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Generation of Symplectic Groups by Transvections over Local Rings with at Least 3 Residue Classes

HIROYUKI ISHIBASHI

Department of Mathematics, Josai University, 1-1, Keyakidai, Sakado, Saitama, Japan

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1. INTRODUCTION

Let R be a commutative local ring with identity 1 and a unique maximal ideal A. Let V denote a free module of rank n over R with an alternating bilinear form f: $V \times V \to R$. We assume $R/A \neq F_2$. However, V may have rad $V \neq \varnothing$.

A linear automorphism σ of V is called an *isometry* on V if σ satisfies $f(\sigma x, \sigma y) = f(x, y)$ for all x, y in V. The set of all isometries is a subgroup of the automorphism group of V which is called a symplectic group, denoted by $\mathrm{Sp}(V)$. For a submodule U of V we define $U^\perp = \{v \in V \mid f(v, U) = 0\}$ and rad $U = U \cap U^\perp$. Then $U^\perp$ is called the orthogonal complement of U, and rad U the radical of U. For submodules U and W of V, by $U \perp W$ we mean $U \oplus W$ with $f(U, W) = 0$. We define

$$\mathrm{Sp}_0(V) = \{\varphi \in \mathrm{Sp}(V) \mid \varphi = 1 \text{ on rad } V\}.$$

For any element a in R and any element v in V we can define an isometry $T_{a,v}$ called a *transvection* with *coefficient* a and axis $v^\perp$ by the formula

$$T_{a,v} z = z + f(z, v) \cdot a \cdot v, \qquad z \in V.$$

For a subset S of V, $T_{R,S} = \{T_{a,s} \mid a \in R, s \in S\}$, and $T_R(S)$ is the subgroup of $\mathrm{Sp}(V)$ generated by $T_{R,S}$.

Let S be a subset of V. If for x, y in S there exists a sequence $s_1, ..., s_r$ in S such that $s_1 = x$, $s_r = y$, and $f(s_i, s_{i+1}) = \text{unit}$ for $i = 1, ..., r-1$, then we

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say that x is connected to y in S. If any two vectors x, y in S are connected in S, then S is said to be connected.

In 1983, Brown and Humphries [1] showed the following:

> For a finite subset S of $V\backslash\text{rad } V$, $T_R(S)=\text{Sp}_0(V)$ if and only if S spans V and S is connected

under the assumption that R is a field and $R \neq F_2$. In this paper, we shall extend this theorem to a local ring R with $R/A \neq F_2$ in Theorem 3.3.

However, in our proof we do not apply their forest technique which reduces connected sets to forests. We prove the theorems rather algebraically. As a result the proof will be shorter, and as we see later the assumption that S is finite can be dropped (the author knew recently that they had also dropped that assumption on finiteness for S).

As an easy consequence of Theorem 3.3 we shall show in Theorem 3.4 that we can choose n vectors $u, v, x_1, ..., x_{n-2}$ in S such that $T_{R,u}$, $T_{1,v}$, $T_{1,x_1}, ..., T_{1,x_{n-2}}$ generates $T_R(M)$, where $M=\{x \in V \mid f(x, V)=R\}$. In Theorem 3.5 we prove that if R is generated by r elements as an additive group then $T_R(M)$ is generated by $n+r-1$ transvections in $T_{R,M}$.

Note that if R is a field then $T_R(M)=\text{Sp}_0(V)$, and if f: $V \times V \to R$ is non-singular, i.e., the mapping $V \to \text{Hom}_R(V, R)$ given by $x \mapsto f(\ ,x)$ is an isomorphism, then $T_R(M)=\text{Sp}(V)$, and in general $\text{Sp}_0(V) \subset T_R(M) \subset \text{Sp}(V)$. In Section 4 we give some applications of these theorems to the generation of $\text{Sp}_0(L^{\perp\perp})$, where L is an arbitrary hyperbolic space of V.

Finally, we note that throughout this paper our assumption $R/A \neq F_2$ will be used only one time in the end of the proof that (c) implies (a) in Theorem 3.3.

Recently, Brown and Humphries [2] extended their results to the case $R=F_2$.

