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A Discrete Kinetic Approximation of Entropy Solutions to Multidimensional Scalar Conservation Laws

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We present a new relaxation approximation to scalar conservation laws in several space variables by means of semilinear hyperbolic systems of equations with a finite number of velocities. Under a suitable multidimensional generalization of the Whitham relaxation subcharacteristic condition, we show the convergence of the approximated solutions to the unique entropy solution of the equilibrium Cauchy problem. © 1998 Academic Press

1. INTRODUCTION

Let u: $\mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ be the (unique) global entropy solution in the sense of Kružkov [Kr] to the Cauchy problem

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} A_j(u) = 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $\mathbf{A} = (A_1, ..., A_d) \in (\mathrm{Lip}_{loc}(\mathbb{R}))^d$ and $u_0 \in L^\infty(\mathbb{R}^d)$.

In this paper we propose to approximate this solution by considering a special class of discrete kinetic systems. Let $N \geqslant d+1$ be fixed and let $\mathbf{f}^\varepsilon = (f_1^\varepsilon, ..., f_N^\varepsilon)$: $\mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$ be a solution to the Cauchy problem for the following semilinear (diagonal) hyperbolic operator

$$\partial_t \mathbf{f}^\varepsilon + \sum_{j=1}^{d} \mathbf{\Lambda}_j \partial_{x_j} \mathbf{f}^\varepsilon = \frac{1}{\varepsilon} (\mathbf{M}(u^\varepsilon) - \mathbf{f}^\varepsilon), \tag{1.3}$$

with the initial condition

$$\mathbf{f}^\varepsilon(x, 0) = \mathbf{f}_0(x). \tag{1.4}$$

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Here $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, $\mathbf{\Lambda}_j = \operatorname{diag}(\lambda_{1j}, ..., \lambda_{Nj}) \in \mathscr{M}^{N \times N}$, and the function u^ε is defined by

$$u^\varepsilon := \sum_{i=1}^{N} f_i^\varepsilon; \tag{1.5}$$

the function $\mathbf{M} = (M_1, ..., M_N) : \mathbb{R} \to \mathbb{R}^N$ is Lipschitz continuous with $\mathbf{M}(0) = 0$ and $\mathbf{f}_0 = (f_{01}, ..., f_{0N}) \in (L^\infty(\mathbb{R}^d))^N$.

To connect problem (1.3)–(1.4) with problem (1.1)–(1.2), let us make some assumptions on the function $\mathbf{M}$ in system (1.3).

Definition 1.1. Let $I \subseteq \mathbb{R}$ be a fixed interval. A Lipschitz continuous function $\mathbf{M} = \mathbf{M}(u) : I \to \mathbb{R}^N$ is a (local) Maxwellian Function for Eq. (1.1) and with respect to the interval I if the following conditions are verified:

$$\sum_{i=1}^{N} M_i(u) = u, \qquad \text{for any} \quad u \in I; \tag{1.6}$$

$$\sum_{i=1}^{N} \lambda_{ij} M_i(u) = A_j(u), \quad j = 1, ..., d \qquad \text{for any} \quad u \in I. \tag{1.7}$$

In the following we shall assume that the function $\mathbf{M}$ is a Maxwellian Function; then the system (1.3) can be considered as a BGK approximation for Eq. (1.1), see [Ce, CoP, Pe1, GR], and conditions (1.6) and (1.7) imply its consistency with the hyperbolic operator (1.1). In fact, if we sum the N equations in (1.3) we obtain a local conservation law, which is satisfied by every solution of (1.3)

$$\partial_t u^\varepsilon + \sum_{j=1}^{d} \partial_{x_j} \left(\sum_{i=1}^{N} \lambda_{ij} f_i^\varepsilon \right) = 0. \tag{1.8}$$

Fix now the initial data u_0 in (1.1)–(1.2). Consider the sequence of solutions $\mathbf{f}^\varepsilon$ to the Cauchy problem (1.3)–(1.4) with $\mathbf{f}_0^\varepsilon = = \mathbf{M}(u_0)$. Assume that the sequence $\{\mathbf{f}^\varepsilon\}$is (locally) uniformly bounded and there exists a (bounded) function $\mathbf{f}^0$ such that

$$\mathbf{f}^\varepsilon \to \mathbf{f}^0, \tag{1.9}$$

as $\varepsilon \to 0$, in a suitable (strong) topology. Then, setting, $u^0 = \sum_{i=1}^{N} f_i^0$, we have

$$u^\varepsilon \to u^0 \tag{1.10}$$

as $\varepsilon \to 0$, in the same topology and the following identities hold true:

$$\mathbf{f}^0 = \mathbf{M}(u^0), \tag{1.11}$$

$$\partial_t u^0 + \sum_{j=1}^{d} \partial_{x_j} \left(\sum_{i=1}^{N} \lambda_{ij} f_i^0 \right) = 0. \tag{1.12}$$

Hence, observing that from (1.11) and assumption (1.7)

$$\sum_{i=1}^{N} \lambda_{ij} f_i^0 = \sum_{i=1}^{N} \lambda_{ij} M_i(u^0) = A_j(u^0), \qquad j = 1, ..., d, \tag{1.13}$$

we conclude that u^0 is a weak solution to the Cauchy problem (1.1)–(1.2). In this paper we shall establish the validity of the above limit, also by proving that u^0 actually satisfies the entropy conditions (see Definition 4.1 below), under some supplementary stability conditions discussed later.

Now we present the main motivations to consider this kind of approximation. First it is important to observe that there is a very strict connection between our discrete kinetic approximation and the relaxation approximation of conservation laws proposed in [JX], at least in one space dimension. In that paper, the authors proposed to approximate Eq. (1.1) by the system

$$\begin{cases} \partial_t u^\varepsilon + \sum_{j=1}^{d} \partial_{x_j} v_j^\varepsilon = 0, \\ \partial_t v_j^\varepsilon + \alpha_j \, \partial_{x_j} u^\varepsilon = \dfrac{1}{\varepsilon} (A_j(u^\varepsilon) - v_j^\varepsilon), \qquad j = 1, ..., d, \end{cases} \tag{1.14}$$

with $\alpha_j > 0$, $(j = 1, ..., d)$. Notice that actually this approximation was proposed for general hyperbolic systems, just taking $u, v \in \mathbb{R}^k$ $(k \geqslant 1)$ in (1.1) and (1.4). In one space dimension $(d = 1)$ we can easily put the approximation (1.14) under the form (1.3) just choosing $N = 2$, $\lambda_{21} = -\lambda_{11} = \sqrt{\alpha_1} > 0$, and taking the Maxwellian function $\mathbf{M}$ as

$$M_1(u) = \frac{1}{2}\left(u - \frac{A(u)}{\sqrt{\alpha_1}}\right), \qquad M_2 = \frac{1}{2}\left(u + \frac{A(u)}{\sqrt{\alpha_1}}\right).$$

Recall that uniform bounds, L^1 stability and convergence to the unique entropy solution of the correspondent one dimensional conservation law for this approximation were first given in [Na]. Similar results, by using finite difference approximation, can be found in [AN1, Yo2]. L^∞ bounds and convergence almost everywhere, but with no stability estimates, were independently established in [CR] by using the compensated compactness framework. The L^1 stability of traveling wave solutions to problem (1.14) (always for $\mathrm{d}=\mathrm{k}=1$) was established in [MN].

Our approximation framework shares most of the advantages of the relaxation approximation: simple formulation even for general multidimensional systems of conservation laws and easy numerical implementation (see [AN2]), hyperbolicity (then finite speed of propagation), regular approximating solutions. Actually the main advantage, especially in the multidimensional case, of both the approximations lies in the possibility of avoiding the resolution of the local Riemann problems in the design of numerical schemes. Moreover our framework, unlike the relaxation approximation, presents a special property: all the approximating problems are in diagonal form, which is highly recommended for numerical and theoretical purposes. As a matter of fact the relaxation approximation (1.14) does not fit in our framework (as shown in Remark 2.4 below) and in particular there is no diagonal form for the system (1.14) for $d>1$.

