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Existence of Solutions to $u'' + u + g(t, u, u') = p(t), u(0) = u(\pi) = 0$

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Existence and multiplicity results for the boundary value problem

$$\begin{cases} u'' + u + g(t, u, u') = p(t), & 0 < t < \pi, \\ u(0) = u(\pi) = 0 \end{cases}$$

are presented. The proofs are based on the alternative method, a connectedness result, the contraction mapping principle, and a detailed analysis of the bifurcation equation utilizing, e.g., a generalization of the mean value theorem for integrals. We shall obtain results with g bounded or unbounded, having finite limits at $\pm\infty$ or without limits, thus extending some recent results in the literature. The proofs offer a constructive way to find the bounds for $\bar{p}$ and to find numerically the number of solutions and the approximative solutions. © 2001 Academic Press

Key Words: boundary value problem; resonance; existence; multiple solutions.

1. INTRODUCTION

The two-point boundary value problem

$$\begin{gathered} u'' + u + g(u') = p(t), \quad 0 < t < \pi, \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

where g and p are continuous functions, has been studied (e.g., by Canada and Drabek [2] and Habets and Sanchez [3]). The existence results of [2] are completed in [3] by a multiplicity result in terms of conditions for $\bar{p}$ in the decomposition of p,

$$p(t) = \bar{p} \sin t + \tilde{p}(t), \tag{1.2}$$

where $\bar{p} \in \mathbb{R}$ and $\tilde{p}$ is orthogonal to $\sin t$. The proof is carried out using mainly topological degree and homotopy arguments.

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We shall present existence and multiplicity results for the boundary value problem

$$u'' + u + g(t, u, u') = p(t), \quad 0 < t < \pi,$$
$$u(0) = u(\pi) = 0. \tag{1.3}$$

The proofs are based on the alternative method (as in [2] and [3]), a connectedness result of [6], the contraction mapping principle, and a detailed analysis of the bifurcation equation utilizing, e.g., a generalization of the mean value theorem for integrals [5]. We shall obtain results for (1.3) with g bounded or unbounded, having finite limits at $\pm\infty$ or without limits.

The proofs offer a constructive way to find the bounds for $\bar{p}$ and to find numerically both the number of solutions and the approximative solutions.

2. THE ALTERNATIVE METHOD

Denote $\phi(t) = \sqrt{\frac{2}{\pi}} \sin t$ and $Lu = u'' + u$. Let k be a modified Green's function satisfying (as a function of t)

$$\begin{aligned} &Lk(t, s) = \delta(t - s) - \phi(t)\phi(s) \\ &k(0, s) = k(\pi, s) = 0 \\ &\int_0^\pi k(t, s)\phi(t)\, dt = 0. \end{aligned} \tag{2.1}$$

The problem (1.3) is equivalent to the pair of equations

$$u_\lambda(t) = \lambda\phi(t) + \int_0^\pi k(t, s)\left[\tilde{p}(s) - g\left(s, u_\lambda(s), u'_\lambda(s)\right)\right] ds, \tag{2.2}$$

$$\bar{\delta}(\lambda) = \bar{p} - \int_0^\pi g\left(t, u_\lambda(t), u'_\lambda(t)\right)\phi(t)\, dt = 0. \tag{2.3}$$

Here, for simplicity, we write

$$p(t) = \bar{p}\phi(t) + \tilde{p}(t) \tag{2.4}$$

instead of (1.2).

That a solution of (2.2)–(2.3) is a solution of (1.3) is easily verified by applying L to the integral equation (2.2) and using the given orthogonality conditions. The proof that a solution of (1.3) satisfies (2.2)–(2.3) is a standard one using the Lagrange identity, symmetry of $k(t, s)$, and the orthogonality conditions.

3. THE RESULTS

We shall use the following assumptions for g: $[0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

(g1) g is continuous and bounded;

(g2) g is continuous and satisfies the Lipschitz condition $|g(t, u, v) - g(t, \bar{u}, \bar{v})| \le M|u - \bar{u}| + N|v - \bar{v}|$, $u, \bar{u}, v, \bar{v} \in \mathbb{R}$, where $M^2 + 4N^2 < 9/2$.