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2. PRELIMINARIES

To simplify the notation from now on we write xy instead of $f(x, y)$ for x, y in V.

Two vectors x and y in V are called a hyperbolic pair if $xy=1$, and a plane $H=Rx \oplus Ry$ is called a hyperbolic plane. Clearly, if u and v are connected then $H=Ru \oplus Rv$ forms a hyperbolic plane. We define an isometry

Δ on H by $\Delta: x \to y$ and $y \to -x$. For a unit a in R, $\Phi(a)$ is an isometry on H defined by $\Phi(a): x \to ax$ and $y \to a^{-1}y$. We know that if H is a hyperbolic plane then $V = H \perp H^{\perp}$.

LEMMA 2.1. *If* $H = Ru \oplus Rv$ *is a hyperbolic plane then* $\mathrm{Sp}(H) = T_R(\{u, v\})$.

Proof. For $v' = (uv)^{-1} v$ we have $uv' = 1$ and $T_R(\{u, v\}) = T_R(\{u, v'\})$. Hence we may assume $uv = 1$. Put $G = T_R(\{u, v\})$. Then $\Delta = T_{1,v} T_{1,u} T_{1,v}$ is in G and $\Phi(a) = \Delta^{-1} T_{a,v} T_{a^{-1},u} T_{a,v}$ is also in G for any unit a in R.

Take any σ in $\mathrm{Sp}(H)$ and let $\sigma u = au + bv$ for a, b in R. Since $R = uV = (\sigma u)(\sigma V) = (\sigma u) V$, a or b is a unit. Using Δ, we may assume a is a unit. Then, $\Phi(a^{-1}) T_{-a^{-1}b,v} \sigma$ fixes u and so maps v to $v + cu$ for some c in R. Hence, $T_{c,u} \Phi(a^{-1}) T_{-a^{-1}b,v} \sigma = 1$. Q.E.D.

From now on, if σ is an isometry in $\mathrm{Sp}(H)$, then we shall use the same notation σ for its natural extension $\sigma \perp 1$.

3. MAIN THEOREM

A vector x in V is called a maximal vector if $xV = R$ and the set of all maximal vectors of V is denoted by M in this paper.

LEMMA 3.1. (a) *If* $M \neq \emptyset$, *then* M *is a connected spanning set of* V.
(b) $T_R(M)$ *acts transitively on* M.

Proof. Take any two vectors x and y in M. Then, for some two vectors u and v in M we have $xu =$ unit and $yv =$ unit. Let $z = u$ if $yu =$ unit, $z = v$ if $xv =$ unit, and $z = u + v$ if otherwise. Then, $xz =$ unit and $zy =$ unit. So M is connected. Set $a^{-1} = xz$ and $b^{-1} = zy$. Then, $T_{a,z-x} x = z$ and $T_{b,y-z} z = y$. Thus, (b) holds. Finally, we show that M spans V. Suppose that there would exist a vector z which is not contained in the module spaned by M. Then clearly z would not be a maximal vector. Therefore, for any x in M, $x + z$ would be a maximal vector as a sum of maximal one and not maximal one. But, $x + z$ is not in M, a contradiction. Q.E.D.

Let A be the unique maximal ideal of R. Then we have three canonical homomorphisms $R \to R/A$, $V \to V/AV$ and $\mathrm{Sp}(V) \to \mathrm{Sp}(V/AV)$. For these three maps we shall use the same notation π or $-$. In the above we have $\bar{a}\bar{v} = \overline{av}$ and $\bar{\sigma}\bar{x} = \overline{\sigma x}$ for a in R, v in V and σ in $\mathrm{Sp}(V)$. Therefore, $\bar{V} = V/AV$ is regarded as an n-dimensional alternating space over the field $\bar{R} = R/A$.

LEMMA 3.2. *Let* U *be a submodule of* V. *If* $\bar{U} = \bar{V}$ *then* $U = V$.

