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K_2 OF VON NEUMANN REGULAR RINGS

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0. Introduction

Matsumoto's theorem [5, 11.1, p. 93] gives a presentation of K_2 of a field in terms of generators corresponding to pairs of non-zero elements of the field and three relations which are satisfied by these generators (the relations (S1), (S1′) and (S4) given below). Our purpose here is to give a similar elementary presentation for K_2 of a commutative ring which is regular in the sense of von Neumann. This will be accomplished following the program begun in [3]: We exhibit functors which

(1) are of finite type in the sense of [3, p. 489], that is, commute with finite direct products and arbitrary direct limits;

(2) admit a natural transformation to K_2 which is an isomorphism when applied to fields.

Then by [3, Proposition 2, p. 490] the natural transformation is an isomorphism when applied to commutative rings which are regular in the sense of von Neumann. (Readers of [3] should be warned that the functor Us introduced there does not satisfy (1) above, and that Corollary 2.8 below should be substituted for [3, Corollary 3(c), p. 490]; see also [4].)

In Section 1 we introduce a functor based on the Matsumoto presentation of K_2, but modified by additional relations so that it commutes with finite products, and we show that it satisfies (1) and (2). In fact, this new functor is universal with respect to transformations from the Matsumoto functor to a functor of finite type.

In Section 2 we introduce a functor based on the identities (H1)–(H5) of [6, p. 283], which hold for elements of K_2 arising from pairs of elements a, b of a commutative ring such that $1 + ab$ is a unit, and show that it also satisfies (1) and (2).

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We thus arrive at two new descriptions of K_2 of a commutative von Neumann regular ring; these are summarized in Corollary 2.8, with the presentations of the functors being described in Definitions 1.1 and 2.1. It should be noted that the functor described in Definition 2.1 agrees with K_2 in a number of other cases [7].

All the rings in this paper are commutative and usually denoted by R. We want to acknowledge a useful letter from W. van der Kallen, H. Maazen and J. Stienstra pointing out the gap in [3] which led to this investigation.

1.

Definition 1.1. Let $E(R)$ denote the abelian group generated by pairs $\{a, b\}$, where a and b are units of R, and subject to the relations

(S1) $\{ab, c\} = \{a, c\}\{b, c\}$,

(S1′) $\{a, bc\} = \{a, b\}\{a, c\}$,

where a, b and c are units of R, and

(E2) $\{(a-1)e+1, (1-b)e+b\} = 1$,

(E3) $\{(a-1)e+1, (b-1)e+1\} = 1$ if $ae+be = e$,

where a and b are units of R, and e is an idempotent. The elements

$$(a-1)e+1, \qquad (1-b)e+b, \qquad (a-1)e+1, \qquad (b-1)e+1$$

are all units of R (for example, $((a-1)e+1)^{-1} = (a^{-1}-1)e+1$, so the relations (E2) and (E3) are well-defined. We note further that (E2) is obviously satisfied if $e = 0$ or $e = 1$, and (E3) if $e = 0$.

Replacing e by $1-e$ in (E2) gives the identity

(E2′) $\{(1-a)e+a, (b-1)e+1\} = 1$.

If $e = 1$, (E3) gives the identity

(S4) $\{a, b\} = 1$ if $a+b = 1$.

As in [3, p. 490], we will let $\mathrm{Us}(R)$ denote the abelian group generated by pairs $\{a, b\}$, where a and b are units of R subjected to the relations (S1), (S1′) and (S4). It is clear that there is a surjective homomorphism $q : \mathrm{Us}(R) \to E(R)$ which is functorial in R. Obviously $E(R)$ is a functor. We want to show that it is of finite type. Our first task is to show that $E(R)$ commutes with finite products.

Proposition 1.2. *Let R_1 and R_2 be commutative rings. Then the induced map*

$$\psi : E(R_1 \times R_2) \to E(R_1) \times E(R_2)$$

defined by

$$\psi(\{(a, b), (c, d)\}) = (\{a, c\}, \{b, d\})$$

is an isomorphism.