If g satisfies (g1), then by Schauder's fixed-point theorem the integral equation (2.2) has at least one solution u_λ for any given $\lambda \in \mathbb{R}$. If, on the other hand, g satisfies (g2), then it can be shown that for a fixed $\lambda \in \mathbb{R}$ the right-hand side of Eq. (2.2) defines an operator which is a contraction mapping on $H^1[0, \pi]$ and hence has a unique fixed point u_λ. In both cases we can calculate

$$\tilde{\delta}(\lambda) = \int_0^\pi g\big(t, u_\lambda(t), u'_\lambda(t)\big)\phi(t)\, dt.$$

In the case of (g1) $\tilde{\delta}$ may be multivalued. Denote

$$a = \inf\{\tilde{\delta}(\lambda) : \lambda \in \mathbb{R}, \quad u_\lambda \text{ is a solution of (2.2)}\},$$
$$b = \sup\{\tilde{\delta}(\lambda) : \lambda \in \mathbb{R}, \quad u_\lambda \text{ is a solution of (2.2)}\}.$$

If $\bar{p} \in (a, b)$ and g satisfies (g2), then there exist $\tilde{\delta}(\lambda_1)$ and $\tilde{\delta}(\lambda_2)$ such that $\tilde{\delta}(\lambda_1) < \bar{p} < \tilde{\delta}(\lambda_2)$, and it can be shown that $\tilde{\delta}(\lambda)$ is (Lipshitz) continuous, which implies that $\tilde{\delta}(\lambda) = \bar{p}$ for a $\lambda \in (\lambda_1, \lambda_2)$ (or for a $\lambda \in (\lambda_2, \lambda_1)$); i.e., problem (1.3) has a solution. Also in the case of (g1), as shown in [2], using a result of ([1], Theorem 3.1), problem (1.3) has a solution. Hence, we can state the following result, which essentially is due to Canada and Drabek [2].

THEOREM 3.1. *If g satisfies* (g1) *or* (g2), *then there exists an interval* $[a, b]$ *such that problem* (1.3) *has* (i) *at least one solution if* $\bar{p} \in (a, b)$ *and* (ii) *no solution if* $\bar{p} \notin [a, b]$.

Remark 3.1. In the case of dependence only on the derivative $g(t, u, v) = g(t, v)$, the inequality $M^2 + 4N^2 < 9/2$ can be replaced by the inequality $N < 3/2$ and in the case $g(t, u, v) = g(t, u)$ by the inequality $M < 3$. Also, we could replace the constants M and N by suitable square integrable functions. As for the case $\bar{p} \in \{a, b\}$, we refer to [3].

EXAMPLE 3.1. Consider the problem

$$u'' + u + \sinh^{-1} u' = \bar{p}\phi(t) + \tilde{p}(t), \qquad u(0) = u(\pi) = 0. \tag{3.1}$$