Proof. Let $\{x_1, ..., x_n\}$ be a base for V. Suppose $\{\bar{u}_1, ..., \bar{u}_n\}$ spans $\bar{V}$ for u_i in U. Express $u_1 = a_1 x_1 + \cdots + a_n x_n$. Then, at least one a_i, say a_1, is a unit. Hence, $\{u_1, u_2, ..., x_n\}$ spans V. Next, express $u_2 = b_1 u_1 + b_2 x_2 + \cdots + b_n x_n$. Then, at least one of $b_2, ..., b_n$ is a unit, since $\bar{u}_1$ and $\bar{u}_2$ are linearly independent in $\bar{V}$. Repeat this method. Q.E.D.

THEOREM 3.3. *For a subset $S \neq \emptyset$ in M,* (a), (b), *and* (c) *are equivalent.*

(a) *$T_R(S)$ acts transitively on M.*

(b) *$T_R(S) = T_R(M)$.*

(c) *S is a connected spanning set of V.*

Proof. That (a) implies (b) is clear, using $T_{d,y} T_{c,x} T_{d,y}^{-1} = T_{c, T_{d,y}x}$.

Next, we show that (b) implies (c). Take any x in M. Then, there exists y in M such that $yx = \text{unit}$. Therefore, if we take σ in $T_R(S)$ with $\sigma = T_{1,x}$, then the identity $\sigma y = T_{1,x} y$ implies that x is contained in the space U spanned by S. Thus, $M \subseteq U$. Since M spans V by Lemma 3.1, we have $U = V$, i.e., S spans V.

Now suppose that S were not connected. Thus, S can be expressed as a disjoint union of two non-empty subsets B and C with $BC = \{0\}$ modulo A. If B would span V then $CV = \{0\}$ modulo A, a contradiction. So, the space U spanned by B does not contain C. By Lemma 3.2 we can take s in B and t in $C \backslash U$ with $\bar{t} \notin \bar{U}$. Clearly $T_R(S)$ can not carry s to t. This is a contradiction, since by (b) of Lemma 3.1 $T_R(M)$ is transitive on M.

Finally, we show that (c) implies (a). If S is infinite then we can choose a finite subset S' of S spanning V, since the rank of V is finite. Further, by adding only a finite number of elements S'' of S to S', we can make $S' \cup S''$ connected. Thus, we may assume that S is finite.

Now, since S is connected, there exists u and v in S with $uv = \text{unit}$. Put $H = Ru \oplus Rv$. We write $D = S \backslash \{u, v\}$. If $D \neq \emptyset$, we define $B = \{s \in D \mid Hs = R\}$ and $C = D \backslash B$.

Since S is connected, if $C \neq \emptyset$ then $B \neq \emptyset$ and there exist s in B and t in C with $st = \text{unit}$. Write $t' = T_{1,s} t$. Then, by $T_{1,s} T_{c,t} T_{1,s}^{-1} = T_{c,t'}$, we see $T_R(S) = T_R((S \backslash t) \cup t')$. Further, $(S \backslash t) \cup \{t'\}$ still spans V and is connected. Moreover $Ht' = R$. This allows us, exchanging t and t' in S and repeating this method, to assume $C = \emptyset$, i.e., $B = D$.

Now, we split $V = H \perp H^{\perp}$. Set $D = \{s_i \mid i \in I\}$ and express for each i in I,

$$s_i = a_i u + b_i v + x_i, \qquad a_i, b_i \in R, \quad x_i \in H^{\perp}.$$

Then $\{x_i\}$ span $H^{\perp}$, since S spans V. Further, by $D = B$, we have $a_i = \text{unit}$ or $b_i = \text{unit}$ for each i in I. Note that for any σ in $\mathrm{Sp}(H)$, $\sigma T_{c,s_i} \sigma^{-1} = T_{c,\sigma s_i}$ is in $T_R(S)$, since $\mathrm{Sp}(H) = T_R(\{u, v\})$ by Lemma 2.1. In particular, since any maximal vector in H can be carried to u by a suitable σ in $\mathrm{Sp}(H)$, $\{T_{1,u+x_i} \mid i \in I\}$ are in $T_R(S)$. Let

$$t_i = u + x_i, \qquad i \in I.$$

Now, take any x in M. We show that x can be carried to u by a product of elements of $T_R(S)$, which completes our proof.

