} + e(b) \frac{G_2(b)}{(b-a)/2} \] (35c)

\[ k_3^D(a) = \lim_{t \to a^-} \sqrt{2(a-x)}D_t(x,0) = e(a) \frac{G_1(a)}{(b-a)/2} - e(a) \frac{G_2(a)}{(b-a)/2} \] (35d)

The functions \( G_i(a) \) and \( G_i(b) \) can be computed specifically by normalizing the parameters of the integral equations as

\[ \bar{x} = \frac{x-c}{a_0}, \quad \bar{t} = \frac{t-c}{a_0}, \quad \bar{h} = \frac{h-c}{a_0} \] (36)

The single-valued conditions Eq. (10) become

\[ \int_{-1}^1 f_1(\bar{t}) \, d\bar{t} = \int_{-1}^1 f_2(\bar{t}) \, d\bar{t} = 0 \] (37)

After employing Gauss–Chebyshev integration formula (Erdogan et al., 1973), Eq. (35) can be rewritten as

\[ f_i(\bar{t}) = \frac{F_i(\bar{t})}{\sqrt{(1+\bar{t})(1-\bar{t})}}, \quad i = 1, 2 \] (38)

where \( f_i(\bar{t}) \) and \( F_i(\bar{t}) \) are related to \( g_i(\bar{t}) \) and \( G_i(\bar{t}) \) in Eqs. (34) and (38), respectively. By using Chebyshev polynomials, the stress and electric displacement intensity factors can be rewritten as

\[ k_3(b) = -c_0\exp(\bar{h})\sqrt{a_0}F_1(1) - c_0\exp(\bar{h})\sqrt{a_0}F_2(1) \] (39a)

\[ k_3(a) = c_0\exp(a)\sqrt{a_0}F_1(-1) + c_0\exp(a)\sqrt{a_0}F_2(-1) \] (39b)

\[ k_3^D(b) = -c_0\exp(\bar{h})\sqrt{a_0}F_1(1) + c_0\exp(\bar{h})\sqrt{a_0}F_2(1) \] (39c)

\[ k_3^D(a) = c_0\exp(a)\sqrt{a_0}F_1(-1) - c_0\exp(a)\sqrt{a_0}F_2(-1) \] (39d)
The unknown crack tip values of $F_i(1)$ and $F_i(-1)$, ($i = 1, 2$) are obtained from quadratic extrapolation by using the values of $F_i$ at nodes 2, 3, 4 and $n - 1$, $n - 2$, $n - 3$ respectively. Here $n$ is the number of collocation points along crack line.

6. Permeable crack problem

For the permeable crack problem, the electric field is continuous across the crack surface. The conditions (6b) and (6d) on the line $y = 0$ for impermeable crack of previous section should be replaced by the following two conditions:

$$
\phi(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq h \quad (40a)
$$
$$
D_y(x, 0) = D_z(x, 0) = D(x) \quad \text{for} \quad a \leq x \leq b \quad (40b)
$$

where $D_z(x, 0)$ denotes the electric displacement of the spaces of the crack itself. For this type of condition, only one function $g_1(x)$ is needed for antiplane displacement. Following similar procedures of previous section, the stress and electric displacement on the crack surface are

$$
\tau_{yz}(x, 0) = \tau(x) = c_0 e^{b} \frac{1}{\pi} \int_a^b \left[ \frac{1}{t-x} + h_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_1(t) \, dt \quad (41a)
$$
$$
D_y(x, 0) = e_0 e^{b} \frac{1}{\pi} \int_a^b \left[ \frac{1}{t-x} + h_1(x, t) + k_2(x, t) + k_3(x, t) \right] g_1(t) \, dt \quad (41b)
$$

The functions $h_i(x, t)$ and $k_j(x, t)$ ($i = 2, 3$) are completely the same as those of Eqs. (17), (16b), (16c), (24a), (24b), (30a) and (30b) depending on the different boundary conditions. The stress intensity factors and the electric displacement intensity factors are expressed as

$$
k_3(b) = -c_0 e^{b} \sqrt{a_0} F_1(1) \quad (42a)
$$
$$
k_3(a) = c_0 e^{a} \sqrt{a_0} F_1(-1) \quad (42b)
$$
$$
k_3^I(b) = -e_0 e^{b} \sqrt{a_0} F_1(1) \quad (42c)
$$
$$
k_3^I(a) = e_0 e^{a} \sqrt{a_0} F_1(-1) \quad (42d)
$$

It has been proved that the normalized stress intensity factors in permeable crack case are the same as those in impermeable crack case if the crack problem is symmetric with respect to the crack line (Li and Duan, 2001). In the following discussion, we only consider the case of impermeable crack.