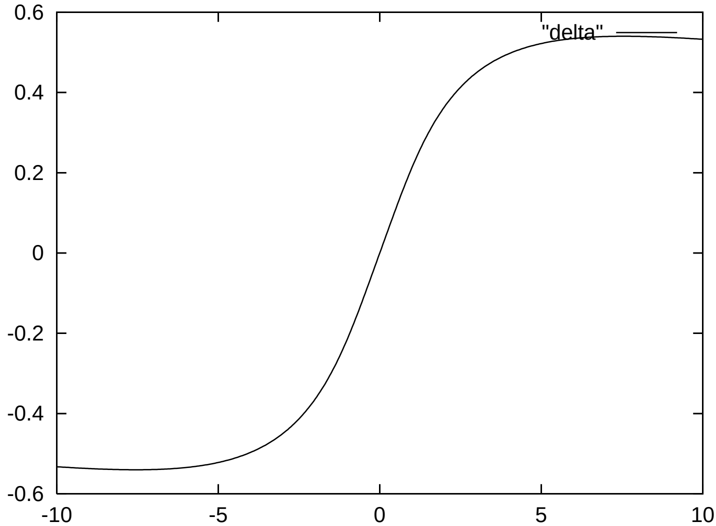

FIGURE 1

The function $g(t, u, v) = \sinh^{-1} v$ satisfies (g2) with $M = 0$ and $N = 1$; hence we can apply Theorem 3.1. The curve $\tilde{\delta}(\lambda)$, which is found numerically for $\tilde{p}(t) \equiv 0$, is shown in Fig. 1. We have $(a, b) \approx (-0.54, 0.54)$. In the general case, if $\|p\|_\infty$ is small enough, then following the proof in [4, p. 795], it can be shown that (1.3) has a (small) solution. Thus, in that case we know that the interval (a, b) is nonempty. Note that g is not bounded. The interval (a, b) depends on $\tilde{p}$ and g. If we have some additional information, we may obtain a priori bounds for (a, b).

PROPOSITION 3.1. *Assume that* $g(t, u, v) = g(t, v)$ *satisfies* $(g2)$ *with* $M = 0$ *and* $N < 3/2$,

$$\int_0^\pi g(t, c\cos t)\sin t \; dt = 0 \qquad \text{for all } c \in \mathbb{R}, \tag{a}$$

and that

$$|\tilde{p}(t) - g(t, v)| \leq m(t), \qquad t \in [0, \pi], \;\; v \in \mathbb{R}, \tag{b}$$

for an $m \in L_1^+[0, \pi]$. *Then* $[a, b] \subset [-d, d]$, *where*

$$d = N \int_0^\pi \int_0^\pi m(s)|k_t(t, s)|\phi(t)\, dt\, ds,$$

i.e., problem (1.3) *does not have a solution if* $|\bar{p}| > d$.

Proof. We have

$$u_\lambda(t) = \lambda\phi(t) + \int_0^\pi k(t,s)\big[\tilde{p}(s) - g\big(s, u'_\lambda(s)\big)\big]\, ds$$

and

$$u'_\lambda(t) = \lambda\phi'(t) + \tilde{u}'_\lambda(t),$$

where $\tilde{u}'_\lambda(t) = \int_0^\pi k_t(t,s)[\tilde{p}(s) - g(s, u'_\lambda(s))]\, ds$ satisfies, by (b),

$$|\tilde{u}'_\lambda(t)| \le \int_0^\pi |k_t(t,s)| m(s)\, ds. \tag{c}$$

We can write

$$g\big(t, u'_\lambda(t)\big) = g(t, \lambda\phi'(t)) + w(t),$$

where

$$w(t) = g\big(t, \lambda\phi'(t) + \tilde{u}'_\lambda(t)\big) - g(t, \lambda\phi'(t))$$

satisfies, by the Lipschitz condition, the inequality

$$|w(t)| \le N|\tilde{u}'_\lambda(t)|. \tag{d}$$

Now, by using (a), (c), and (d) we obtain

$$\begin{aligned} |\tilde{\delta}(\lambda)| &= \Big|\int_0^\pi g(t, u'_\lambda(t))\phi(t)\, dt\Big| = \Big|\int_0^\pi w(t)\phi(t)\, dt\Big| \\ &\le \int_0^\pi N|\tilde{u}'_\lambda(t)|\phi(t)\, dt \le N\int_0^\pi \int_0^\pi |k_t(t,s)| m(s)\, ds\, \phi(t)\, dt = d, \end{aligned}$$

which means that problem (1.3) cannot have a solution if $|\bar{p}| > d$. ∎

EXAMPLE 3.2. Consider the boundary value problem

$$u'' + u + \sin u' = \bar{p}\phi(t), \qquad u(0) = u(\pi) = 0. \tag{3.2}$$

The function $g(t, u, v) = \sin v$ satisfies (g2) with $M = 0$ and $N = 1$, and condition (a), condition (b) with $m(t) \equiv 1$ when $\tilde{p}(t) \equiv 0$ and $d \approx 1.1776$. Hence, by Proposition 3.1 we deduce that problem (3.2) does not have a solution if $|\bar{p}| > 1.1776$. In Example 3.3 we will find numerically a smaller lower estimate for $|\bar{p}|$ for the nonexistence.

