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The geometric realizations of the decompositions of 3-orbifold fundamental groups

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Abstract

We introduce a type of generalized orbifold called an "orbifold composition". We study their topology and the extensions and deformations of the maps between them. As the main goal, we obtain the theorems which yield the geometric realizations of amalgamated free products and HNN extensions of 3-orbifold fundamental groups. They are extensions of results of Feustel (1972; 1973) and Feustel and Gregorac (1973).

Keywords: Orbifold fundamental group; Orbifold composition; Amalgamated free product; HNN extension

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0. Introduction

We can say that there are three principal results in the classical 3-manifold theory. The first one is Waldhausen's classification theorem on Haken manifolds (1968). The second one is the theorem on the geometric realization of the decomposition of the fundamental group by Feustel [5,6] and Feustel and Gregorac [7]. The last one is the torus decomposition theorem by Jaco and Shalen [11] and Johannson [10]. In each case, the authors use mainly "cut-and-paste" methods, that is, the methods of modifications of mappings, and cuttings and pastings of manifolds along certain surfaces.

In [22], Thurston addressed the conjecture that each piece of the torus decomposition described above admits some geometric structure, and proved that Haken manifolds admit a hyperbolic structure. His work originated the modern 3-manifold theory, which

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is strongly related to differential geometry, especially to hyperbolic geometry. Solving the Smith Conjecture, Thurston used orbifolds, which are a kind of generalized manifold.

It is quite natural to extend results for manifolds to those for orbifolds. Indeed, Satake proved the Gauss–Bonnet theorem for orbifolds [19], which first introduced the notion of orbifolds. Let us consider the extensions of the above classical results for 3-manifolds. Bonahon and Siebenmann [1] proved the toric orbifold decomposition theorem. As for Waldhausen's classification theorem for orbifolds, Zimmermann [25] showed its analogue under the assumption of the existence of geometric decompositions. Takeuchi [21] did this for finitely good orbifolds, and Takeuchi and Yokoyama [23] classified a larger class of orbifolds than the class classified in [21].

The remaining result is the geometric realization of the decomposition of the orbifold fundamental group, which is the subject of this paper. In [21,23,24], the authors proved some useful theorems. We use these, prove some others, and obtain the following two results:

Theorem 7.6. *Let M be a compact, orientable, and irreducible* 3*-orbifold. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose S algebraically splits $\pi_1^{orb}(M)$ as an amalgamated free product $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

Theorem 7.10. *Let M be a compact, orientable, and irreducible* 3*-orbifold. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose S algebraically splits $\pi_1^{orb}(M)$ as an HNN extension $\langle A, t \mid t^{-1}H_1t = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

The statements of Theorems 7.6 and 7.10 are completely parallel to those of Feustel and Gregorac's theorems, which are as follows:

Theorem 0.1 [5,6]. *Let M be a compact, orientable, and irreducible* 3*-manifold. Let S be a closed and orientable* 2*-manifold which is not the* 2*-sphere. Suppose S algebraically splits $\pi_1(M)$ as an amalgamated free product $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

Theorem 0.2 [7]. *Let M be a compact, orientable, and irreducible* 3*-manifold. Let S be a closed and orientable* 2*-manifold which is not the* 2*-sphere. Suppose S algebraically splits $\pi_1(M)$ as an HNN extension $\langle A, t \mid tH_1t^{-1} = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

In this paper, for the reader's convenience, we review some basic facts on orbifolds in Section 1, and on group actions on trees in Section 2.

In Section 3, an orbifold composition is defined which is made from several orbifolds by attaching them together through certain orbi-maps. In addition, we study coverings and the fundamental group of an orbifold composition.

In Section 4, we focus on the universal covering. Let X be an n-orbifold composition and X^0, X^1 be two suborbifold compositions derived from X by cutting open along an $(n-1)$-suborbifold of X. We construct the universal covering of X by the "tree construction" and show that $\pi_1^{orb}(X)$ is the free product of $\pi_1^{orb}(X^0)$ and $\pi_1^{orb}(X^1)$ with an amalgamation. The HNN extension case is also investigated.