Proof. Let $R = R_1 \times R_2$. We define maps

$$\phi_i : E(R_i) \to E(R), \quad i = 1, 2,$$

by

$$\phi_1(\{a, b\}) = \{(a, 1), (b, 1)\},$$

$$\phi_2(\{a, b\}) = \{(1, a), (1, b)\}.$$

It is straightforward to verify that these maps are well-defined. For example, for the relation (E2) and ϕ_1 we have

$$\phi_1(\{(a-1)e + 1, (1-b)e + b\} =$$

$$= \{((a, 1) - 1)(e, 0) + 1, (1 - (b, 1))(e, 0) + (b, 1)\},$$

and the right-hand side is trivial in $E(R)$ by (E2) since $(e, 0)$ is an idempotent of R. Combining ϕ_1 and ϕ_2 gives a map

$$\phi : E(R_1) \times E(R_2) \to E(R)$$

defined by

$$\phi(\{a, b\}, \{c, d\}) = \{(a, 1), (b, 1)\} \{(1, c), (1, d)\}.$$

We then have that $\psi\phi$ is the identity and hence ϕ is one-to-one.

Next we show that ϕ is onto, and thus an isomorphism with ψ as its inverse, which will complete the proof. In $E(R)$, we have

$$(*) \quad \{(a,b), (c,d)\} = \{(a, 1), (c, 1)\}\{(a, 1), (1, d)\}\{(1, b), (c, 1)\}\{(1, b), (1, d)\}.$$

We claim that the middle two factors in $(*)$ are trivial. For, in R let $x = (a, 1)$,

$y = (1,d)$ and $e = (1,0)$. Then e is idempotent and

$$(x-1)e+1 = x, \qquad (1-y)e+y = y\ ,$$

so by (E2), $\{x,y\} = 1$. Similarly, by letting $z = (1,a)$ and $w = (b, 1)$, with e as above, by (E2′) we also find $\{z, w\} = 1$. Thus (∗) reduces to

$$\{(a,b),(c,d)\} = \phi(\{a,b\},\ \{c,d\})\ ,$$

and hence ϕ is onto. □

Corollary 1.3. *E is a functor of finite type.*

Proof. E obviously commutes with arbitrary direct limits, and by induction and Theorem 1.2 it also commutes with finite products. □

The map $\mathrm{Us} \to K_2$ given by symbols will factor through E. In fact, we will show that any natural transformation from Us to a functor of finite type factors through E.

By a *partition* of R we mean a set $e_1, ..., e_k$ of pairwise orthogonal non-zero idempotents of R summing to 1. If $e_1, ..., e_k$ and $f_1, ..., f_l$ are both partitions of R, the second is a *refinement* if for each i, $f_i e_j$ is non-zero for exactly one j, which we denote by $j(i)$, and we have a ring homomorphism $Re_{j(i)} \to Rf_i$ given by multiplication by f_i.

If F is a functor from the category of commutative rings to the category of abelian groups, we define a functor $\hat{F}$ as follows: $\hat{F}(R)$ is the direct limit, over the partitions $e_1, ..., e_k$ of R ordered by refinement, of $F(Re_1) \times ... \times F(Re_k)$. If h: $F \to G$ is a natural transformation, we have an induced natural transformation $\hat{h}$: $\hat{F} \to \hat{G}$.

It is easy to see, but not necessary for our purposes here, that $\hat{F}(R)$ is the group of global sections of the sheaf F associated to F on the Boolean spectrum $X(R)$ of R constructed in [3, Proposition 1, p. 489].

It is clear that if F commutes with finite products, then $F = \hat{F}$.

Lemma 1.4. *The map $\hat{q}$: $\hat{\mathrm{Us}} \to \hat{E} = E$ induced from the surjection q : $\mathrm{Us} \to E$ is an isomorphism.*

Proof. We already know that for any R the composite $\mathrm{Us}(R) \to \hat{\mathrm{Us}}(R) \to E(R)$ is onto and hence the second factor is onto. Now let e be any idempotent of R, and consider the relations (E2) and (E3) for this e. By the remarks following Definition 1.1, these relations hold when $e = 0$ or 1, either trivially or as a consequence of (S4). Thus (E2) and (E3) are satisfied in $\mathrm{Us}(Re) \times \mathrm{Us}(R(1-e))$ and hence in $\hat{\mathrm{Us}}(R)$. Thus there exists a map of $E(R)$ to $\hat{\mathrm{Us}}(R)$ inverse to $\hat{q}$. □

Proposition 1.5. *Let F be a functor of finite type and let h : Us $\to F$ be a natural transformation. Then there is a natural transformation h': $E \to F$ such that*

$$\begin{array}{ccc} \mathrm{Us} & \xrightarrow{\ h\ } & F \\ & \searrow \quad \nearrow_{h'} & \\ & E & \end{array}$$

commutes.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Us} & \xrightarrow{\ h\ } & F \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Us}} & \xrightarrow{\ \hat{h}\ } & \hat{F} \end{array}$$

By Lemma 1.4, $\widehat{\mathrm{Us}} = E$, and since F is of finite type, we have $\hat{F} = F$, and the result follows. □

Corollary 1.6. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Us} & \longrightarrow & K_2 \\ & \searrow \quad \nearrow & \\ & E & \end{array}$$

(As remarked above, Corollary 1.6 in fact has an easy direct proof.)