As is well known for general relaxation problems also, approximation (1.3) needs for suitable stability conditions to produce the correct limits. In the framework of general quasilinear hyperbolic relaxation problems this condition is known as the subcharacteristic condition, see [Wh, Li, CLL, JX, Na]. Here we can argue in the spirit of the Chapman–Enskog analysis (see Section 2) to find a formal stability condition for (1.3), namely

$$\sum_{i=1}^{N}\left(\sum_{j=1}^{d}\lambda_{ij}\xi_j\right)^2 M_i'(u)\geqslant\left(\sum_{j=1}^{d}A_j'(u)\,\xi_j\right)^2, \tag{1.15}$$

for every $\xi\in\mathbb{R}^d$ and every $u\in I$. Actually to prove convergence results we need the following slightly stronger version of condition (1.15):

(*Monotonicity Condition*) Every component of the Maxwellian Function $\mathbf{M}$ is monotone nondecreasing on the interval I.

The plan of the paper is as follows: in Section 2 we discuss the first order expansion of our approximation in the spirit of that of Chapman–Enskog for the kinetic theory of Boltzmann equations. Then we give our monotonicity condition, which implies the dissipativity of this first order correction, and some examples of different choices of the matrices of velocities $\mathbf{\Lambda}_j$ and the local Maxwellian function $\mathbf{M}$.

Sections 3 and 4 are devoted to prove the convergence of our approximation. In particular we show that our monotonicity condition implies special comparison and stability properties of system (1.3) and in particular that the evolution operator associated to problem (1.3)–(1.4) is contractive in the L^1 norm and the system is quasimonotone, see [HN, Na] and Section 3 below. Therefore we are able to prove that under our monotonicity condition problem (1.3)–(1.4) has a unique uniformly bounded solution for any $\varepsilon > 0$ and the sequence of solutions forms a compact subset of $C([0, T]; L^1_{loc}(\mathbb{R}^d))^N$, for any $T > 0$. Then we prove the convergence of the sequence u^ε to the (unique) entropy solution of problem (1.1)–(1.2). Entropy conditions are established by using a complete family of kinetic entropy functions such that for any of those functions a H–Theorem for system (1.3) holds. For the relaxation case ($d = 1$, $N = 1$) some different classes of entropy functions were considered in [Na, Ji].

Let us now recall some basic references concerning the kinetic approximation of conservation laws. The fluid dynamical limit of kinetic equations is a classical problem in mathematical physics. In particular Euler equations can be formally obtained as the fluid dynamical limit of Boltzmann equations, see [Ce, CIP] and references therein. Actually this limit has been rigorously established only as long as the limit solutions are regular. The rigorous theory of kinetic approximations for solutions with shocks is more recent and mainly developed when the limit equation is scalar. The first result of convergence of a fractional step BGK approximation with continuous velocities, with an entropy conditions for the limit (weak) solution, was proved in [Br1] (see also [GM]). Another convergence result was given later in [PT], using a continuous velocities BGK model. A related kinetic formulation can be found in [LPT1]. Let us also recall that some results have been established for special systems or partially kinetic approximations [LPT2, JPP, BC, LPS]. Related numerical schemes can be found in [CoP, Pe2, Pe3]; for a general overview and many other references see [GR].

The case of discrete velocities models has been also considered by many people, see the review paper [PI]. In particular we mention the studies on the fluid dynamical limit for the Broadwell model [CaP, Xi]. Convergence for various relaxation models was investigated in [CL, CLL, CR, Ja, LN, TW, WX, Yo1, Yo2]. Let us also remark that monotonicity tools, similar to those of the present paper, but for a different relaxation approximation to problem (1.1)–(1.2), were used in a recent independent work by Katsoulakis and Tzavaras [KT].

Finally let us discuss some numerical aspects related to our approach. A lot of computational work has been done in the last ten years in the very closed framework of lattice Boltzmann and BGK models, see [QSO] and references therein. Let us also mention the relaxation schemes of [JX, AN1] and the monotone schemes of [Br2], the latter being an example of numerical (relaxed, i.e., $\varepsilon = 0$) first order discretization of our construction. A quite complete investigation on second order relaxation and discrete kinetic schemes, directly issued by the approximation (1.3), for general systems of conservation laws in several space variables is developed in [AN2].

2. MONOTONE MAXWELLIAN FUNCTIONS AND THE CHAPMAN–ENSKOG ANALYSIS

In this section we discuss the stability conditions for the discrete kinetic approximation (1.3). Since the local equilibrium for that system is given by the hyperbolic system (1.1), it is natural to seek a dissipative first–order approximation to (1.3), which is the analogue of the compressible Navier–Stokes equations in the classical kinetic theory. In principle we could try to use the theory developed in a more general context in [CLL]. Unfortunately it is easy to realize that their main assumption, namely the existence of a strictly convex dissipative entropy for the relaxing system (1.3), which verifies in particular the requirement (iii) of Definition 2.1 of [CLL], is not satisfied in the present case and we need a different construction.

Let $\mathbf{f}^\varepsilon$ be a sequence of solutions to (1.3)–(1.4) parametrized by ε, for a fixed initial data $\mathbf{f}_0$, which for simplicity we can choose as a local equilibrium, i.e., $\mathbf{f}_0(x) = \mathbf{M}(u_0)$ for some $u_0 \in L^\infty(\mathbb{R}^d)$. Set

$$u^\varepsilon := \sum_{i=1}^N f_i^\varepsilon, \qquad v_j^\varepsilon := \sum_{i=1}^N \lambda_{ij} f_i^\varepsilon, \quad j = 1, ..., d.$$

Then, from (1.3) and the compatibility assumptions (1.6)–(1.7), we have

$$\begin{cases} \partial_t u^\varepsilon + \sum_{j=1}^d \partial_{x_j} v_j^\varepsilon = 0 \\ \partial_t v_j^\varepsilon + \sum_{l=1}^d \partial_{x_l} \left(\sum_{i=1}^N \lambda_{ij} \lambda_{il} f_i^\varepsilon \right) = \frac{1}{\varepsilon} (A_j(u^\varepsilon) - v_j^\varepsilon), \qquad j = 1, ..., d. \end{cases} \tag{2.1}$$

Consider a formal expansion of $\mathbf{f}^\varepsilon$ in the form

$$\mathbf{f}^\varepsilon = \mathbf{M}(u^\varepsilon) + \varepsilon \mathbf{f}_1^\varepsilon + O(\varepsilon^2). \tag{2.2}$$

Then

$$\begin{aligned} v_j^\varepsilon &= A_j(u^\varepsilon) - \varepsilon \left(\partial_t v_j^\varepsilon + \sum_{l=1}^d \partial_{x_l} \left(\sum_{i=1}^N \lambda_{ij} \lambda_{il} f_i^\varepsilon \right) \right) \\ &= A_j(u^\varepsilon) - \varepsilon \left(\partial_t v_j^\varepsilon + \sum_{l=1}^d \partial_{x_l} \left(\sum_{i=1}^N \lambda_{ij} \lambda_{il} M_i(u^\varepsilon) \right) \right) + O(\varepsilon^2). \end{aligned} \tag{2.3}$$

Reporting in (2.1) yields

$$\begin{aligned} &\partial_t u^\varepsilon + \sum_{j=1}^d \partial_{x_j} A_j(u^\varepsilon) \\ &\quad = \varepsilon \sum_{j=1}^d \partial_{x_j} \left(\partial_t v_j^\varepsilon + \sum_{l=1}^d \partial_{x_l} \left(\sum_{i=1}^N \lambda_{ij} \lambda_{il} M_i(u^\varepsilon) \right) \right) + O(\varepsilon^2). \end{aligned} \tag{2.4}$$

Now, dropping the higher order terms in (2.3), we have

$$\partial_t v_j^\varepsilon = A_j'(u^\varepsilon)\,\partial_t u^\varepsilon + O(\varepsilon) = -\sum_{l=1}^{d} A_j'(u)\,A_l'(u)\,\partial_{x_l} u^\varepsilon + O(\varepsilon). \tag{2.5}$$

Then, up to the higher order terms in (2.4), we obtain

$$\partial_t u^\varepsilon + \sum_{j=1}^{d} \partial_{x_j} A_j(u^\varepsilon) = \varepsilon \sum_{j=1}^{d} \partial_{x_j}\left(\sum_{l=1}^{d} B_{jl}(u^\varepsilon)\,\partial_{x_l} u^\varepsilon\right) \tag{2.6}$$

with

$$B_{jl}(u) := \sum_{i=1}^{N} \lambda_{ij}\lambda_{il} M_i'(u^\varepsilon) - A_j'(u)\,A_l'(u). \tag{2.7}$$

Therefore we find the following stability condition.