Section 5 concerns orbi-maps. We study the fixed points of a spherical subgroup of the deck transformation group of the universal covering of a 3-orbifold. Lemma 5.9 gives sufficient conditions for the extensions of orbi-maps from a discal 2-orbifold, spherical 2-orbifold, or the double of a ballic 3-orbifold. By this lemma we can do extensions and constructions of orbi-maps under almost the same conditions as in the manifold case. From this point of view the lemma is valuable in itself. In addition, its proof has the interesting implication that we may examine group actions through the topology of orbifolds.

Theorem 6.1 states that each component of the inverse image of a certain 2-suborbifold of X by an orbi-map from a 3-orbifold M to X is an incompressible 2-suborbifold of M, where X is an orbifold composition with some conditions on extendability of orbi-maps. We also prove some theorems (Theorems 6.2 and 6.3) which are used to decrease the number of components.

In the concluding section, we state and prove the main theorems, which enable us to realize the decompositions of the fundamental groups. Let us present an overview of the proof of Main Theorem 7.6, to see how effective our preparation has been.

(i) Recall that the fundamental group $\pi_1^{orb}(M)$ of a 3-orbifold M is decomposed as

$$\langle A_1 * A_2 \mid H_1 = H_2, \phi \rangle,$$

where H is isomorphic to the fundamental group $\pi_1^{orb}(S)$ of a closed, orientable and nonspherical 2-orbifold S. First we take $S \times I$ and the orbi-covering M_i associated with A_i and construct an orbifold composition X by attaching them. Sections 4 and 5.2 are used here. This newly constructed space X plays a role analogous to that of an Eilenberg–MacLane space.

(ii) Make an orbi-map $f : M \to X$ which induces an isomorphism from $\pi_1^{orb}(M)$ to $\pi_1^{orb}(X)$. For this, we need theorems from Sections 4 and 5.

(iii) Each component of the inverse image of S by f is an incompressible 2-suborbifold by Theorem 6.1. We decrease the numbers of these components by using Theorems 6.2 and 6.3 repeatedly. Finally, the inverse image has only one component, F, which actually realizes the decomposition of $\pi_1^{orb}(M)$.

The techniques developed in [21,23,24], and this paper, should prove very useful in the study of 3-orbifolds by cut-and-paste methods.

1. Preliminaries on orbifolds

Throughout this paper, all orbifolds are connected unless otherwise stated. For basic facts on orbifolds, see [22,1,4,21]. We review some theorems required in using cut-and-paste methods for 3-orbifolds. Theorems 1.1–1.3 are derived from equivariant theorems. (See [8,9,17,18,24].)

Theorem 1.1 (Loop theorem). *Let M be a good 3-orbifold with boundaries. Let F be a connected 2-suborbifold in ∂M. If* $\mathrm{Ker}(\pi_1^{orb}(F) \to \pi_1^{orb}(M)) \neq 1$, *then there exists a discal 2-suborbifold D properly embedded in M such that $\partial D \subset F$ and ∂D does not bound any discal 2-suborbifold in F.*

Theorem 1.2 (Dehn's lemma). *Let M be a good 3-orbifold with boundaries. Let γ be a simple closed curve in $\partial M - \Sigma M$ such that the order of $[\gamma]$ is n in $\pi_1^{orb}(M)$. Then there exists a discal suborbifold $D^2(n)$ properly embedded in M with $\partial D^2(n) = \gamma$.*

Theorem 1.3 (Sphere theorem). *Let M be a good 3-orbifold. Let $p : \widetilde{M} \to M$ be the universal cover of M. If $\pi_2(\widetilde{M}) \neq 0$, then there exists a spherical suborbifold S in M such that $[\widetilde{S}] \neq 0$ in $\pi_2(\widetilde{M})$, where $\widetilde{S}$ is any component of $p^{-1}(S)$.*

The next corollary is derived directly from Theorem 1.3.