Theorem 1.7. *Let R be a regular ring in the sense of von Neumann. Then*:
(1) Us(R) $\to K_2(R)$ *is surjective*;
(2) $E(R) \to K_2(R)$ *is an isomorphism.*

Proof. The first statement follows from the second since Us(R) $\to E(R)$ is onto.

For the second, we consider the case where R is a field. Then Us(R) = $E(R)$ and by [5, Theorem 11.1, p. 93], Us(R) = $K_2(R)$. Thus the natural transformation $E \to K_2$ of Corollary 1.6 is an isomorphism for fields. Since K_2 is of finite type and, by Corollary 1.3, E is of finite type, the result follows from [3, Proposition 2, p. 490]. □

Part (1) of Theorem 1.7 was proven by other techniques in [1, p. 6].

2.

In this section we construct another functor of finite type which also gives a presentation of K_2 of a von Neumann regular ring.

Definition 2.1. Let $D(R)$ denote the abelian group generated by pairs $\langle a, b\rangle$, where a and b are elements of R with $1 + ab$ a unit, subject to the relations

(D1) $\quad \langle a, b\rangle \langle -b, -a\rangle = 1,$

where $1 + ab$ is a unit of R,

(D2) $\quad \langle a, b + c\rangle = \langle a, b\rangle \langle a, c(1 + ab)^{-1}\rangle$

where $1 + ab$ and $1 + a(b + c)$ are units of R,

(D3) $\quad \langle a, bc\rangle\langle b, ca\rangle\langle c, ab\rangle = 1,$

where $1 + abc$ is a unit of R.

If in (D3) we replace b by a and c by -1, we get

$$\langle a, -1\rangle\langle 1, -a\rangle\langle -1, a\rangle = 1,$$

with $1 - a$ a unit of R; by (D1) the product of the first two factors is trivial, and hence we have the relation

(D1′) $\quad \langle -1, a\rangle = 1,$

with $1 - a$ a unit of R.

If in (D3) we replace b by $-b$ and c by -1, we get

$$\langle a, b\rangle\langle -b, -a\rangle\langle -1, -ab\rangle = 1,$$

with $1 + ab$ a unit of R, and so (D1) and (D1′) are equivalent in the presence of (D3).

If in (D2) we replace a, b, c by their negatives, we have

$$\langle -a, -b-c\rangle = \langle -a, -b\rangle\langle -a, -c(1 + ab)^{-1}\rangle$$

with $1 + ab$ and $1 + a(b + c)$ units of R. Taking inverses and applying (D1) yields

$$\langle b+c, a\rangle = \langle b, a\rangle\langle c(1+ab)^{-1}, a\rangle .$$

Renaming, we have

$$\text{(D2}')\qquad \langle a+b, c\rangle = \langle a, c\rangle\langle b(1+ac)^{-1}, c\rangle,$$

with $1+ac$ and $1+c(a+b)$ units of R.

Obviously $D(R)$ is a functor. We want to show that it is of finite type. Our first task is to show that $D(R)$ commutes with finite products.

Proposition 2.2. *Let R_1 and R_2 be commutative rings. Then the induced map*

$$\psi : D(R_1 \times R_2) \to D(R_1) \times D(R_2)$$

defined by

$$\psi(\langle(a,b),(c,d)\rangle) = (\langle a,c\rangle\langle b,d\rangle)$$

is an isomorphism.