Proposition 2.1. *The first-order approximation to system* (1.3) *takes the form* (2.6) *and it is dissipative provided that the following condition is verified,*

$$\sum_{j,\,l=1}^{d} B_{jl}(u)\,\xi_j \xi_l \geqslant 0, \tag{2.8}$$

for every $\xi \in \mathbb{R}^k$ and every u belonging to some fixed interval $I \subseteq \mathbb{R}$.

Let us note that the expansion (2.6) cannot be considered in any way as a rigorous asymptotic description of system (1.3). Actually to prove our rigorous convergence results we need a slightly stronger version of condition (2.8).

Definition 2.2. Let $I \subseteq \mathbb{R}$ be a fixed interval. A Lipschitz continuous function $\mathbf{M} = \mathbf{M}(u) : I \to \mathbb{R}^N$ is a Monotone Maxwellian Function (**MMF**) for Eq. (1.1) and with respect to the interval I if conditions (1.6) and (1.7) are verified and moreover

$$M_i \text{ is a monotone (increasing) function on } I, \text{ for every } i = 1, ..., N. \tag{2.9}$$

Proposition 2.3. *Let M be a* **MMF** *according to Definition* 2.2. *Then M verifies inequality* (2.8).

Proof. It is enough to rewrite condition (2.8) as

$$\sum_{i=1}^{N}\left(\sum_{j=1}^{d}\lambda_{ij}\xi_j\right)^2 M_i'(u)$$
$$\geqslant\left(\sum_{j,\,l=1}^{d}A_j'(u)\,A_l'(u)\,\xi_j\xi_l\right)$$
$$=\left(\sum_{j=1}^{d}A_j'(u)\,\xi_j\right)^2=\left(\sum_{i=1}^{N}\left(\sum_{j=1}^{d}\lambda_{ij}\xi_j\right)M_i'(u)\right)^2. \tag{2.10}$$

From (1.6) and (2.9) we have that $0\leqslant M_i'(u)\leqslant 1$ and

$$\sum_{i=1}^{N}M_i'(u)=1.$$

Hence inequality (2.10), and then condition (2.8), follows by the discrete Jensen inequality. ∎

The remainder of the section is devoted to presenting some examples of different approximations according to the choices of the matrices of velocities $\mathbf{\Lambda}_j$ and the local Maxwellian function $\mathbf{M}$.

(a) Take $d=1$, $N=2$. Consider the conservation law

$$\partial_t u+\partial_x A(u)=0 \tag{2.11}$$

and the approximating discrete Boltzmann system, for $\lambda_1<\lambda_2$,

$$\begin{cases}\partial_t f_1^\varepsilon+\lambda_1\,\partial_x f_1^\varepsilon=\dfrac{1}{\varepsilon}\,(M_1(u^\varepsilon)-f_1^\varepsilon),\\[2ex] \partial_t f_2^\varepsilon+\lambda_2\,\partial_x f_2^\varepsilon=\dfrac{1}{\varepsilon}\,(M_2(u^\varepsilon)-f_2^\varepsilon),\end{cases} \tag{2.12}$$

where $u^\varepsilon:=f_1^\varepsilon+f_2^\varepsilon$. From (1.6)–(1.7) the Maxwellian function for this problem is given by

$$M_1(u)=\frac{\lambda_2 u-A(u)}{\lambda_2-\lambda_1}\qquad M_2(u)=\frac{\lambda_1 u-A(u)}{\lambda_1-\lambda_2} \tag{2.13}$$

and the condition (2.9) reads now

$$\lambda_1\leqslant A'(u)\leqslant\lambda_2,\qquad \text{for}\quad u\in I, \tag{2.14}$$

for some fixed interval $I\subseteq\mathbb{R}$.

Setting $v^\varepsilon = \lambda_1 f_1^\varepsilon + + \lambda_2 f_2^\varepsilon$, we rewrite system (2.12) in a relaxation form:

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon + \partial_x ((\lambda_1 + \lambda_2)\, v^\varepsilon - \lambda_1 \lambda_2 u^\varepsilon) = \dfrac{1}{\varepsilon} (A(u^\varepsilon) - v^\varepsilon) \end{cases} \tag{2.15}$$

Hence the condition (2.14) is just the subcharacteristic condition for system (2.15) as in [Wh, Li]. In particular for $\lambda_2 = -\lambda_1 = \lambda > 0$ we recover the relaxation approximation (1.14) of Eq. (2.11), see [JX, Na].

(b) In the numerical approximation of Eq. (2.11) it could be also useful to deal with more velocities, say $N > d + 1$. This formally corresponds to more accurate approximation schemes, see [Br2, AN2]. Take $N = 3$, $d = 1$. It is easy to show that a convergent approximation is given, for $\lambda_1 < \lambda_2 < \lambda_3$, by the following **MMF**,

$$\begin{aligned} M_1(u) &= \frac{\lambda_3 u - A(u) + (\lambda_2 - \lambda_3)\, M_2(u)}{\lambda_3 - \lambda_1}, \\ M_3(u) &= \frac{\lambda_1 u - A(u) + (\lambda_2 - \lambda_1)\, M_2(u)}{\lambda_1 - \lambda_3}, \end{aligned} \tag{2.16}$$

for any Lipschitz continuous function $M_2(u)$ such that

$$0 \leqslant M_2'(u) \leqslant \inf \left(\frac{A'(u) - \lambda_1}{\lambda_2 - \lambda_1}, \frac{\lambda_3 - A'(u)}{\lambda_3 - \lambda_2} \right), \tag{2.17}$$

for $u \in I$, for some $I \subseteq \mathbb{R}$. For $\lambda_3 = -\lambda_1 = \lambda > 0$, and $\lambda_2 = 0$, we can choose

$$M_1' = \frac{(-A')_+}{\lambda}, \qquad M_2' = 1 - \frac{|A'|}{\lambda}, \qquad M_3' = \frac{(A')_+}{\lambda}. \tag{2.18}$$

This choice corresponds, in the relaxation limit, to the Engquist–Osher numerical scheme, see [Br2]. Its relaxation formulation is given now by

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \\ \partial_t v^\varepsilon + \partial_x z^\varepsilon = \dfrac{1}{\varepsilon} (A(u^\varepsilon) - v^\varepsilon) \\ \partial_t z^\varepsilon + \lambda^2\, \partial_x v^\varepsilon = \dfrac{1}{\varepsilon} (\lambda^2 (u^\varepsilon - M_2(u^\varepsilon)) - z^\varepsilon), \end{cases} \tag{2.19}$$

where $v^{\varepsilon} = \lambda(f_3^{\varepsilon} - f_1^{\varepsilon}))$ and $z^{\varepsilon} = \lambda^2(f_3^{\varepsilon} + f_1^{\varepsilon})f)$. In this example it is also possible to verify that condition (2.9) is actually strictly stronger than condition (2.8). In fact for $\lambda_3 = -\lambda_1 = \lambda > 0$, and $\lambda_2 = 0$, condition (2.9) reads

$$0 \leqslant M_2'(u) \leqslant 1 - \frac{|A'(u)|}{\lambda},$$

while condition (2.8) becames

$$M_2'(u) \leqslant 1 - \left(\frac{A'(u)}{\lambda}\right)^2.$$

In particular no monotonicity assumptions are required on the function M_2.