7. Results and discussions

In the following numerical calculations, the material properties of PZT-4 with $c_0 = 25.6$ GPa, $e_0 = 12.7$ C/m$^2$, $\varepsilon_0 = 6.46 \times 10^{-9}$ C/Vm are used. For convenient discussion, both the stress and electric displacement intensity factors are normalized as follows:

$$
k_a = \frac{k_3(a)}{\tau_0 \sqrt{a_0}} = \frac{k_3^I(a)}{D_0 \sqrt{a_0}} \quad (43a)
$$
$$
k_b = \frac{k_3(b)}{\tau_0 \sqrt{a_0}} = \frac{k_3^I(b)}{D_0 \sqrt{a_0}} \quad (43b)
$$
The shear stresses and electric displacements applied on the crack surfaces are $\tau_0 = 4.2$ MPa and $D_0 = 0.002$ C/m$^2$, respectively. The combined effects of boundary conditions, nonhomogeneous parameter $\beta$ and geometric configuration on the intensity factors will be discussed separately.

### 7.1. Effects of crack eccentricity

For three cases of boundary conditions, the variations of normalized intensity factors with $c/h$ at two crack tips are plotted respectively in Fig. 2(a)–(c) at different normalized nonhomogeneous parameter $\beta a_0$ when $h = 5$ cm and $a_0 = 2/3$ cm. The results show that the normalized intensity factors are greater as the crack tip approaches the traction-free/electrically opened boundary surface than that approaches the clamped/electrically closed boundary. This is because the crack tips tend to open more easily for the former boundary conditions. For larger $\beta a_0$ the factors at the weaker material side ($k_a$) become smaller and those at the stronger side ($k_b$) become larger. For positive $\beta a_0$ it shows that $k_b > k_a$ because of stiffer

![Fig. 2. Variations of normalized intensity factors at crack tips with $c/h$ ($a_0 = 2/3$ cm, $h = 5$ cm). (a) Case 1, (b) Case 2 and (c) Case 3.](image-url)
material properties. In Case 3 with B–B boundary conditions, the factor $k_b$ at right crack tip is strongly affected by the edge condition and nonhomogeneous parameter thus cause acute rise. In conclusion, we have seen that the edge boundary conditions and the nonhomogeneous parameter significantly control the magnitudes of stress and electric displacement intensity factors.

Consider a special case when $\beta a_0 = 0$ and $c/h = 0.5$, which represents a homogeneous strip containing a center crack. The factors $k_a$ and $k_b$ are equal as expected.

7.2. Effects of crack length

In this section, we study the effects of crack length when the crack is located at the center of the strip. For $h = 5\, \text{cm}$ and $c = 2.5\, \text{cm}$, Fig. 3(a)–(c) plot the variations of the normalized intensity factors with $c/a_0$ for three boundary conditions, respectively. Note that the crack length $a_0$ in normalized Eqs. (43a) and (43b) is a variable here.

Fig. 3. Variations of normalized intensity factors with $c/a_0$ ($c = 2.5\, \text{cm}, h = 5\, \text{cm}$). (a) Case 1, (b) Case 2 and (c) Case 3.
(a) $\beta = 0$: homogeneous piezoelectric material

In the case of homogeneous piezoelectric materials, the trend of variation is similar to the homogeneous elastic medium problem. Increasing the crack length will reduce the normalized intensity factors as the crack tip approaches the clamped/electrically closed edge surface. The opposite trend is true as the crack tip approaches the traction-free/electrically opened edge surface. Similarly to Section 7.1, these two results are strongly affected by the edge conditions. When the crack length is small enough, say $c/a_0 > 5$, the normalized factors approach unity. It can be seen as a very short crack located within a large plane thus the edge effects can almost be ignored.

(b) $\beta > 0$: nonhomogeneous piezoelectric material

As it has been mentioned in the Section 7.1, the edge boundary conditions and the nonhomogeneous parameter are two important factors that affect stress and electric displacement intensity factors. As the crack approaches the traction-free/electrically opened edge surface, the normalized factors $k_a$ will gradually decrease if the material property gradient of FGPM are strong enough, say $\beta h = 3.75$ or

Fig. 4. Variations of normalized intensity factors with $h/c$ ($c = 1$ cm, $a_0 = 2/3$ cm). (a) Case 1, (b) Case 2 and (c) Case 3.
7.5. This phenomenon can be seen from the variations of \( k_a \) in Fig. 3(b) and (c). On the other hand, consider the curves of \( k_b \) with \( \beta h = 7.5 \) in Fig. 3(a) and (b), they still increase as the crack approaches the clamped/electrically closed edge surface. These opposite tendencies are due to that the effect of the nonhomogeneous parameter is stronger than the edge conditions. For a small crack (larger ratio of \( c/a_0 \)), the normalized factor approaches to a nearly fixed value depending only the nonhomogeneous parameter \( \beta h \) and the edge effect may also be neglected.

