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Existence of Spectral Values for Irreducible C_0-Semigroups

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A C_0-semigroup (T_t) of positive linear operators on a Banach lattice E is called irreducible if it leaves no closed lattice ideals $\neq \{0\}$, E invariant. It is shown that all known (and some recently discovered) conditions on E and/or an irreducible cyclic semigroup (T^n) implying that $\sigma(T) \neq \{0\}$, have precise analogs for irreducible semigroups, to the effect that the generator A has nonvoid spectrum and, if (T_t) is eventually compact or A has compact resolvent, that the spectral bound $s(A)$ is an eigenvalue of A with positive eigenvector.

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Let E denote a (real or complex) Banach lattice (for notation, we follow the terminology used in [6]). A semigroup S of positive linear operators on E is called *irreducible* (resp. *band irreducible*) if no closed lattice ideals (resp. bands) other than $\{0\}$, E are invariant under every $T \in S$. A single (linear) operator $T \geqslant 0$ on E is called *(band)irreducible* if it leaves no closed ideal (band) $\neq \{0\}$, E invariant, i.e., iff the cyclic semigroup $(T^n)_{n \in \mathbb{N}}$ is irreducible (band irreducible). By contrast, for a C_0-semigroup $(T_t)_{t \geqslant 0}$ on E to be irreducible (band irreducible) it is sufficient but not necessary that some operator T_t $(t > 0)$ be irreducible (band irreducible). Since every band in a Banach lattice is a closed ideal, it is clear that band irreducibility is the weaker concept. Various examples of irreducible operators and semigroups can be found in [2] and [5].

Supposing now (T_t) to be an irreducible or band irreducible C_0-semigroup on E, we are concerned with conditions on E and/or (T_t) implying the spectral bound $s(A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$ of the generator A of (T_t) to be finite (or equivalently, the finite spectrum of A to be nonvoid). The corresponding question for cyclic semigroups $(T^n)_{n \in \mathbb{N}}$ has been treated in [5] (see also [6, V, Sect. 6]); however, the results obtained there have been considerably improved by de Pagter's recent theorem [4], to the

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effect that every compact irreducible operator $T \geqslant 0$ on E has strictly positive spectral radius $r(T)$ (and, consequently, a nonzero eigenvalue). This result has been extended (under mild additional conditions) to band irreducible operators [2, 7], where also a substantial extension of the so-called Theorem of Ando–Krieger (cf. [8, p. 621]) can be found. On the other hand, an example given in [5, Sect. 3] shows that an irreducible positive operator on $L^p[0, 1]$ $(1 \leqslant p < +\infty)$ can be topologically nilpotent (i.e., have $r(T) = 0$), while G. Greiner [3] has constructed an irreducible C_0-semigroup on the same spaces whose generator has void (finite) spectrum.

It will be shown below (Theorem B) that all conditions known to imply that a (band) irreducible operator T has strictly positive spectral radius, have precise analogs for C_0-semigroups. For the convenience of the reader and to put our results in perspective, we shall list these conditions in the following theorem. Let us recall also that if E is an order complete (= Dedekind complete) Banach lattice, an operator contained in the band $(E' \otimes E)^{\perp\perp}$ of $\mathscr{L}^r(E)$ is called an *abstract kernel operator*. (Here $\mathscr{L}^r(E)$ denotes the order complete Banach lattice of all order bounded operators on E, under the norm $T \to \|T\|_r := \| \, |T| \, \|$.) In many concrete cases (e.g., for many Köthe function spaces), the abstract kernel operators are just those defined by measurable kernels in the usual way (see [6, IV. 9]).

THEOREM A. *Let E denote a Banach lattice* ($\dim E > 1$), *and let* $T \geqslant 0$ *denote an irreducible operator on E. Then each of the following conditions on E and/or T implies that* $r(T) > 0$:

(i) $E = C_0(U)$, *U locally compact.*

(ii) *The positive cone* E_+ *contains an extreme ray.*

(iii) *Some power* T^p $(p \in \mathbb{N})$ *is compact.*

(iv) *E is order complete and some power* T^p $(p \in \mathbb{N})$ *is an abstract kernel operator.*

In particular, in case (iii) $r(T)$ *is an eigenvalue of T.*

Under conditions (i) and (ii), the assertion that $r(T) > 0$ was proved in [5] (see also [6, V. 6.1]) while the sufficiency of (iii) and (iv) is due to de Pagter [4, Theorem 3 and Proposition 5]).