(c) Next let us present a general procedure to construct Monotone Maxwellian Functions for a given equation. Let $N \geqslant d+1$ and choose d vectors $\lambda_j = (\lambda_{1j}, ..., \lambda_{Nj})$, with $j = 1, ..., d$, such that

$$\sum_{i=1}^{N} \lambda_{ij} = 0, \qquad j = 1, ..., d \tag{2.20}$$

$$\sum_{i=1}^{N} \lambda_{il}\lambda_{ij} = 0, \qquad j, l = 1, ..., d, \qquad \textit{for} \quad j \neq l. \tag{2.21}$$

This is always possible by taking first the vector $v = (1, ..., 1) \in \mathbb{R}^N$ and then any orthogonal basis of the orthogonal space to the vector v. A (vector valued) function $\mathbf{M}$ which satisfies conditions (1.6) and (1.7) is now given by

$$\mathbf{M}(u) = \frac{u}{N} v + \sum_{j=1}^{d} \frac{A_j(u)}{|\lambda_j|^2} \lambda_j. \tag{2.22}$$

Hence condition (2.9) is satisfied whenever

$$1 \geqslant -N \sum_{j=1}^{d} \frac{A_j'(u)}{|\lambda_j|^2} \lambda_{ij}, \qquad i = 1, ..., N, \tag{2.23}$$

which generalizes to the multidimensional case the subcharacteristic condition.

(d) We can give also an example with non-orthogonal velocity vectors. Let $N = d+1$ and fix $\lambda > 0$. Set

$$\lambda_{ij} = -\lambda\delta_{ij}, \qquad i = 1, ..., d, \qquad \lambda_{Nj} = \lambda;$$

a correspondent Maxwellian function is given now by

$$M_N(u) = \frac{1}{N}\left(u + \frac{1}{\lambda}\sum_{j=1}^{d} A_j(u)\right)$$

$$M_i(u) = M_N(u) - \frac{1}{\lambda} A_i(u), \qquad i = 1, ..., d.$$

The monotonicity is recovered just by taking

$$\lambda \geqslant \sup_{|u| \leqslant B} \sup_{j=1, ..., d} (2d-1)\,|A_j'(u)|$$

for a suitable value of $B > 0$. In particular, in view of Theorem 3.1 below, we can choose $B = \|u_0\|_\infty$.

(e) Let us conclude our presentation by an extension of example (b) to the multidimensional case. Take $N = 2d+1$ and set, for $j = 1, ..., d$,

$$\lambda_{ij} = \begin{cases} \lambda_i \delta_{ij}, & i = 1, ..., d, \\ 0, & i = d+1, \\ -\lambda_{i-(d+1)} \delta_{ij}, & i = d+2, ..., 2d+1. \end{cases} \tag{2.24}$$

Here the constant values λ_i $(i = 1, ..., d)$ will be chosen later. For the derivatives of the components of the Maxwellian function $\mathbf{M}$ we choose

$$\begin{aligned} M_i'(u) &= \frac{1}{\lambda_i}(A_i'(u))_+, \qquad i = 1, ..., d \\ M_{d+1}'(u) &= 1 - \sum_{j=1}^{d} \frac{|A_j'(u)|}{\lambda_j}, \\ M_i'(u) &= \frac{1}{\lambda_{i-(d+1)}}(-A_{i-(d+1)}'(u))_+, \qquad i = d+2, ..., 2d+1. \end{aligned} \tag{2.25}$$

According to this choice, the function $\mathbf{M}$ is a **MMF** on a suitable interval $I \subseteq \mathbb{R}$, if

$$1 \geqslant \sum_{j=1}^{d} \frac{|A_j'(u)|}{\lambda_j}$$

for any $u \in I$.

Remark 2.4. Concerning the relaxation approximation (1.14) of [JX] to Eq. (1.1), we would like to point out that this approximation does not fit in our framework in the multidimensional case and then we are not able to show its convergence as $\varepsilon \downarrow 0$ by using our methods. In fact it is possible to prove, by a simple explicit computation, that it is impossible to diagonalize the left hand side of system (1.14), if $d \geqslant 2$.

More precisely, take a function $\mathbf{f} = (f_1, ..., f_N)$ and some constants $\lambda_{ij} \in \mathbb{R}$ ($i = 1, ..., N$; $j = 1, ..., d$; $N \geqslant d+1$) such that

$$\partial_t f_i + \sum_{j=1}^{d} \lambda_{ij} f_i = 0. \tag{2.26}$$

Let $b_{ij} \in \mathbb{R}$ ($i = 1, ..., N$; $j = 0, ..., d$) be some real coefficients such that the functions

$$\begin{aligned} u &= \sum_{i=1}^{N} b_{i0} f_i, \\ v_j &= \sum_{i=1}^{N} b_{ij} f_i, \qquad j = 1, ..., d \end{aligned} \tag{2.27}$$

are solutions of the homogeneous counterpart of system (1.14), i.e.,

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} \partial_{x_j} v_j = 0 \\ \partial_t v_j + \alpha_j \partial_{x_j} u = 0 \qquad (j = 1, ..., d). \end{cases} \tag{2.28}$$

Then it is easy to check that there exists a unique $k \in \{1, ..., d\}$, such that $b_{ik} \neq 0$ for some $i \in \{1, ..., N\}$, which gives $v_j = 0$ if $j \neq k$, In fact, from (2.27)–(2.28) we have

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} v_j = \sum_{i=1}^{N} \left(b_{i0} \partial_t f_i + \sum_{j=1}^{d} b_{ij} \partial_{x_j} f_i \right) = 0,$$

$$\partial_t v_j + \alpha_j \partial_{x_j} u = \sum_{i=1}^{N} (b_{ij} \partial_t f_i + \alpha_j b_{i0} \partial_{x_j} f_i) = 0,$$

for $j = 1, ..., N$. So, by using (2.26) and the independence of $\partial_{x_j} f_i$, we obtain the following conditions on the coefficients, for any $i = 1, ..., d$, $j = 1, ..., N$:

$$b_{ij} \lambda_{ij} = a_j b_{i0}, \qquad b_{ij} = b_{i0} \lambda_{ij}, \qquad b_{ij} \lambda_{il} = 0, \quad l \neq j. \tag{2.29}$$

From (2.29) it is easy to conclude that there exists no more than a unique value $k \in \{1, ..., d\}$ such that $\alpha_l = \lambda_{il} = b_{il} = 0$ if $l \neq k$, for any $i \in \{1, ..., N\}$, and $\alpha_k = \lambda_{ik}^2$, $b_{ik} \neq 0$ for some $i \in \{1, ..., N\}$.

3. GLOBAL EXISTENCE AND UNIFORM ESTIMATES

In this section we shall prove that if **M** is a Monotone Maxwellian function (**MMF**) with respect to a suitable interval I, which only depends on the L^∞ norm of the initial data, the Cauchy problem (1.3)–(1.4) possesses a globally bounded solution. Let us observe that in this section our arguments are completely independent from condition (1.7). In fact, for any fixed function **M**, which satisfies (1.6) and (2.9), we can construct a flux-function $\mathbf{A} = (A_1, ..., A_N)$ such that (1.7) holds, just by setting $A_j(u) = \sum_{i=1}^N \lambda_{ij} M_i(u)$ $(j = 1, ..., d)$. Our main results are the following.