7.3. Edge effects

In Section 7.1, we have seen the edge effects as the crack moved from left to right. Here we consider the edge effect by another way. Under the conditions \( a_0 = 2/3 \) cm and \( c = 1 \) cm, the distance between crack tip \( b \) and right edge surface is varied by changing the width of the strip. Fig. 4 plot the results of normalized intensity factors varied with the normalized length \( h/c \). It shows that the factors are significantly influenced only within the smaller ratio of \( h/c \) (i.e., due to the boundary effects). Beyond this range the factors are almost constant for a given \( \beta a_0 \). Again, traction-free/electrically opened boundary condition will raise the intensity factors and the opposite is true for clamped/electrically closed. The factor \( k_a \) is slightly affected by the conditions of right boundary edge surface.

7.4. A crack in a functionally graded piezoelectric medium

Finally we discuss the special reduced case in which both boundary edges of the strip move to infinity. The problem reduces to an infinite functionally graded piezoelectric medium containing a crack. Fig. 5 shows the variations of normalized intensity factors with nonhomogeneous parameter \( \beta a_0 \) when \( a_0 = 2/3 \) cm. Erdogan (1985) had solved the same problem but for pure nonhomogeneous elastic material. The curves for \( \beta a_0 < 0 \) are numerically the same as Fig. 2 of Erdogan (1985).

![Fig. 5. Variations of normalized intensity factors with \( \beta a_0 \) for a crack in an infinite medium (\( a_0 = 2/3 \) cm).](image)
8. Conclusions

The fracture behavior of an impermeable/permeable eccentric crack within a functionally graded piezoelectric strip has been studied. Under the antiplane shear and in plane electric displacement, the problem can be reduced to a set of singular integral equations. The stress and electric displacement intensity factors are obtained by using Gauss–Chebyshev polynomials. The roles of the boundary conditions, geometric configurations and the nonhomogeneous parameters $\beta$ have been discussed in detail to show their effects on the intensity factors. For impermeable crack case, both the stress and electric displacement intensity factors depend on the applied mechanical and electric loads. However, the factors for the permeable crack depend only on the mechanical loads. The higher intensity factors occur at the crack tips near to the traction-free/electrically opened edge as expected. Also as in the nonhomogeneous elastic material, it is found that higher intensity factors occur at those crack tips with stiffer material properties. If the material properties of gradient of FGPM are strong enough, it may further exceed the effect of the edge boundary condition and thus cause opposite trends on the intensity factors.

Appendix A

The functions $R_{i}(x)$ ($i = 1, 2, \ldots, 12$) in Eqs. (8), (21) and (27) are as follows:

\[ R_{1}(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right)A_{1}(\rho) \right\} d\rho \]  
\[ R_{2}(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right) \left[ \frac{e_{0}}{\varepsilon_{0}} A_{1}(\rho) + B_{1}(\rho) \right] \right\} d\rho \]  
\[ R_{3}(x) = R_{7}(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right)A_{1}(\rho) e^{-i\sigma \rho} \right\} d\rho \]  
\[ R_{4}(x) = R_{8}(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right) \left[ \frac{e_{0}}{\varepsilon_{0}} A_{1}(\rho) + B_{1}(\rho) \right] e^{-i\sigma \rho} \right\} d\rho \]  
\[ R_{5}(x) = R_{9}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right) (ip) \left[ \left( c_{0} + \frac{e_{0}^{2}}{\varepsilon_{0}} \right) A_{1}(\rho) + e_{0} B_{1}(\rho) \right] \right\} d\rho \]  
\[ R_{6}(x) = R_{10}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right) (ip)(-\varepsilon_{0}) B_{1}(\rho) \right\} d\rho \]  
\[ R_{11}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right) \left[ \frac{e_{0}}{\varepsilon_{0}} A_{1}(\rho) + e_{0} B_{1}(\rho) \right] e^{-i\sigma \rho} \right\} d\rho \]  
\[ R_{12}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{x}{x^{2} + \rho^{2} + i\beta \rho} \right) \left[ \frac{e_{0}}{\varepsilon_{0}} A_{1}(\rho) + e_{0} B_{1}(\rho) \right] e^{-i\sigma \rho} \right\} d\rho \]

References