Theorem A can be extended to cover band irreducible operators, as follows: While condition (i) seems to have no precise analog, (ii) carries over easily if T is order σ-continuous (= sequentially order continuous). Under the same additional assumption, conditions (iii) and (iv) carry over, as was proved recently by Grobler [2] (see also [7], where the same was shown under an additional condition on E). We summarize:

COROLLARY A. *If $T \geqslant 0$ is band irreducible and order σ-continuous, then each of conditions* (ii), (iii), (iv) *above implies that* $r(T) > 0$.

Let us note that in (iii) and (iv) it can be assumed without loss of generality that $p = 1$. In fact, it suffices to select some real number $\lambda > r(T)$ and consider the operator $S = T^p(\lambda - T)^{-p}$, which is irreducible (resp. band irreducible) whenever T is, and for which $r(S) > 0$ iff $r(T) > 0$ (spectral mapping theorem).

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We now assume $(T_t)_{t \geqslant 0}$ to be a strongly continuous semigroup (C_0-semigroup) of positive linear operators on an arbitrary Banach lattice E. As before, we denote by A the generator of (T_t) and by $s(A)$ the spectral bound $\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$. In case A has void (finite) spectrum we let, as usual, $s(A) = -\infty$.

THEOREM B. *Let E denote any Banach lattice* ($\dim E > 1$), *and let (T_t) denote an irreducible C_0-semigroup on E. Then each of the following conditions on E and/or (T_t) implies that $s(A) > -\infty$:*

(i) *$E = C_0(U)$, U locally compact.*

(ii) *The positive cone E_+ contains an extreme ray.*

(iii) *For some $t > 0$, T_t is compact.*

(iii′) *The generator A has compact resolvent.*

(iv) *E is order complete and for some $t > 0$, T_t is an abstract kernel operator.*

In particular, in cases (iii) *and* (iii′) *the generator A of (T_t) has the eigenvalue $s(A)$ with corresponding positive eigenvector.*

The corollary of Theorem A has the following analog:

COROLLARY B. *Let (T_t) be band irreducible and let T_t be order σ-continuous for each $t > 0$. Then each of conditions* (ii), (iii), (iv) *of Theorem* B *implies that $s(A) > -\infty$. Under condition* (iii), *$s(A)$ is an eigenvalue of A with positive eigenvector; the same is true under* (iii′) *provided that E has a separating order σ-continuous dual.*

The proofs of Theorem B and Corollary B are based on Theorem A, Corollary A, and the following four lemmata.

LEMMA 1. *If (T_t) is irreducible then $T_t \neq 0$ for all $t > 0$.*

Proof. We first note that by [1], for any $\lambda > s(A)$ and $t > 0$ the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ is given by

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T_s \, ds = R_t(\lambda, A) + e^{-\lambda t} T_t R(\lambda, A),$$

where $R_t(\lambda, A) := \int_0^t e^{-\lambda s} T_s \, ds$ and the integral over $\mathbb{R}_+$ is to be understood as an improper Riemann integral in the strong operator topology. Hence we obtain the basic inequality

$$T_t R(\lambda, A) \leqslant e^{\lambda t} R(\lambda, A) \qquad (\lambda > s(A),\ t \geqslant 0). \tag{1}$$

Now suppose that $T_t = 0$ for some $t > 0$, and let $\varepsilon := \inf\{t : T_t = 0\}$. Since $T_t \to \operatorname{id} E$ for $t \downarrow 0$, we have $\varepsilon > 0$. By definition of ε, $T_{\varepsilon/2} \neq 0$; we choose $z > 0$ so that $T_{\varepsilon/2} z =: y > 0$. Letting $w := R(\lambda, A)\, y$ for any fixed $\lambda > s(A)$, from (1) we obtain $T_t w \leqslant e^{\lambda t} w$ for all $t > 0$. Since $w > 0$, w must be a quasi-interior point of E_+. On the other hand, we have

$$T_{\varepsilon/2} w = T_{\varepsilon/2} R(\lambda, A)\, T_{\varepsilon/2} z = R(\lambda, A)\, T_\varepsilon z = 0,$$

since (T_t) commutes with $R(\lambda, A)$. It follows that $T_{\varepsilon/2} = 0$, and this contradicts the definition of ε. ∎