Write

$$x = cu + dv + y, \qquad c, d \in R, \quad y \in H^{\perp}.$$

If neither c nor d are units, then for some x_j we have $yx_j = \text{unit}$, since $x \in M$. Therefore, mapping x by T_{1,t_j}, we may assume $c = \text{unit}$. Further, mapping x by a suitable element in $\mathrm{Sp}(H)$, we may assume

$$x = v + y.$$

Let p be a unit such that $p - 1$ is also a unit, in fact such p exists, since $R/A \neq F_2$. Write $y = \sum_{i \in I} e_i x_i$, $e_i \in R$. Take any i in I. If xt_i is not a unit then $(\Phi(p^{-1})\, x)\, t_i$ is a unit. So we may assume $xt_i = \text{unit}$. Then, putting $e = -(xt_i)^{-1} e_i$, $T_{e,t_i} x$ has a maximal vector as its H part and has zero as a coefficient on x_i. Thus, repeating this method for all i in I, we may assume that x is a maximal vector in H, and so x is carried to u by a suitable element in $\mathrm{Sp}(H)$. Q.E.D.

R. Brown and S. P. Humphries' results now immediately follow from our theorem, because if R is a field then $M = V \backslash \mathrm{rad}\, V$ and $\mathrm{Sp}_0(V) = T_R(M)$.

THEOREM 3.4. *Let S be a subset of M which is connected and spans V. Then, there exist n vectors $\{u, v, x_1, ..., x_{n-2}\}$ in S such that $\{T_R(u), T_{1,v}, T_{1,x_1}, ..., T_{1,x_{n-2}}\}$ generates $T_R(M)$. Further, we can choose u arbitrarily in S and v also arbitrarily among those vectors in S with $uv = \text{unit}$.*

Proof. First we show that we can choose n vectors in S which are connected and span V.

Take any connected vectors u and v in S, i.e., $uv = \text{unit}$. Clearly $\{\bar{u}, \bar{v}\}$ are connected and free in $\bar{V}$. Set $S' = \{u, v, x_1, ..., x_i\}$, $x_i \in S$ and suppose that $\bar{S}'$ is connected and free. Let W be the space spanned by S'. If $\bar{W} \nsubseteq \bar{V}$ then by Lemma 3.2 we can choose a vector x_{i+1} in S such that $\bar{x}_{i+1}$ is not contained in $\bar{W}$ and $\bar{x}_{i+1}$ is connected to $\bar{S}'$. Thus, we can choose n vectors $S'' = \{u, v, x_1, ..., x_{n-2}\}$ in S such that $\bar{S}''$ is connected and $\bar{S}''$ is a base for $\bar{V}$. Then, S'' is connected and by Lemma 3.2 it spans V.

Therefore, by Theorem 3.3, $\{T_{R,u}, T_{R,v}, T_{R,x_1}, ..., T_{R,x_{n-2}}\}$ generates $T_R(M)$. Here, we note that for any two vectors s, t in V if $st = \text{unit}$ then, writing $a = (st)^{-2}$, we have

$$T_{a,s} T_{1,t} T_{R,s} T_{1,t^{-1}} T_{a,s^{-1}} = T_{R,st \cdot t} = T_{R,t}.$$

Therefore, by the connectedness of $\{u, v, x_1, ..., x_{n-2}\}$, we have our theorem. Q.E.D.

THEOREM 3.5. *If $M \neq \emptyset$ and R is generated by r elements as an additive group, then $T_R(M)$ is generated by $n+r-1$ transvections in $T_{R,M}$.*

Proof. By Lemma 3.1, M is a connected spanning set of V. Hence, applying Theorem 3.4, for some n vectors $x_1, ..., x_n$ in M, $T_R(M)$ is generated by $T_{R,x_1}, T_{1,x_2}, ..., T_{1,x_n}$. Further, since $T_{a,x} T_{b,x} = T_{a+b,x}$ for any a, b in R and x in V, we see that T_{R,x_1} is generated by r transvections. Thus, we have the theorem. Q.E.D.

4. GENERATION OF $\mathrm{Sp}_0(L \perp \mathrm{rad}\, V)$

For a subset X of V we define a subgroup $\mathrm{Sp}(V, X)$ of $\mathrm{Sp}(V)$ as

$$\mathrm{Sp}(V, X) = \{\psi \in \mathrm{Sp}(V) \mid \psi = 1 \text{ on } X\}.$$

According to this notation, the subgroup $\mathrm{Sp}_0(V)$ defined in the Introduction is $\mathrm{Sp}(V, \mathrm{rad}\, V)$.