THEOREM 3.2. *Assume that g satisfies (g1) or (g2) and that $c \in (a, b)$ is a limit point of both $\{\tilde{\delta}(\lambda) : \lambda \in (-\infty, d]\}$ and $\{\tilde{\delta}(\lambda) : \lambda \in [d, \infty)\}$ for a $d \in \mathbb{R}$. Then, if $\bar{p} \in (a, b)\setminus\{c\}$, problem (1.3) has at least two solutions.*

Proof. Suppose first that (g1) holds. Since g is continuous and bounded, then for any closed bounded interval $I = [\alpha, \beta]$ there exists a closed bounded convex subset B of $H^1[0, \pi]$ such that the mapping T,

$$T(u, \lambda) = \lambda\phi(t) + \int_0^\pi k(t, s) \times \left[\tilde{p}(s) - g\big(s, u(s), u'(s)\big)\right] ds, \qquad \lambda \in I,\ u \in B,$$

is a compact continuous mapping from $B \times I$ into B. Then, by [6, Fixed Point Theorem, p. 341], there exists a connected set $S \subset B \times I$ of fixed points of T, and S meets both $B \times \{\alpha\}$ and $B \times \{\beta\}$. Now, for any $\bar{p} \in (a, b)\backslash\{c\}$ we can find $\tilde{\delta}(\lambda_1)$, $\tilde{\delta}(\lambda_2)$ and $\tilde{\delta}(\lambda_3)$ such that $\lambda_1 < \lambda_2 < \lambda_3$ and $\tilde{\delta}(\lambda_1) < \bar{p} < \tilde{\delta}(\lambda_2)$, $\tilde{\delta}(\lambda_3) < \bar{p} < \tilde{\delta}(\lambda_2)$. Hence, as a continuous real valued function on a connected set S_1 associated with the interval $I_1 = [\lambda_1, \lambda_2]$, $(u, \lambda) \to \bar{p} - \tilde{\delta}(\lambda)$ assumes the value 0 on S_1; i.e., problem (1.3) has a solution. The same conclusion holds true for the interval $I_2 = [\lambda_2, \lambda_3]$; i.e., problem (1.3) has at least two solutions.

If g satisfies (g2), then $\tilde{\delta}(\lambda)$ is single valued and Lipschitz continuous and the proof is obvious. ∎

In [3] it is proved that if g satisfies (g1) and is locally Lipschitz continuous then the conclusion of Theorem 3.2 holds true, provided that $g(t, u, v) = g(v)$ has finite limits $g(-\infty)$ and $g(+\infty)$ with $p \in (a, b)\backslash\{l\}$, $l = g(-\infty) + g(\infty)$. In the following example we have $g(v) = \sin v$, which does not have limits at $\pm\infty$ but $\lim_{|\lambda|\to\infty} \tilde{\delta}(\lambda) = 0$, and we can apply Theorem 3.2.

EXAMPLE 3.3. Consider the boundary value problem

$$u'' + u + \sin u' = \bar{p}\phi(t), \qquad u(0) = u(\pi) = 0. \tag{3.2}$$

The function $g(t, u, v) = \sin v$ satisfies (g2) with $M = 0$ and $N = 1$, and we have

$$\begin{aligned}
\tilde{\delta}(\lambda) &= \int_0^\pi \sin\big(\lambda\phi'(t) + \tilde{u}'_\lambda(t)\big)\phi(t)\,dt \\
&= \int_0^\pi \sin\left(\lambda\sqrt{\frac{2}{\pi}}\cos t + \tilde{u}'_\lambda(t)\right)\sqrt{\frac{2}{\pi}}\sin t\ dt \\
&= \frac{1}{2}\int_0^\pi \cos\left(\lambda\sqrt{\frac{2}{\pi}}\cos t + \tilde{u}'_\lambda(t) + t\right) dt \\
&\quad - \frac{1}{2}\int_0^\pi \cos\left(\lambda\sqrt{\frac{2}{\pi}}\cos t + \tilde{u}'_\lambda(t) - t\right) dt.
\end{aligned}$$