LEMMA 2. *Let (T_t) be irreducible on E, and for any fixed $\lambda > s(A)$ and $t \geqslant 0$ define $V_t := T_t R(\lambda, A) = T_t(\lambda - A)^{-1}$. Then:*

(a) *For any $x \in E$, $x > 0$, $V_t x$ is a quasi-interior point of E_+ (in particular, V_t is an irreducible operator).*

(b) *For all $t > 0$, $x > 0$ implies $T_t x > 0$.*

Proof. We first prove (a) for $t = 0$. From inequality (1) above we obtain, for any $x > 0$, the invariance of the closed ideal generated by $R(\lambda, A)\, x$ under the semigroup (T_t); hence $R(\lambda, A)\, x$ must be quasi-interior to E_+. Multiplying both sides of (1) by the operator T_s ($\geqslant 0$) and exchanging s and t, we obtain for all $s \geqslant 0$, $t \geqslant 0$,

$$T_s V_t \leqslant e^{\lambda s} V_t. \tag{2}$$

Now if $x > 0$ is given and if $y := V_t x$, we obtain $T_s y \leqslant e^{\lambda s} y$ for all $s \geqslant 0$. Therefore, the closed ideal generated by $V_t x$ is invariant under (T_t); irreducibility now implies that either $V_t x$ is quasi-interior to E_+, or else that $V_t x = 0$. But $0 = V_t x = T_t[R(\lambda, A)\, x]$ implies that $T_t = 0$ which contradicts Lemma 1. This proves (a); it is now also clear that (b) holds. ∎

Note. The proofs of Lemmata 1 and 2 are arranged so that under the general hypothesis of Corollary B, both lemmata can be seen to remain valid if "irreducible" is replaced by "band irreducible," "closed ideal" by "band," and "quasi-interior point of E_+" by "weak order unit of E."

LEMMA 3. *Let S, T be positive operators on a Banach lattice E. If T is weakly compact and order (σ-)continuous, then ST is order (σ-)continuous.*

Proof. Let D denote a non-void directed ($\geqslant$) subset of E. We have to show that $\inf D = 0$ implies $\inf ST(D) = 0$; in the case of order σ-continuity of T we assume D to be countable. Now we have $\inf T(D) = 0$ by order ($\sigma-$)continuity; weak compactness of T implies that $y = \lim T(D)$ exists weakly (hence in norm, [6, II.5.9, Corollary]). But $y = \inf T(D)$, since E_+ is closed, and thus $\lim T(D) = 0$. This implies $\lim ST(D) = 0$, which in turn implies $0 = \inf ST(D)$. ▮

LEMMA 4. *Let (T_t) be a semigroup such that each T_t $(t > 0)$ is order (σ-)continuous, and suppose that E has a separating order (σ-)continuous dual. Then for each $\lambda > s(A)$, $R(\lambda, A) = (\lambda - A)^{-1}$ is order (σ-)continuous.*

Proof. It will be enough to prove the assertion concerning order σ-continuity. Let (x_n) be a decreasing sequence in E with $\inf_n x_n = 0$, and let $0 \leqslant x' \in E'_0$ be arbitrary (E'_0 = order σ-continuous dual of E). For $\lambda > s(A)$ we have, by [1],

$$\langle R(\lambda, A)\, x_n, x' \rangle = \int_0^\infty e^{-\lambda s} \langle T_s x_n, x' \rangle \, ds.$$

By hypothesis, the integrand converges to 0 pointwise on $\mathbb{R}_+$; the dominated convergence theorem implies $\lim_n \langle R(\lambda, A)\, x_n, x' \rangle = 0$. Thus if $v \in E_+$ satisfies $v \leqslant R(\lambda, A)\, x_n$ for all n, it follows that $\langle v, x' \rangle = 0$ for all $x' \in (E'_0)_+$; since E'_0 separates E, it follows that $v = 0$ and hence that $\inf_n R(\lambda, A)\, x_n = 0$. ▮

Proof of Theorem B. For the proof of Theorem B we recall that for any λ in the resolvent set of A (in particular, for $\lambda > s(A)$) the mapping $\alpha \to 1/(\lambda - \alpha)$ defines a bijection of $\sigma(A)$ onto $\sigma(R(\lambda, A))$, where it is understood that the point ∞ of the Riemann sphere is considered an element of $\sigma(A)$ in case A is unbounded. (It is natural in this context, though not necessary for positive semigroups, to consider complex Banach lattices; cf. [6, II.11].)