Proof. Let $R_1 \times R_2$. We define maps

$$\phi_i : D(R_i) \to D(R), \quad i = 1,2,$$

by

$$\phi_1(\langle a,b\rangle) = \langle(a,0),(b,0)\rangle, \qquad \phi_2(\langle a,b\rangle) = \langle(0,a),(0,b)\rangle.$$

It is easy to check that the ϕ_i are well-defined. Combining ϕ_1 and ϕ_2 gives a map

$$\phi : D(R_1) \times D(R_2) \to D(R)$$

defined by

$$\phi(\langle a, b\rangle, \langle c, d\rangle) = \langle(a, 0), (c, 0)\rangle\langle(0, b), (0, d)\rangle.$$

It is clear then that $\psi\phi$ is the identity, and hence ϕ is one-to-one. We will now show that ϕ is onto and thus an isomorphism, which will complete the proof. In $D(R)$ we have

$$\langle(a, b), (c, d)\rangle = \langle(a, b), (c, 0) + (0, d)\rangle = \langle(a, b), (c, 0)\rangle\langle(a, b), (0, d)\rangle.$$

The second equality follows from (D2) since $1 + (a, b)(c, 0) = (1 + ac, 1)$ is a unit of R. By (D2′),

$$\langle(a,b),\ (c,0)\rangle = \langle(a,0)(c,0)\rangle\langle(0,b),(c,0)\rangle\,,$$
$$\langle(a,b),(0,d)\rangle = \langle(a,0),(0,d)\rangle\langle(0,b),(0,d)\rangle.$$

Now apply (D3) to $(a,0),(1,0),(0,b)$; we have that

$$\langle(a,0),0\rangle\langle(1,0),0\rangle\langle(0,b),(a,0)\rangle = 1.$$

We claim that the first two factors are trivial. For by (D1) with $a = b = 0$ we have $\langle 0,0\rangle^2 = 1$, so that by (D3) with $b = c = 0$,

$$\langle a,0\rangle\langle 0,0\rangle\langle 0,0\rangle = 1$$

and hence $\langle a,0\rangle = 1$. We conclude that

$$\langle(0,b),(a,0)\rangle = 1.$$

In a similar fashion,

$$\langle(a,0),(0,b)\rangle = 1$$

Combining these calculations, we have that

$$\langle(a,b),(c,d)\rangle = \langle(a,0),(c,0)\rangle\langle(0,b),(0,d)\rangle = \phi(\langle a,c\rangle,\ \langle b,d\rangle),$$

and hence ϕ is onto. □

Corollary 2.3. *D is a functor of finite type.*

Proof. D obviously commutes with arbitrary direct limits, and by induction and Proposition 2.2 it also commutes with finite products. □

Next we will define a natural transformation $\mathrm{Us} \to D$.

Lemma 2.4. *The function p: $\mathrm{Us} \to D$ given by*

$$p(\{u,v\}) = \langle v^{-1}(u-1),\ v\rangle$$

is a well-defined natural transformation.

Proof. We need to verify that the defining relations (S1), (S1′) and (S4) of Us are satisfied.

For (S1), we need to show that if u, v, w are units of R, then

$$\langle w^{-1}(uv-1), w\rangle = \langle w^{-1}(u-1), w\rangle\langle w^{-1}(v-1), w\rangle;$$

this follows from (D2′) with

$$a = w^{-1}(u-1), \qquad b = w^{-1}u(v-1), \qquad c = w.$$

For (S1′), we need to show that if u, v, w are units of R, then

$$\langle v^{-1}w^{-1}(u-1), vw\rangle = \langle v^{-1}(u-1), v\rangle\langle w^{-1}(u-1), w\rangle\,.$$

First note that if we replace b and c by $-b$ and $-c$ in (D3), we have the relation

$$\langle a, bc\rangle\langle -b, -ca\rangle\langle -c, -ab\rangle = 1\,.$$

Then by (D1) it follows that

$$\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle.$$

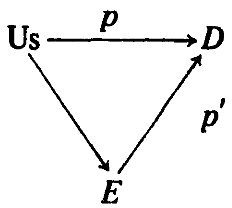
Now let $a = v^{-1}w^{-1}(u-1)$, $b = v$, $c = w$ in this last relation; this gives the desired result.

For (S4) we need to show that if u and $1-u$ are units of R, then

$$\langle u^{-1}((1-u)-1), u\rangle = 1,$$

i.e., that $\langle -1, u\rangle = 1$. But this is the relation (D1′). □

Corollary 2.5. *There is a natural transformation p': $E \to D$ such that the diagram*



commutes.

Proof. Apply Proposition 1.5 to the transformation p of Lemma 2.4. □

We want to show that p' is an isomorphism for a von Neumann regular ring. Since D and E are both of finite type, it will be sufficient to do this for fields.