The functions $\theta_\lambda(t) = \tilde{u}'_\lambda(t) + t$ and $\bar{\theta}_\lambda(t) = \tilde{u}'_\lambda(t) - t$, and their derivatives are continuous and bounded uniformly in $\lambda \in \mathbb{R}$. Since

$\phi'(t) = \sqrt{\frac{2}{\pi}} \cos t$ has a finite number of critical points on $[0, \pi]$ and $q(u) = \cos u$ is 2π-periodic with a mean value 0, then by Lemma 2 of [5],

$$\int_0^\pi \cos\left(\lambda\sqrt{\frac{2}{\pi}} \cos t + \theta_\lambda(t)\right) dt \to 0$$

$$\text{and} \qquad \int_0^\pi \cos\left(\lambda\sqrt{\frac{2}{\pi}} \cos t + \bar{\theta}_\lambda(t)\right) dt \to 0,$$

as $|\lambda| \to \infty$. Hence $\lim_{|\lambda|\to\infty} \tilde{\delta}(\lambda) = 0$, which by Theorem 3.2 implies that problem (3.2) has at least two solutions if $\bar{p} \in (a, b)\backslash\{0\}$. Again, following the proof in [4, p. 795], it can be shown that problem (3.2) has a (small) solution if $\|p\|_\infty$ is small enough, and consequently the interval (a, b) in this case is nonempty. We have found numerically that $(a, b) \approx (-0.36, 0.36)$. The curve $\tilde{\delta}(\lambda)$ is shown in Fig. 2.

Example 3.3 also serves as an example for the following more general result.

COROLLARY 3.1. *Assume that* $g\colon \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous with Lipschitz constant* $N < \frac{3}{2}$ *and* T*-periodic with the mean value* $\bar{g} = \frac{1}{T}\int_0^T g(u)\,du$. *Then the problem*

$$u'' + u + g(u') = \bar{p}\phi(t) + \tilde{p}(t), \qquad u(0) = u(\pi) = 0, \tag{3.3}$$

has at least two solutions, if $\bar{p} \in (a, b)\backslash\{2\sqrt{\frac{2}{\pi}}\bar{g}\}$.

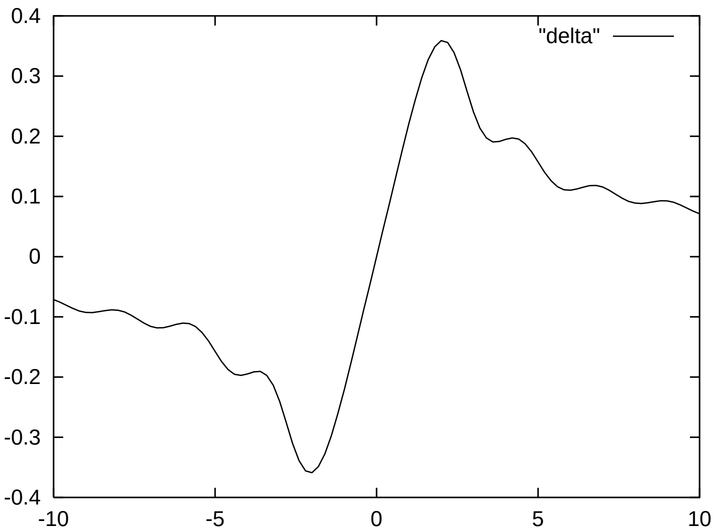

FIGURE 2

Proof. The function $g(t, u, v) = g(v)$ satisfies (g2) with the condition given in Remark 3.1, and following the proof of Lemma 2 in [5] we obtain

$$\lim_{|\lambda|\to\infty} \tilde{\delta}(\lambda) = \lim_{|\lambda|\to\infty} \int_0^\pi g\big(\lambda\phi'(t) + \tilde{u}'_\lambda(t)\big)\phi(t)\,dt$$
$$= \bar{g}\int_0^\pi \phi(t)\,dt = 2\sqrt{\frac{2}{\pi}}\,\bar{g}. \tag{3.4}$$