An orthogonal direct sum $L = H_1 \perp \cdots \perp H_r$ of hyperbolic planes $H_1, ..., H_r$ is called a hyperbolic space. Clearly we have $L^\perp = L^{\perp\perp\perp}$ and $V = L \perp L^\perp$. For a hyperbolic space L of V we have

$$\mathrm{rad}\, V = L^\perp \cap L^{\perp\perp} = \mathrm{rad}\, L^\perp = \mathrm{rad}\, L^{\perp\perp}$$

and

$$L^{\perp\perp} = L \perp \mathrm{rad}\, V.$$

Next, we shall show that two groups $\mathrm{Sp}(V, L^\perp)$ and $\mathrm{Sp}_0(L^{\perp\perp})$ are isomorphic. Let σ be in $\mathrm{Sp}(V, L^\perp)$. Then, since $(\sigma L^{\perp\perp})\, L^\perp = \sigma L^{\perp\perp} \sigma L^\perp = \{0\}$, we have $\sigma L^{\perp\perp} \subset L^{\perp\perp}$. Replacing σ with σ^{-1}, we have $\sigma L^{\perp\perp} = L^{\perp\perp}$. This means that the restriction $\sigma|_{L^{\perp\perp}}$ is an isometry in $\mathrm{Sp}(L^{\perp\perp})$. Since σ fixes $L^\perp \cap L^{\perp\perp} = \mathrm{rad}\, L^{\perp\perp}$, $\sigma|_{L^{\perp\perp}}$ is in $\mathrm{Sp}_0(L^{\perp\perp})$. Thus, we have a restriction map

$$\chi\colon \quad \mathrm{Sp}(V, L^\perp) \to \mathrm{Sp}_0(L^{\perp\perp})$$

defined by $\chi(\sigma) = \sigma|_{L^{\perp\perp}}$.

THEOREM 4.1. *For a hyperbolic space L of V, the restriction map $\chi\colon \mathrm{Sp}(V, L^\perp) \to \mathrm{Sp}_0(L^{\perp\perp})$ is a group isomorphism.*

Proof. Since the injectivity and the group homomorphism of χ is clear, we prove the surjectivity of χ. Take any σ' in $\mathrm{Sp}_0(L^{\perp\perp})$. We shall show that $\sigma = (\sigma'|_L) \perp (1_{L^\perp})$ is in $\mathrm{Sp}(V, L^\perp)$. Then $\chi\sigma = \sigma'$ is clear.

First, by the definition of σ, σ is a linear map on V. Since $\sigma = \sigma'$ on $L^{\perp\perp}$, we have

$$\begin{aligned}\sigma V &= \sigma(L \perp L^{\perp}) = \sigma L \perp \sigma L^{\perp} = \sigma L \perp L^{\perp} \subset \sigma L^{\perp\perp} + L^{\perp}\\ &= \sigma' L^{\perp\perp} + L^{\perp} = L^{\perp\perp} + L^{\perp} = V.\end{aligned}$$

Thus, σ is surjective. To show the injectivity of σ we prove that $\operatorname{Ker}\sigma = \{0\}$. We have $V = L \perp L^{\perp}$. Let $z = x + y$ be in $\operatorname{Ker}\sigma$ for $x \in L$ and $y \in L^{\perp}$. Then, since $\sigma z = \sigma' x + y = 0$, we have $y = -\sigma' x$. Since $x \in L \subset L^{\perp\perp}$ and $\sigma' L^{\perp\perp} = L^{\perp\perp}$, y is in $L^{\perp\perp}$. And so z is also in $L^{\perp\perp}$. However, since $\sigma = \sigma'$ on $L^{\perp\perp}$ and σ' is injective, z must be zero. Thus, σ is injective on V.

That σ preserves the form $f\colon V \times V \to R$ is easy to see and hence σ is in $\operatorname{Sp}(V, L^{\perp})$. Q.E.D.