In particular, then, $s(A) = -\infty$ means $\sigma(A) = \{\infty\}$ or equivalently, $\sigma(R(\lambda, A)) = \{0\}$. Thus to show that $s(A) > -\infty$, it suffices to prove that $R(\lambda, A)$ is not topologically nilpotent. Using Theorem A and the well-known fact that irreducibility of (T_t) is equivalent to irreducibility of $R(\lambda, A)$ for any $\lambda > s(A)$, we consider the four conditions of Theorem B in order.

(i) and (ii) Since $R(\lambda, A)$ is ($\geqslant 0$ and) irreducible for $\lambda > s(A)$, Theorem A(i), (ii) imply that $r(R(\lambda, A)) > 0$.

(iii) and (iii′) If T_t or $R(\lambda, A)$ is compact, then the operator $V_t := T_t R(\lambda, A)$ is compact, and irreducible by Lemma 2. Theorem A (iii) implies that $r(V_t) > 0$. On the other hand, from Formula (1) (proof of Lemma 1) we obtain $V_t \leqslant e^{\lambda t} R(\lambda, A)$. Since the spectral radius is an isotone function on the cone of positive operators in $\mathscr{L}(E)$, it follows that $r(R(\lambda, A)) \geqslant e^{-\lambda t} r(V_t) > 0$.

(iv) If T_t is an abstract kernel operator then so is V_t. In fact, denote by I the lattice ideal of $\mathscr{L}^r(E)$ generated by $E' \otimes E$; since $(E' \otimes E) \circ R(\lambda, A) \subset E' \otimes E$ and since $R(\lambda, A) \geqslant 0$, it follows that $I \circ R(\lambda, A) \subset I$. Now $T_t \in (E' \otimes E)^{\perp\perp}$ means that $T_t = \sup_\alpha T_\alpha$ for a directed ($\leqslant$) family of positive operators $T_\alpha \in I$; because of $T_t R(\lambda, A) = \sup_\alpha T_\alpha R(\lambda, A)$ we obtain $V_t \in (E' \otimes E)^{\perp\perp}$. Since by Lemma 2, V_t is irreducible, we conclude from Theorem A(iv) that $r(V_t) > 0$ and hence, as above, that $r(R(\lambda, A)) > 0$.

To prove the final assertion of Theorem B, consider condition (iii) first. We use the well-known identity

$$(1 - e^{-\lambda t} T_t)\, x = (\lambda - A) \int_0^t e^{-\lambda s} T_s x \, ds,$$

which is valid for all $x \in E$, $\lambda \in \mathbb{C}$, and the equality $r(T_t) = e^{\omega t}$ where ω denotes the growth bound of (T_t). Let T_t be compact; since $\omega \geqslant s(A) > -\infty$, $e^{\omega t}$ is an eigenvalue with positive eigenvector x_0 of T_t. Now $y := \int_0^t e^{-\omega s} T_s x_0 \, ds > 0$ by Lemma 2(b), and $(\omega - A)\, y = 0$. Therefore, $\omega = s(A)$ has the claimed property.

Under condition (iii′), since $R(\lambda_0, A)$ is compact (for any $\lambda_0 > s(A)$), the spectral radius $r > 0$ of $R(\lambda_0, A)$ is an eigenvalue with eigenvector $x_0 > 0$. A simple calculation shows that x_0 is an eigenvector of A for the eigenvalue $\lambda_0 - (1/r)$, which necessarily equals $s(A)$ (cf. the remark at the beginning of the proof). ∎

Proof of Corollary B. The proof of this corollary is basically the same as for Theorem B; first of all we note that under the general hypothesis of Corollary B, each of the operators $V_t = T_t R(\lambda, A)$ is band irreducible (Lemma 2 and subsequent note). However, some additional arguments are necessary because, in order to employ Corollary A, we have to show that V_t is order σ-continuous, or at least that V_t dominates a band irreducible, order σ-continuous positive operator. It will then be sufficient to prove that $r(V_t) > 0$ in each case; as before we take up conditions (ii)–(iv) in order.