THEOREM 3.1. *Let $u_0 \in L^\infty(\mathbb{R}^d)$ be fixed and set $\mathbf{f}_0 = \mathbf{M}(u_0)$. Assume that* **M** *is a* **MMF** *on the interval $I := \{u \in \mathbb{R} \mid |u| \leqslant \|u_0\|_\infty\}$. Then, for any $\varepsilon > 0$, there exists a (unique) global solution $\mathbf{f}^\varepsilon \in C([0, \infty); L^1_{loc}(\mathbb{R}^d))^N \cap L^\infty(\mathbb{R}^d \times \mathbb{R}_+)^N$ to the Cauchy problem* (1.3)–(1.4). *Moreover the following estimates hold*:

$$M_i(-\|u_0\|_\infty) \leqslant f_i^\varepsilon \leqslant \leqslant M_i(\|u_0\|_\infty,) \qquad i = 1, ..., N; \tag{3.1}$$

$$\|u^\varepsilon\|_\infty \leqslant \|u_0\|_\infty. \tag{3.2}$$

Set $\lambda_i = (\lambda_{i1}, ..., \lambda_{id}) \in \mathbb{R}^d$, $i = 1, ..., N$. For any open set Ω and $t > 0$, let us denote

$$\Omega_t^\lambda(s) := \{y \in \mathbb{R}^d \mid \exists x \in \Omega, \exists j \in \{1, ..., N\} \text{ s.t.: } y = x + (t - s)\lambda_j\}$$

for any $0 \leqslant s \leqslant t$. Clearly, $\Omega = \Omega_t^\lambda(t)$.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *let $\tilde{\mathbf{f}}^\varepsilon$ be another global solution with the same properties, associated to the initial condition $\tilde{\mathbf{f}}_0$. Then*

(a) *if $\mathbf{f}_0(x) \leqslant \tilde{\mathbf{f}}_0(x)$ for almost every $x \in \mathbb{R}^d$, then $\mathbf{f}^\varepsilon(x, t) \leqslant \tilde{\mathbf{f}}^\varepsilon(x, t)$ for almost every $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$;*

(b) *for any $\Omega \subseteq \mathbb{R}^d$ and $0 \leqslant s \leqslant t$*

$$\sum_{i=1}^N \int_\Omega |f_i^\varepsilon(x, t) - \tilde{f}_i^\varepsilon(x, t)| \, dx \leqslant \sum_{i=1}^N \int_{\Omega_t^\lambda(s)} |f_i^\varepsilon(x, s) - \tilde{f}_i^\varepsilon(x, s)| \, dx. \tag{3.3}$$

Let us observe that it is easy to extend the present results, as well as the results of the next section, to the case of initial data not in equilibrium, i.e., $\mathbf{f}_0 \neq \mathbf{M}(u_0)$, just by arguing as in [Na].

To prove Theorems 3.1 and 3.2 we need to recall some general results concerning semilinear (diagonal) hyperbolic systems. The proofs can be found in [HN] (in the general quasilinear weakly coupled case). Consider the Cauchy problem

$$\partial_t \mathbf{f} + \sum_{j=1}^{d} \mathbf{\Lambda}_j \partial_{x_j} \mathbf{f} = \mathbf{G}(\mathbf{f}) \tag{3.4}$$

with the initial condition

$$\mathbf{f}(x, 0) = \mathbf{f}_0(x). \tag{3.5}$$

Here $\mathbf{f} := (f_1, ..., f_N) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$, $\mathbf{\Lambda}_j = \operatorname{diag}(\lambda_{1j}, ..., \lambda_{Nj}) \in \mathscr{M}^{N \times N}$, the function $\mathbf{G}(\mathbf{f}) = (g_1(\mathbf{f}), ..., g_N(\mathbf{f})) : \mathbb{R}^N \to \mathbb{R}^N$ is a given Lipschitz continuous function, and $\mathbf{f}_0 = (f_{01}, ..., f_{0N}) \in L^\infty(\mathbb{R}^d)^N$.

Definition 3.3. A function $\mathbf{f} \in L^\infty(\mathbb{R}^d \times (0, T))^N$ $(T > 0)$ is a (weak) solution of the Cauchy problem (3.4)–(3.5) if, for all $\varphi \in C_0^\infty((\mathbb{R}^d \times (0, T))$ and every $i = 1, ..., N$, it holds

$$\iint \left[f_i \left(\partial_t \varphi + \sum_{j=1}^{N} \lambda_{ij} \partial_{x_j} \varphi \right) + g_i(\mathbf{f}) \varphi \right] \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.6}$$

and, for any open set $\Omega \subseteq \mathbb{R}^d$,

$$\lim_{T \to 0^+} \frac{1}{T} \int_0^T \int_\Omega |f_i(x, t) - f_{0i}(x)| \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{3.7}$$

Proposition 3.4. *For any $\mathbf{f}_0 \in L^\infty(\mathbb{R}^d)^N$, there is $T > 0$ (only depending on $\|\mathbf{f}_0\|_\infty$) such that there exists a unique (weak) solution $\mathbf{f}$ of (3.4)–(3.5) in $\mathbb{R}^d \times (0, T)$ and $\mathbf{f} \in C([0, T); L^1_{loc}(\mathbb{R}^d)^N)$. Moreover, there are only two possibilities: either $\mathbf{f} \in L^\infty(\mathbb{R}^d \times (0, T))^N$ for any $T > 0$, or there exists $T^* < +\infty$ such that, for any $T < T^*$, $\mathbf{f}$ is defined on $\mathbb{R}^d \times (0, T)$ and*

$$\lim_{T \to T^*_-} \|\mathbf{f}\|_{L^\infty(\mathbb{R}^d \times (0, T))} = +\infty.$$

One of the main tools in this paper is the monotonicity properties of some special systems. These properties hold under the so-called quasimonotonicity of the source term $\mathbf{G}$ (see [HN, Sm, PW]).

Proposition 3.5. *Let $\mathbf{f}$ and $\tilde{\mathbf{f}}$ be two weak solutions of problem (3.4)–(3.5) in $\mathbb{R}^d \times (0, T)$ for the initial data $\mathbf{f}_0$ and $\tilde{\mathbf{f}}_0$, respectively. Let $Q \subseteq \mathbb{R}^N$ be an interval (with non empty interior) such that* *each component $g_i$of* $\mathbf{G}$ *is nondecreasing in* f_j, *for* $i \neq j$, *for any* $f \in Q$ (3.8)

$$\mathbf{f}, \tilde{\mathbf{f}} \in Q \quad a.e.\ in \quad \mathbb{R}^d \times (0, T). \tag{3.9}$$

If $\mathbf{f}_0 \leqslant \tilde{\mathbf{f}}_0$ *for almost every* $x \in \mathbb{R}^d$, *then* $\mathbf{f} \leqslant \tilde{\mathbf{f}}$ *for almost every* $(x, t) \in \mathbb{R}^d \times (0, T)$.

Proof of Theorem 3.1. Set

$$M_i^{\#}(u) = \begin{cases} M_i(-\|u_0\|_\infty), & u < -\|u_0\|_\infty, \\ M_i(u), & |u| \leqslant \|u_0\|_\infty, \\ M_i(\|u_0\|)_\infty, & u > \|u_0\|_\infty, \end{cases} \tag{3.10}$$

for $i = 1, ..., N$. Assume also $u_0 \in \mathrm{Lip}_{loc}(\mathbb{R}^d)$ and let $\mathbf{f}^{\#} \in \mathrm{Lip}_{loc}(\mathbb{R}^d \times (0, \infty))^N$ the solution of problem (1.3)–(1.4) with $\mathbf{M}$ replaced by $\mathbf{M}^{\#}$ and $\mathbf{f}_0 = \mathbf{M}(u_0)$. For simplicity in this proof we shall omit the index ε. Let

$$T^\delta = \sup\{T \geqslant 0 \mid |u^{\#}(x, t)| \leqslant \|u_0\|_\infty + \delta/2, \qquad \text{a.e. in} \quad \mathbb{R}^d \times (0, T)\}$$

where $u^{\#} = \sum_{i=1}^N f_i^{\#}$. Clearly $T^\delta > 0$. Moreover, according to Definition 1.1 and thanks to the present assumptions, the functions $M_i^{\#}$ are monotone (nondecreasing) on $\mathbb{R}$. Then the system (1.3), with $\mathbf{M}^{\#}$, verifies the assumptions of Proposition 3.5. In particular its right-hand side is quasi-monotone in the strip $\mathbb{R}^d \times (0, T^\delta)$, i.e., it verifies condition (3.8).