Lemma 2.6. *Let R be a field. Then the map $p(R)$: $\mathrm{Us}(R) \to D(R)$ is an isomorphism.*

Proof. We construct an inverse to $p(R)$. Let $f: D(R) \to \mathrm{Us}(R)$ be defined by

$$f(\langle a,b\rangle) = \{1+ab, b\} \quad \text{if } b \neq 0,$$
$$f(\langle a,0\rangle) = 1.$$

We need to verify that the defining relations (D1), (D2) and (D3) of D are satisfied.

First note that if a or b is 0, $f(\langle a, b\rangle) = 1$ since $f(\langle a, 0\rangle) = 1$ by definition and $f(\langle 0,b\rangle) = \{1,b\} = 1$.

For (D1), we need to check that the relation $\langle a, b\rangle\langle -b, -a\rangle = 1$ is preserved. If a or b is zero, this is true. If neither are zero, the left-hand side is carried to

$$\{1+ab, b\}\{1+ab, -a\} = \{1+ab, -ab\} = 1.$$

For (D2), we need to check that the relation

$$\langle a, b+c\rangle = \langle a, b\rangle\langle a, c(1+ab)^{-1}\rangle$$

is preserved. Again, this is clear if any one of a, b or c is zero. Thus assume all are non-zero. Then the right-hand side goes to

$$\{1+ab, b\}\{(1+a(b+c))(1+ab)^{-1}, c(1+ab)^{-1}\}.$$

If $b+c=0$, the left-hand side goes to zero and the right-hand side becomes

$$\begin{aligned}\{1+ab, b\}\{(1+ab)^{-1}, -b(1+ab)^{-1}\} &= \{(1+ab)^{-1}, b^{-1}\}\{(1+ab)^{-1}, -b(1+ab)^{-1}\}\\ &= \{(1+ab)^{-1}, -(1+ab)^{-1}\},\end{aligned}$$

which is trivial since $\{u, -u\} = 1$ in $\mathrm{Us}(R)$ for any field. If $b+c \neq 0$, the left-hand side goes to $\{1+a(b+c), b+c\}$, and we need to prove that

$$\{1+a(b+c), b+c\} = \{1+ab, b\}\{1+a(b+c)(1+ab)^{-1}, c(1+ab)^{-1}\}.$$

But by skew symmetry this is just the (1,2)-identity of [1, p. 21] (cf. [2, Proposition 1.6(b)]) which holds in $\mathrm{Us}(R)$ for any field R.

For (D3) we need to check that the relation

$$\langle a, bc\rangle\langle b, ca\rangle\langle c, ab\rangle = 1$$

is preserved by f. This is clear if any one of a, b or c is zero. If all are non-zero, the left-hand side of the relation is carried to

$$\begin{aligned}\{1+abc, bc\}\{1+abc, ac\}\{1+abc, ab\} &= \{1+abc, (abc)^2\}\\ &= \{1+abc, -abc\}^2 = 1.\end{aligned}$$

It then follows that f is well-defined. Now

$$pf(\langle a,b\rangle) = \langle b^{-1}((1+ab)-1), b\rangle = \langle a,b\rangle \quad \text{for } b \neq 0,$$

and

$$pf(\langle a,0\rangle) = 1 = \langle a,0\rangle;$$

while

$$fp(\{u,v\}) = \{1+v^{-1}(u-1)v, v\} = \{u,v\}.$$

Thus f is indeed inverse to p, and the result follows. □

Theorem 2.7. *Let R be a regular ring in the sense of von Neumann. Then*:
(1) $\mathrm{Us}(R) \to D(R)$ *is surjective*;
(2) $E(R) \to D(R)$ *is an isomorphism.*

Proof. The first statement follows from the second since $\mathrm{Us}(R) \to E(R)$ is onto. For the second, by Lemma 2.6, $p' : E \to D$ is an isomorphism for fields, and hence since E is of finite type by Corollary 1.3 and D is of finite type by Corollary 2.3, it follows by [3, Proposition 2, p. 490] that p' is an isomorphism when R is von Neumann regular.

Corollary 2.8. *Let R be a regular ring in the sense of von Neumann. Then $D(R)$, $E(R)$ and $K_2(R)$ are all isomorphic.*

More generally, it would be of interest to know exactly when some of the functors studied here agree. In particular, we would like to raise the following two questions:
(1) For which rings R is the natural map $E(R) \to D(R)$ an isomorphism?
(2) For which rings R is the natural map $\mathrm{Us}(R) \to E(R)$ an isomorphism?

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