By the theorem, we shall identify these two groups. We note that $\operatorname{Sp}_0(L^{\perp\perp}) = \operatorname{Sp}(L \perp \operatorname{rad} V, \operatorname{rad} V)$.

LEMMA 4.2. *Let u, v, and w be vectors in M. Suppose $uv = uw = 1$. Then, we can map w to v by a product of two transvections in $T_R(M)$ without moving u.*

Proof. Write $H = Ru \oplus Rv$ and $V = H \perp H^{\perp}$. Express $w = au + v + z$, $a \in R$, $z \in H^{\perp}$. Then $T_1(u+z)\, T_{a-1}(u)$ maps w to v without moving u. Q.E.D.

THEOREM 4.3. *For a hyperbolic space L of V let M' be the set of maximal vectors of $V' = L \perp \operatorname{rad} V$, i.e., $M' = \{x \in V' \mid xV' = R\}$. Then,*

$$T_R(M') = \operatorname{Sp}(V, L^{\perp}).$$

Proof. As we mentioned above $V' = L^{\perp\perp}$. Further, by Theorem 4.1 we can identify $\operatorname{Sp}(V, L^{\perp})$ with $\operatorname{Sp}_0(L^{\perp\perp})$. So we show $T_R(M') = \operatorname{Sp}_0(V')$.

Now, since $M'(\operatorname{rad} V') = \{0\}$, $T_R(M')$ fixies rad V'. Hence, $T_R(M') \subset \operatorname{Sp}_0(V')$. To show the converse we prove that for any σ in $\operatorname{Sp}_0(V')$, σ is a finite product of elements of $T_R(M')$. Let $\{u, v\}$ be a hyperbolic pair in L. Hence $uv = \sigma u \sigma v = 1$. In particular, $\{u, \sigma u\} \subset M'$. Therefore, we have θ in $T_R(M')$ such that $\theta\sigma u = u$, because $T_R(M')$ acts transitively on M' by (b) of Lemma 3.1.

Further, by Lemma 4.2 we can carry $\theta\sigma v$ to v by ρ in $T_R(M')$ without moving u. Thus, setting $\sigma' = \rho\theta\sigma$, we see $\sigma' = 1$ on $H = Ru \oplus Rv$. Split $V' = H \perp H^{\perp}$. Then $\sigma' H^{\perp} = H^{\perp}$. Therefore, we can apply the above argument to $\sigma'|_{H^{\perp}}$. Thus, repeating this method we have $\rho_1\theta_1, \ldots, \rho_r\theta_r$ in $T_R(M')$ such that $\rho_r\theta_r \cdots \rho_1\theta_1\sigma = 1$. Q.E.D.

THEOREM 4.4. *Let L, V', and M' be the same objects as in Theorem 4.3. Then, for any subset $S \neq \emptyset$ in M', the following are equivalent:*

(a) *$T_R(S)$ acts transitively on M',*

(b) $T_R(S) = \mathrm{Sp}(V, L^{\perp})$,

(c) *S spans V' and S is connected.*

Proof. This is a trivial consequence of Theorems 3.3 and 4.3. Q.E.D.

THEOREM 4.5. *Let L, V', and M' be the same objects as in Theorem* 4.3, *and let V' be a free module of $m = \mathrm{rank}\ V'$. Suppose that $\phi \neq S \subset M'$ and S is a connected spanning set of V'. Then S contains m vectors $\{u, v, x_1, \ldots, x_{m-2}\}$ such that $\mathrm{Sp}(V, L^{\perp})$ is generated by $\{T_{R,u}, T_{1,v}, T_{1,x_1}, \ldots, T_{1,x_{m-2}}\}$, where we can choose $\{u, v\}$ as an arbitrarily hyperbolic pair in M'.*

Proof. Use Theorems 3.4 and 4.3. Q.E.D.

THEOREM 4.6. *Let L, V', and M' be the same objects as in Theorem* 4.3. *If V' is a free module of $m = \mathrm{rank}\ V'$ and R is generated by r elements as an additive group, then $\mathrm{Sp}(V, L^{\perp})$ is generated by $m + r - 1$ transvections in $T_{R,M'}$.*

Proof. Use Theorems 3.5 and 4.3. Q.E.D.

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