Consider now the associated system of ordinary differential equations

$$\begin{cases} \dot{p}_i = \dfrac{1}{\varepsilon} (M_i^{\#}(v) - p_i), \\ p_i(0) = p_{0i}, \qquad i = 1,‘ ..., N, \end{cases} \tag{3.11}$$

for $v = \sum_{i=1}^N p_i$. If $|v(0)| \leqslant \|u_0\|_\infty$, then the global solution v is explicitly given by

$$v(t) = v(0) = \sum_{i=1}^N p_{0i}, \qquad p_i = e^{-t/\varepsilon} p_{0i} + (1 - e^{-t/\varepsilon})\, M_i(v(0)). \tag{3.12}$$

Let $p_i^{\pm}$ be the solution corresponding to the initial data

$$p_{0i}^{\pm} = M_i(\pm \|u_0\|_\infty)$$

and set $v^{\pm} = \sum_{i=1}^N p_i^{\pm}$. Then

$$v^{\pm}(t) = v^{\pm}(0) = \sum_{i=1}^N M_i(\pm \|u_0\|_\infty) = \pm \|u_0\|_\infty$$

and

$$p_i^{\pm}(t) = M_i(\pm \|u_0\|_\infty),$$

for any $i = 1, ..., N$.

Hence we can apply Proposition 3.5 to obtain by comparison

$$M_i(-\|u_0\|_\infty) \leqslant f_i^{\#} \leqslant M_i(\|u_0\|_\infty), \qquad i = 1, ..., N \tag{3.13}$$

and

$$|u^{\#}(x, t)| \leqslant \|u_0\|_\infty \tag{3.14}$$

for every $(x, t) \in \mathbb{R}^d \times (0, T^\delta)$. Therefore, by standard continuation arguments, $T^\delta = +\infty$ and (3.13), (3.14) hold in $\mathbb{R}^d \times (0, \infty)$ and on the range of $u^{\#}$ we have that $\mathbf{M} = \mathbf{M}^{\#}$. Then we obtain estimates (3.1)–(3.2) and the global existence for the solutions of our original problem. Finally it is easy to extend our result to general initial data by density arguments. ∎

Proof of Theorem 3.2. Let us omit in this proof the index ε. Part (a) is an easy consequence of the monotonicity arguments used in the previous proof and Proposition 3.5. To prove (b) we use the Duhamel formula

$$\begin{aligned} f_i(x, t) = {} & e^{-(t-s)/\varepsilon} f_i(x-(t-s)\,\lambda_i, s) \\ & + \int_s^t \frac{1}{\varepsilon}\, e^{-(t-\tau)/\varepsilon} M_i(u(x-(t-s-\tau)\,\lambda_i, \tau))\, \mathrm{d}\tau \end{aligned} \tag{3.15}$$

for $i = 1, ..., N$, $x \in \mathbb{R}^d$, $t \geqslant s \geqslant 0$. A similar formula holds for $\tilde{\mathbf{f}}$. Then, integrating both the identities against the function $\operatorname{sgn}(f_i - \tilde{f}_i)$ over a fixed open region $\Omega \subseteq \mathbb{R}^d$ and taking the sum on the indexes $i = 1, ..., N$, we obtain

$$\begin{aligned} & \sum_{i=1}^N \int_\Omega |f_i(x, t) - \tilde{f}_i(x, t)|\, \mathrm{d}x \\ & \leqslant e^{-(t-s)/\varepsilon} \sum_{i=1}^N \int_{\Omega_t^\lambda(s)} |f_i(x, s) - \tilde{f}_i(x, s)|\, \mathrm{d}x \\ & \quad + \int_s^t \frac{1}{\varepsilon}\, e^{-(t-\tau)/\varepsilon} \int_{\Omega_t^\lambda(\tau)} \sum_{i=1}^N |M_i(u(x, \tau)) - M_i(\tilde{u}(x, \tau))|\, \mathrm{d}x\, \mathrm{d}\tau. \end{aligned}$$

Let us observe now that, since $u, \tilde{u} \in I$, we can apply the monotonicity properties of the function $\mathbf{M}$,

$$\sum_{i=1}^{N} |M_i(u) - M_i(\tilde{u})| = \left(\sum_{i=1}^{N} \int_0^1 M_i'(\alpha u + (1-\alpha)\tilde{u})\, d\alpha \right) |u - \tilde{u}|$$
$$= |u - \tilde{u}| \leqslant \sum_{i=1}^{N} |f_i - \tilde{f}_i|,$$

since $\sum_{i=1}^{N} M_i'(u) = 1$, for almost every $u \in I$.

Set

$$\mu^t(s) := \int_{\Omega_t^\lambda(s)} \sum_{i=1}^{N} |f_i(x,t) - \tilde{f}_i(x,t)|\, dx$$

for $0 \leqslant s \leqslant t$. We have

$$\mu^t(t) \leqslant e^{-(t-s)/\varepsilon} \mu^t(s) + \int_s^t \frac{1}{\varepsilon} e^{-(t-\tau)/\varepsilon} \mu^t(\tau)\, d\tau \tag{3.16}$$

which implies

$$\mu^t(t) \leqslant \mu^t(s) \tag{3.17}$$

for any $t \geqslant s \geqslant 0$. This concludes the proof. ∎

4. CONVERGENCE TO THE ENTROPY SOLUTION

In this section we establish the convergence, as $\varepsilon \downarrow 0$, of the sequence $\mathbf{f}^\varepsilon$ of solutions of problem (1.3)–(1.4) to a limit function $\mathbf{f}$. Let $u = \sum_{i=1}^{N} f_i = \lim_{\varepsilon \to 0} u^\varepsilon$ (in a suitable topology). We also prove that $\mathbf{f}$ is a Maxwellian distribution, i.e., $\mathbf{M}(u) = \mathbf{f}$, and u is the (unique) entropy solution of problem (1.1)–(1.2) according to the following now classical definition [Kr].

DEFINITION 4.1. A function $u \in L^\infty(\mathbb{R}^d \times (0,T))$ $(T > 0)$ is an entropy solution of problem (1.1)–(1.2) if:

(i) for any $k \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$, $\varphi \geqslant 0$, we have

$$\iint \left\{ |u-k|\, \partial_t \varphi + \operatorname{sgn}(u-k) \sum_{j=1}^{d} (A_j(u) - A_j(k))\, \partial_{x_j} \varphi \right\} dx\, dt \geqslant 0 \tag{4.1}$$

(ii) for any interval $I \subseteq \mathbb{R}^d$

$$\lim_{T \to 0^+} \frac{1}{T} \int_0^T \int_I |u(x, t) - u_0(x)| \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{4.2}$$

Our convergence theorem is the following.

THEOREM 4.2. *Let $u_0 \in L^\infty(\mathbb{R}^d)$ be fixed and set $\mathbf{f}_0 = \mathbf{M}(u_0)$. Assume that* $\mathbf{M}$ *is a* **MMF** *on the interval $I := \{u \in \mathbb{R} \mid |u| \leqslant \|u_0\|\}$. Let $\mathbf{f}^\varepsilon \in C([0, \infty); L^1_{loc}(\mathbb{R}^d)^N) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}_+)^N$ be the solution of problem* (1.3)–(1.4) *($\varepsilon > 0$) given by Theorem* 3.1. *Let $u \in C((0, \infty); L^1_{loc}(\mathbb{R}^d))$ be the (unique) entropy solution of problem* (1.1)–(1.2). *Then, as $\varepsilon \to 0^+$,*

$$u^\varepsilon := \sum_{i=1}^N f_i^\varepsilon \to u \qquad \textit{in} \quad C([0, \infty); L^1_{loc}(\mathbb{R}^d)), \tag{4.3}$$

$$v_j^\varepsilon := \sum_{i=1}^N \lambda_{ij} f_i^\varepsilon \to A_j(u) \qquad \textit{in} \quad C([0, \infty); L^1_{loc}(\mathbb{R}^d)), \tag{4.4}$$

for $j = 1, ..., N$.

The proof of this theorem will follow after a sequence of preliminary results.

PROPOSITION 4.3. *Under the assumptions of Theorem* 4.2, *let $\mathbf{f}^\varepsilon$ be the solution of problem* (1.3)–(1.4). *Then, for any bounded open set $\Omega \subseteq \mathbb{R}^d$, there exist a positive constant $h_0 > 0$ and a continuous nondecreasing function $\omega \in C((0, h_0])$, not depending on ε and with $\omega(0) = 0$, such that, for every $t \geqslant 0$*

$$\int_\Omega \sum_{i=1}^N |f_i^\varepsilon(x+h, t) - f_i^\varepsilon(x, t)| \, dx \leqslant \omega(|h|), \tag{4.5}$$

for any $h \in \mathbb{R}^d$, $|h| \leqslant h_0$.