(ii) Let u generate an extreme ray of E_+ (i.e., let $u > 0$ be an atom of E). By Lemma 2 and the subsequent Note, $V_t u$ is a weak order unit of E.

Hence $V_t u \wedge u = \varepsilon u$ for some $\varepsilon > 0$ which implies $V_t u \geqslant \varepsilon u$. By induction we obtain $V_t^n u \geqslant \varepsilon^n u$ for all $n \in \mathbb{N}$, whence it follows that $\|V_t^n\|^{1/n} \geqslant \varepsilon$ and $r(V_t) \geqslant \varepsilon$.

(iii) If T_t is compact then V_t is compact, and band irreducible by Lemma 2 and the subsequent Note. Lemma 3 assures that $V_t = R(\lambda, A)\, T_t$ is order σ-continuous. The fact that $s(A)$ is an eigenvalue of A with positive eigenvector, is proved exactly as for Theorem B.

(iii′) Since E_0' is required to separate E, Lemma 4 assures that $R(\lambda, A)$ is order σ-continuous. The remaining part of the proof is the same as for Theorem B.

(iv) Suppose that T_t is an abstract kernel operator and recall that, by order completeness of E, the lattice $\mathscr{L}^r(E)$ (order bounded operators) is order complete. The assumption $0 \leqslant T_t \in (E_0' \otimes E)^{\perp\perp}$ (T_t being order σ-continuous) means that $T_t = \sup_\alpha T_\alpha$ where (T_α) is a family of positive operators contained in the ideal J generated by $E_0' \otimes E$. Since $E_0' \otimes E$ is invariant under *left* multiplication by $R(\lambda, A)$, it follows that $R(\lambda, A)\, T_\alpha \in J$ for each α. Now, since $R(\lambda, A)\, T_\alpha \leqslant R(\lambda, A)\, T_t = V_t$, $B := \sup_\alpha R(\lambda, T)\, T_\alpha$ exists in $\mathscr{L}^r(E)$ and $B \in (E_0' \otimes E)^{\perp\perp}$. By its definition, B is order σ-continuous; we show that B is band irreducible. In fact, if $x > 0$ then $T_t x > 0$ (Lemma 2 and subsequent note) and hence there exists α such that $T_\alpha x > 0$. Thus by Lemma 2 again, $R(\lambda, A)\, T_\alpha x$ is a weak order unit of E. Therefore, Bx is a weak order unit of E; in particular, B is band irreducible. Hence by Corollary A we have $r(B) > 0$; this implies $r(V_t) \geqslant r(B) > 0$. ∎

References

1. G. Greiner, J. Voigt, and M. Wolff, On the spectral bound of the generator of semigroups of positive operators, *J. Operator Theory* **5** (1981), 245–256.
2. J. J. Grobler, Band irreducible operators. *Indag. Math.* **48** (1986), 405–409.
3. R. Nagel (Ed.), One-Parameter Semi-Groups of Positive Operators," Lecture Notes in Math., No. 1184, Springer-Verlag, Berlin/Heidelberg/New York, 1986.
4. B. de Pagter, Irreducible compact operators, *Math. Z.* **192** (1986), 149–153.
5. H. H. Schaefer, Topologische Nilpotenz irreduzibler Operatoren, *Math. Z.* **117** (1970), 135–140.
6. H. H. Schaefer, "Banach Lattices and Positive Operators," Grundl. Math. Wiss., Vol. 215, Springer-Verlag, Berlin/Heidelberg/New York, 1974.
7. H. H. Schaefer, On theorems of de Pagter and Ando Krieger. *Math. Z.* **192** (1986), 155–157.
8. A. C. Zaanen, "Riesz Spaces II," North-Holland, Amsterdam/New York/Oxford, 1983.