The conclusion follows then from Theorem 3.2. ∎

The result (3.4) is valid for any ϕ for which ϕ' has a finite number of critical points and $\phi' \in C^1[0, \pi]$. Thus we could derive similar results, e.g., for the boundary value problem,

$$u'' + u + g(u') = p(t), \qquad u'(0) = u'(\pi) = 0, \tag{3.5}$$

in which case the null space is spanned by $\cos t$. We will not go into details here.

COROLLARY 3.2. *Let $g(t, u, v) = g(u, v)$ satisfy (g2) and assume that the limits $g(\infty, -\infty) = \lim_{u\to\pm\infty, v\to-\infty} g(u, v)$ and $g(\infty, \infty) = \lim_{u\to\pm\infty, v\to\infty} g(u, v)$ exist and are finite. Then the problem*

$$u'' + u + g(u, u') = \bar{p}\phi(t) + \tilde{p}(t), \qquad u(0) = u(\pi) = 0,$$

has at least two solutions, if $\bar{p} \in (a, b)\backslash\{l\}$, where $l = g(\infty, -\infty) + g(\infty, \infty)$.

Proof. Following the idea of [3], without loss of generality we may and do suppose henceforth that $l = 0$, i.e., $g(\infty, -\infty) = -g(\infty, \infty)$. Indeed, by letting

$$h(u, v) = g(u, v) - \frac{g(\infty, -\infty) + g(\infty, \infty)}{2} = g(u, v) - \frac{l}{2}$$

and $q(t) = p(t) - l/2$ we obtain an equivalent problem,

$$u'' + u + h(u, u') = q(t), \qquad u(0) = u(\pi) = 0,$$

with h and q satisfying the hypotheses of g and p of the theorem and $h(\infty, -\infty) = -h(\infty, \infty)$.

By Theorem 3.2 it suffices to show that $\lim_{|\lambda|\to\infty} \tilde{\delta}(\lambda) = 0$. But this follows from the Lebesque convergence theorem and the condition $g(\infty, -\infty) = -g(\infty, \infty)$ because for any u_λ we have

$$\tilde{\delta}(\lambda) = \int_0^\pi g\left(\lambda\sqrt{\frac{2}{\pi}}\sin t + \tilde{u}_\lambda(t),\ \lambda\sqrt{\frac{2}{\pi}}\cos t + \tilde{u}'_\lambda(t)\right)\sqrt{\frac{2}{\pi}}\sin t\ dt,$$

where both $\tilde{u}_\lambda$ and $\tilde{u}'_\lambda$ are bounded on $[0, \pi]$ uniformly in $\lambda \in \mathbb{R}$.

EXAMPLE 3.4. Since $g(t, u, v) = \tan^{-1} v$ satisfies (g2) and has finite limits $\pm\frac{\pi}{2}$ at $\pm\infty$ we conclude, by Corollary 3.2, that the problem

$$u'' + u + \tan^{-1} u' = \bar{p}\phi(t) + \tilde{p}(t), \qquad u(0) = u(\pi) = 0, \tag{3.6}$$

has at least two solutions, if $\bar{p} \in (a, b)\setminus\{0\}$. In [4] it has been proved that if $\|p\|_\infty$ is small, then problem (3.6) has a solution. We have found the curve $\tilde{\delta}(\lambda)$ for $\tilde{p}(t) \equiv 0$ numerically (see Fig. 3), with $a \approx -0.3$ and $b \approx 0.3$.