Proof. The proof is a direct consequence of Theorem 3.2, by taking $\tilde{\mathbf{f}}^\varepsilon(x, t) = \mathbf{f}^\varepsilon(x+h, t)$. ∎

To establish the equicontinuity in time of the sequences u^ε and $\mathbf{f}^\varepsilon$, we need the following interpolation lemma due to Kružkov [Kr].

LEMMA 4.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded convex open set and set $\Omega_{h_0} = \{x \in \mathbb{R}^d \mid d(x, \Omega) < h_0\}$, for some fixed $h_0 > 0$. Let w be a measurable bounded* *function in* $\Omega_{h_0} \times (0, T)$ $(T > 0)$ *and let* $\omega_\Omega \in C([0, h_0))$ *be a nondecreasing function, with* $\omega_\Omega(0) = 0$, *such that for every* $t \in (0, T)$, $|h| \leqslant h_0$

$$\int_{\Omega_{h_0}} |w(x+h, t) - w(x, t)| \, dx \leqslant \omega_\Omega(|h|). \tag{4.6}$$

Assume the following condition holds,

$$\left| \int_\Omega (w(x, t+\tau) - w(x, t)) \, \varphi(x) \, dx \right| \leqslant C_\Omega \tau \, \|\varphi\|_{C^2}, \tag{4.7}$$

for any $t, t+\tau \in (0, T)$ $(\tau > 0)$, *for any* $\varphi \in C_0^2(\Omega)$ *and some constant* $C_\Omega > 0$. *Then for any* $0 \leqslant t \leqslant t+\tau \leqslant T$ *we have*

$$\int_\Omega |w(x, t+\tau) - w(x, t)| \, dx \leqslant \tilde{\omega}_\Omega(\tau), \tag{4.8}$$

where

$$\tilde{\omega}_\Omega(\tau) = C_\Omega \min_{|h| \leqslant h_0} \left[|h| + \omega_\Omega(|h|) + \frac{\tau}{h^2} \right].$$

PROPOSITION 4.5. *Under the assumptions of Theorem* 4.2, *let* $u^\varepsilon = \sum_{i=1}^N f_i^\varepsilon$. *Then for any bounded convex open set* $\Omega \subseteq \mathbb{R}^d$ *there exist a positive constant* $\tau_0 > 0$ *and a continuous nondecreasing function* $\tilde{\omega} \in C([0, \tau_0))$, *not depending on* ε *and with* $\tilde{\omega}(0) = 0$ *such that for every* $0 \leqslant t \leqslant t+\tau$ $(\tau \in (0, \tau_0))$ *it holds*

$$\int_\Omega |u^\varepsilon(x, t+\tau) - u^\varepsilon(x, t)| \, \mathrm{d}x \leqslant \tilde{\omega}(\tau). \tag{4.9}$$

Proof. Using Lemma 4.4 and thanks to Proposition 4.3, we have just to establish the inequality (4.7) for u^ε. In fact, for any $\varphi \in C_0^2(\Omega)$, we have

$$\begin{aligned}
&\left| \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x, t)) \, \varphi(x) \, \mathrm{d}x \right| \\
&\quad = \left| \int_\Omega \left(\int_t^{t+\tau} \partial_t u^\varepsilon(x, s) \, \mathrm{d}s \right) \varphi(x) \, \mathrm{d}x \right| \\
&\quad = \left| \int_\Omega \left(-\int_t^{t+\tau} \sum_{i=1}^N \sum_{j=1}^d \lambda_{ij} \, \partial_{x_j} f_i^\varepsilon(x, s) \, \mathrm{d}s \right) \varphi(x) \, \mathrm{d}x \right| \\
&\quad = \left| \int_\Omega \sum_{i=1}^N \sum_{j=1}^d \lambda_{ij} \left(\int_t^{t+\tau} f_i^\varepsilon(x, s) \, \mathrm{d}s \right) \partial_{x_j} \varphi(x) \, \mathrm{d}x \right| \\
&\quad = C_\Omega \tau \, \|\varphi\|_{C^1(\Omega)}. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 4.6. *Under the assumptions of Theorem 4.2, let* $\mathbf{f}^\varepsilon$ *be the solution of problem* (1.3)–(1.4). *Then, for any bounded convex open set* $\Omega \subseteq \mathbb{R}^d$ *and for any* $\nu > 0$, *there exists a positive constant* $\tau_0 > 0$ *and a continuous nondecreasing function* $\tilde{\omega}^\nu \in C([0, \tau_0))$, *not depending on* ε *and with* $\tilde{\omega}^\nu(0) = 0$, *such that, for every* $\nu \leqslant t \leqslant t + \tau$ $(\tau \in (0, \tau_0))$, *there holds*

$$\int_\Omega \sum_{i=1}^N |f_i^\varepsilon(x, t+\tau) - f_i^\varepsilon(x)| \, dx \leqslant \tilde{\omega}^\nu(\tau), \tag{4.10}$$

for any $0 < \varepsilon < \nu$.

Proof. Thanks to Lemma 4.4 and inequality (4.5), we have only to prove that for any bounded convex open set $\Omega \subseteq \mathbb{R}^d$ there exists a constant $C > 0$ such that

$$\left| \int_\Omega (f_i^\varepsilon(x, t+\tau) - f_i^\varepsilon(x, t)) \, \varphi(x) \, dx \right| \leqslant C\tau \, \|\varphi\|_{C^2} \tag{4.11}$$

for $i = 1, ..., N$, $\nu \leqslant t \leqslant t + \tau$ and for any $\varphi \in C_0^2(\Omega)$.

From the Duhamel formula (3.15) and for any $\varphi \in C_0^2(\mathbb{R}^d)$ we have, omitting for simplicity the index ε,

$$\int f_i(x, t) \, \varphi(x) \, dx = \int e^{-t/\varepsilon} f_{0i}(x) \, \varphi(x + \lambda_i t) \, dx$$
$$+ \int_0^t \frac{1}{\varepsilon} e^{-(t-s)/\varepsilon} \int M_i(u(x, s)) \, \varphi(x + \lambda_i(t-s)) \, dx \, ds$$

for $i = 1, ..., N$. Then

$$\begin{aligned} I(t) &= \left| \int (f_i(x, t+\tau) - f_i(x, t)) \, \varphi(x) \, dx \right| \\ &\leqslant \left| \int (e^{-(t+\tau)/\varepsilon} \varphi(x + \lambda_i(t+\tau)) - e^{-t/\varepsilon} \varphi(x + \lambda_i t)) \, f_{0i}(x) \, dx \right| \\ &\quad + \frac{1}{\varepsilon} \left| \int_0^{t+\tau} e^{-(t+\tau-s)/\varepsilon} \int M_i(u(x, s)) \, \varphi(x + \lambda_i(t+\tau-s)) \, dx \, ds \right. \\ &\quad \left. - \int_0^t e^{-(t-s)/\varepsilon} \int M_i(u(x, s)) \, \varphi(x + \boldsymbol{\lambda}_i(t-s)) \, dx \, ds \right| \\ &= I_1 + \frac{1}{\varepsilon} I_2. \end{aligned}$$

First we have, for $v \leqslant t \leqslant t+\tau$

$$I_1 \leqslant \left| \int (e^{-(t+\tau)/\varepsilon} - e^{-t/\varepsilon})\, \varphi(x+\lambda_i(t+\tau))\, f_{0i}(x)\, \mathrm{d}x \right|$$

$$+ \left| \int e^{-t/\varepsilon} (\varphi(x+\boldsymbol{\lambda}_i(t+\tau)) - \varphi(x+\boldsymbol{\lambda}_i t))\, f_{0i}(x)\, \mathrm{d}x \right|$$

$$\leqslant C \frac{e^{-v/\varepsilon}}{\varepsilon} \tau \, \|f_{0i}\|_\infty \, \|\varphi\|_{C^1}.$$

For the next term we have

$$I_2 \leqslant \left| \int_0^t \int [M_i(u(x, s+\tau)) - M_i(u(x, s))]\, e^{-(t-s)/\varepsilon} \varphi(x+\lambda_i(t-s))\, \mathrm{d}x\, \mathrm{d}s \right|$$

$$+ \left| \int_0^\tau \int M_i(u(x, s))\, e^{-(t+\tau-s)/\varepsilon} \varphi(x+\lambda_i(t-s))\, \mathrm{d}x\, \mathrm{d}s \right|.$$

Then, since M_i is Lipschitz continuous and by using Proposition 4.5, it follows

$$I_2 \leqslant C\tau \, \|\varphi\|_{C^1} \left(\varepsilon + \frac{e^{-t/\varepsilon}}{\varepsilon} \right).$$

Therefore, the conclusion follows by summing up the different estimates. ∎

Next we can estimate the deviation from the equilibrium in the L^1 norm.