Corollary 1.4. *Let M be a good 3-orbifold. If M is irreducible, then for any manifold covering $\widetilde{M}$ of M, $\pi_2(\widetilde{M}) = 0$.*

In the remaining part of this section, we demonstrate several propositions derived from Theorems 1.1–1.3. The proofs are almost the same as in the case of 3-manifolds as found in [23, Theorems 1.5–1.8].

Proposition 1.5. *Let M be a good 3-orbifold, F be a connected and incompressible 2-suborbifold which is 2-sided and properly embedded in M, and N be the orbifold derived from M by cutting open along F. Then, M is irreducible if and only if each component of N is irreducible.*

Proposition 1.6. *Let M be a good and locally orientable 3-orbifold, F be a connected and incompressible 2-suborbifold which is 2-sided and properly embedded in M, and N be the orbifold derived from M by cutting open along F. Then, for any component N' of N,* $\mathrm{Ker}(\pi_1^{orb}(N') \to \pi_1^{orb}(M)) = 1$.

Let M be a good 3-orbifold and F a connected 2-suborbifold which is properly embedded and 2-sided in M. It is clear that if $\mathrm{Ker}(\pi_1^{orb}(F) \to \pi_1^{orb}(M)) = 1$, then F is incompressible in M. Under some additional hypotheses, the converse stands.

Proposition 1.7. *Let M be a good and locally orientable* 3*-orbifold, and F be a connected* 2*-suborbifold which is* 2*-sided and properly embedded in M. If F is incompressible, then* $\mathrm{Ker}(\pi_1^{orb}(F) \to \pi_1^{orb}(M)) = 1$.

Proposition 1.8. *Let M be a good* 3*-orbifold, and F be a connected* 2*-suborbifold which is* 2*-sided and properly embedded in M. Let $p' : M' \to M$ be a covering and F' be a component of $p'^{-1}(F)$. Then*:

(i) *if F' is incompressible in M', then F is incompressible in M,*

(ii) *if M is locally orientable and F is incompressible in M, then F' is incompressible in M'.*

2. Preliminaries on some groups acting on trees

In [20], some fixed point theorems about group actions on trees are proved. Here we use restricted forms as follows.

Let T be a tree, i.e., a connected and simply connected 1-complex, and G be a group simplicially acting on T. Let $n \geqslant 1$ be an integer. Put

$$G_n = \langle a_1, \ldots, a_n \mid a_1^{\alpha_1} = \cdots = a_n^{\alpha_n} = (a_i a_j)^{\beta_{i,j}} = 1,\ 1 \leqslant i < j \leqslant n \rangle,$$

where $\alpha_i, \beta_{i,j} \geqslant 2$ are integers.

Proposition 2.1. *Let $p_1, p_2 \in T$ be fixed points of $g \in G$ and ℓ be the unique simple path from p_1 to p_2. Then any vertex and edge on ℓ are fixed by g.*

Proof. Since p_1, p_2 are fixed points of g, and ℓ is simple, $g(\ell)$ is a simple path from p_1 to p_2. Thus $\ell = g(\ell)$. Observe that any vertex and edge of ℓ are fixed by g. □

Lemma 2.2. *If $G = G_n$, then T has a fixed vertex of G_n or there is an edge E of T such that $G_n(E) = E$ and $G_n|E$ is orientation reversing.*

Proof. This follows directly from [20, Theorem 15, p. 18] and [20, Corollary 2, p. 64]. □

3. Orbifold compositions

From now on, we assume that all orbifolds are good, connected, and locally orientable, unless otherwise stated.