If g depends on t, we replace the assumptions of finite limits of Corollary 3.2 by the assumption

$$\begin{aligned} \lim_{u\to\pm\infty, v\to\infty} g(t,u,v) = g(t,\infty,\infty) &= -g\left(t+\frac{\pi}{2},\infty,-\infty\right) \\ &= \lim_{u\to\pm\infty, v\to-\infty} g\left(t+\frac{\pi}{2},u,v\right) \\ &\qquad \text{uniformly in } t\in\left[0,\frac{\pi}{2}\right] \end{aligned} \tag{3.7}$$

and obtain in a similar way.

COROLLARY 3.3. *Let g satisfy* (g2) *and* (3.7). *Then problem* (1.3) *has at least two solutions, if $\bar{p} \in (a, b)\setminus\{l\}$.*

As we can see from the graphs of $\tilde{\delta}(\lambda)$, in Examples 3.1–3.4 $\tilde{\delta}(\lambda)$ is odd when $\tilde{p}(t) \equiv 0$. Indeed, we have

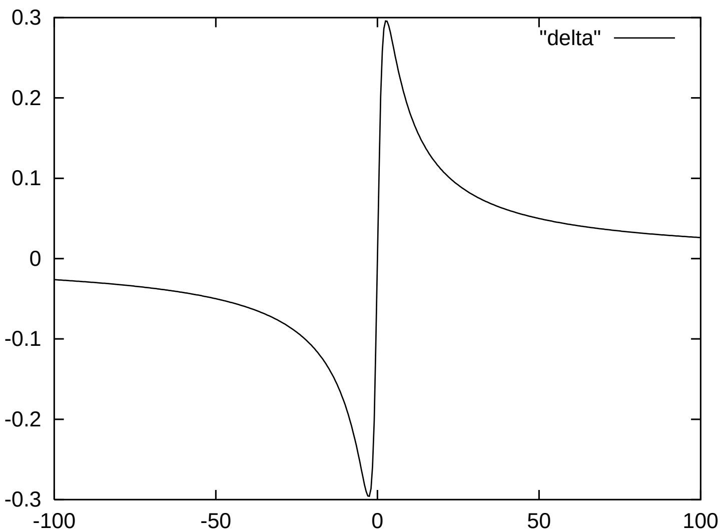

FIGURE 3

THEOREM 3.3. *If g satisfies (g2) and if $g(t, -u, -v) = -g(t, u, v)$, $0 \leq t \leq \pi$, $u, v \in \mathbb{R}$, and if $\tilde{p}(t) \equiv 0$, then $\tilde{\delta}(\lambda)$ is odd and the number of solutions of the boundary value problem*

$$u'' + u' + g(t, u, u') = 0, \qquad u(0) = u(\pi) = 0 \tag{3.8}$$

is odd.

Proof. For each $\lambda \in \mathbb{R}$ the integral equation (2.2) has a unique solution u_λ. From

$$\begin{aligned} -u_\lambda(t) &= -\lambda\phi(t) + \int_0^\pi k(t,s) g\big(s, u_\lambda(s), u'_\lambda(s)\big) ds \\ &= -\lambda\phi(t) - \int_0^\pi k(t,s) g\big(s, -u_\lambda(s), -u'_\lambda(s)\big)\, ds, \qquad 0 \leq t \leq \pi, \end{aligned}$$

it follows by the uniqueness that $u_{-\lambda} = -u_\lambda$, $\lambda \in \mathbb{R}$. Hence

$$\begin{aligned} \tilde{\delta}(-\lambda) &= \int_0^\pi g\big(t, u_{-\lambda}(t), u'_{-\lambda}(t)\big)\phi(t)\, dt \\ &= \int_0^\pi g\big(t, -u_\lambda(t), -u'_\lambda(t)\big)\phi(t)\, dt \\ &= -\int_0^\pi g\big(t, u_\lambda(t), u'_\lambda(t)\big)\phi(t)\, dt \\ &= -\tilde{\delta}(\lambda), \qquad \lambda \in \mathbb{R}. \end{aligned}$$

This proves that $\tilde{\delta}(\lambda)$ is odd and hence that the number of solutions of (3.8) is odd.

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