PROPOSITION 4.7. *Under the assumptions of Theorem* 4.2, *suppose that the initial data* u_0 *is locally of bounded variation. Then there exists a constant* $C > 0$ *such that, for any open set* $\Omega \subseteq \mathbb{R}^d$ *and any* $t > 0$

$$\int_\Omega \sum_{i=1}^N |f_i^\varepsilon - M_i(u^\varepsilon)|\, \mathrm{d}x \leqslant C\varepsilon \sum_{i=1}^N \|f_{0i}\|_{BV(\Omega_i^\lambda(0))}. \tag{4.12}$$

Proof. For any $\varepsilon > 0$ and $i = 1, ..., N$, take smooth initial data. Then we have

$$\partial_t (f_i^\varepsilon - M_i(u^\varepsilon)) + \frac{1}{\varepsilon} (f_i^\varepsilon - M_i(u^\varepsilon))$$

$$= - \sum_{j=1}^d \left(\lambda_{ij}\, \partial_{x_j} f_i^\varepsilon + M_i'(u^\varepsilon) \sum_{k=1}^N \lambda_{kj}\, \partial_{x_j} f_k^\varepsilon \right).$$

Hence, after integration and by standard methods, we obtain that there exists a constant $C>0$ such that

$$\int_\Omega \sum_{i=1}^N |f_i^\varepsilon - M_i(u^\varepsilon)|\,\mathrm{d}x \leqslant C\int_0^t e^{-(t-s)/\varepsilon}\int_\Omega \sum_{i=1}^N \sum_{j=1}^d |\partial_{x_j} f_i^\varepsilon|\,\mathrm{d}x\,\mathrm{d}s.$$

Then, from Proposition 9 and using density arguments, we obtain (4.12). ∎

The previous results were established to prove the compactness of the sequence $\mathbf{f}^\varepsilon$ in $C([0,\infty); (L^1_{loc}(\mathbb{R}^d))^N)$ via the classical Fréchet–Kolmogorov and Ascoli Theorems. Let us study now the consistency of our approximation with the entropy inequalities (4.1). To this purpose we introduce some special kinetic entropy functions in the spirit of [Br1, PT]. Since we need to define the function $\mathbf{M}$ for all $u \in \mathbb{R}$, let us set through the remainder of the paper $\mathbf{M} = \mathbf{M}^\#$, the last function being defined by (3.10).

For every $k \in \mathbb{R}$ take

$$\Phi^k(\mathbf{f}) = \sum_{i=1}^N |f_i^\varepsilon - M_i(k)|$$

and

$$\Psi_j^k(\mathbf{f}) = \sum_{i=1}^N \lambda_{ij}\,|f_i^\varepsilon - M_i(k)|, \qquad j=1, ..., N.$$

Therefore the functions $\Phi^k(\mathbf{f})$ form a family of kinetic entropy functions for system (1.3), with entropy fluxes given by $(\Psi_1^k(\mathbf{f}), ..., \Psi_N^k(\mathbf{f}))$, and we have the following H-Theorem.

PROPOSITION 4.8. *For every solution* $\mathbf{f}^\varepsilon \in C([0,\infty); L^1_{loc}(\mathbb{R}^d)^N)$ *of problem* (1.3)–(1.4), *for every* $k \in \mathbb{R}$, *and for any* $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,\infty))$, $\varphi \geqslant 0$, *we have*

$$\iint \sum_{i=1}^N |f_i^\varepsilon - M_i(k)| \left(\partial_t \varphi + \sum_{j=1}^d \lambda_{ij}\,\partial_{x_j}\varphi\right)\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant \frac{1}{\varepsilon}\left(\iint \left[\sum_{i=1}^N |f_i^\varepsilon - M_i(k)| - |u^\varepsilon - k|\right]\varphi\,\mathrm{d}x\,\mathrm{d}t\right) \geqslant 0. \tag{4.13}$$

Proof. To obtain (4.13), we first multiply Eq. (1.3) by $\mathrm{sgn}(f_i^\varepsilon - M_i(k))\,\varphi(x,t)$, take the sum for $i=1, ..., N$ and integrate over $\mathbb{R}^d \times (0,\infty)$. Then we have

$$\iint \sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)| \left(\partial_t \varphi + \sum_{j=1}^{d} \lambda_{ij} \partial_{x_j} \varphi \right) \mathrm{d}x\, \mathrm{d}t$$

$$= \frac{1}{\varepsilon} \iint \sum_{i=1}^{N} \operatorname{sgn}(\ {}_i^\varepsilon - M_i(k))(f_i^\varepsilon - M_i(u^\varepsilon))\ \varphi\ \mathrm{d}x\, \mathrm{d}t$$

$$= \frac{1}{\varepsilon} \iint \sum_{i=1}^{N} [(|f_i^\varepsilon - M_i(k)| - \operatorname{sgn}(f_i^\varepsilon - M_i(k))(M_i(u^\varepsilon) - M_i(k))]\ \varphi\ \mathrm{d}x\, \mathrm{d}t.$$

On the other hand we have

$$\begin{aligned} \sum_{i=1}^{N} & \operatorname{sgn}(f_i^\varepsilon - M_i(k))(M_i(u^\varepsilon) - M_i(k)) \\ & \leqslant \sum_{i=1}^{N} |M_i(u^\varepsilon) - M_i(k)| \\ & = |u^\varepsilon - k| \leqslant \sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)|. \end{aligned}$$

Then the proof is complete. ▌

Proof of Theorem 4.2. First let us take the initial data $u_0 \in BV_{loc}(\mathbb{R}^d)$. Then, thanks to Theorem 3.1, 3.2, Propositions 4.3, 4.5, 4.6, 4.7, and by standard compactness arguments, there exists $\mathbf{f} \in L^\infty(\mathbb{R}^d \times (0, \infty))^N$ which is the limit as $\varepsilon \downarrow 0$ in $C([0, \infty); (L^1_{loc}(\mathbb{R}^d))^N)$ of $\mathbf{f}^\varepsilon$ and such that, setting $u = \sum_{i=1}^N f_i$, we have the convergences (4.3) and (4.4). In particular, thanks to Propositions 4.5, 4.7, we can prove the convergence in this topology even for $t = 0$. The case of initial data u_0 which are not of bounded variation is considered just following step by step the arguments given in the proof of Theorem 5.1 in [Na]. Hence we obtain the convergence result and moreover, thanks to Proposition 4.7, $\mathbf{f} = \mathbf{M}(u)$. To show that the limit function u verifies the entropy inequalities (4.1) we just remark that for almost every $(x, t) \in \mathbb{R}^d \times (0, \infty)$

$$\sum_{i=1}^{N} |f_i - M_i(k)| = \sum_{i=1}^{N} |M_i(u) - M_i(k)| = |u - k| \tag{4.14}$$

and for any $j = 1, ..., N$,

$$\begin{aligned} \sum_{i=1}^{N} \lambda_{ij} |f_i - M_i(k)| &= \sum_{i=1}^{N} \lambda_{ij} |M_i(u) - M_i(k)| \\ &= \operatorname{sgn}(u - k)(A_j(u) - A_j(k)). \end{aligned} \tag{4.15}$$

The conclusion follows. ▌

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