Definition 3.1. Let I, J be countable sets, X_i $(i \in I)$ be n-orbifolds, Y_j $(j \in J)$ be $(n-1)$-orbifolds. Let $f_j^\varepsilon : Y_j \times \varepsilon \to X_{i(j,\varepsilon)}$ be orbi-maps such that $(f_j^\varepsilon)_*$ are monic where $j \in J$, $i(j,\varepsilon) \in I$, $\varepsilon = 0, 1$. Then we call $X = (X_i, Y_j \times [0,1], f_j^\varepsilon)_{i \in I, j \in J, \varepsilon = 0,1}$ an *n-dimensional orbifold composition*. The maps f_j^ε are called the *attaching maps* of X. Each $X_i, Y_j \times [0,1]$ is called a *component of* X. The equivalence relation $\sim$ in

$$\coprod_{i \in I, j \in J} \big(|X_i| \cup (|Y_j| \times [0,1])\big)$$

is defined to be generated by

$$(y,\varepsilon) \sim \overline{f}_j^{\varepsilon}(y), \qquad \varepsilon = 0, 1, \quad y \in |Y_j|, \quad j \in J.$$

We call the identified space $\coprod_{i\in I, j\in J}(|X_i| \cup |Y_j| \times [0,1])/\sim$ the *underlying space of* X, denoted by $|X|$, and call the identified space

$$\left\{\left(\bigcup_{i\in I} \Sigma X_i\right) \cup \left(\bigcup_{j\in J} \Sigma\bigl(Y_j \times [0,1]\bigr)\right)\right\} \Big/ \sim$$

the *singular set of* X, denoted by ΣX.

From now on, we assume that the underlying space $|X|$ is connected. Note that $|X_i|$ and $|Y_j \times (0,1)|$ are embedded in $|X|$. As in the case of the "mapping cylinder", $f_j^{\varepsilon}(\varepsilon)$ may have intersections and self-intersections.

For an orbifold composition we consider a 1-complex $\mathcal{C}(X)$ as follows: Each vertex corresponds to each component X_i, each edge corresponds to each component $Y_j \times [0,1]$, and a vertex belongs to an edge if and only if for the corresponding components $Y_j \times [0,1]$ and X_i there exists an attaching map between them. The formal definition is given in the following.

Definition 3.2. Let $X = (X_i, Y_j \times [0,1], f_j^{\varepsilon})_{i\in I, j\in J, \varepsilon=0,1}$ be an orbifold composition. Define the identified space $\mathcal{C}(X)$ by $|X|/\approx$ where

$$x \approx y \Leftrightarrow \begin{cases} \text{there is some } i \in I \text{ such that } x, y \in |X_i|/\sim, \text{ or} \\ \text{there are some } j \in J \text{ and } t \in [0,1] \text{ such that } x, y \in |Y_j \times t|/\sim. \end{cases}$$

We call $\mathcal{C}(X)$, each X_i, each $Y_j \times [0,1]$, and each $Y_j \times \frac{1}{2}$, the *associated* 1-*complex*, a *vertex orbifold*, an *edge orbifold of* X, and the *core* of $Y_j \times [0,1]$, respectively.

Next we consider an isomorphism of orbifold compositions as a map which is a componentwise isomorphism and commutes with the attaching maps. See the following definition.

Definition 3.3. Let

$$X = \bigl(X_i, Y_j \times [0,1], f_j^{\varepsilon}\bigr)_{i\in I, j\in J, \varepsilon=0,1}, \qquad X' = \bigl(X'_k, Y'_\ell \times [0,1], g_\ell^{\varepsilon}\bigr)_{k\in K, \ell\in L, \varepsilon=0,1}$$

be orbifold compositions. We say that X and X' are *isomorphic* if there exist a set of maps $\{\varphi_i, \psi_j\}_{i\in I, j\in J}$ and bijections $\eta: I \to K$, $\xi: J \to L$ such that, after changing the orientations of $[0,1]$'s if necessary, the following conditions hold:

(1) for each $i \in I$, φ_i is an isomorphism (of orbifolds) from X_i to $X'_{\eta(i)}$. And for each $j \in J$, ψ_j is an isomorphism (of orbifolds) from $Y_j \times [0,1]$ to $Y'_{\xi(j)} \times [0,1]$,

(2) for each $j \in J$, and $\varepsilon = 0, 1$, $\varphi_{i(j,\varepsilon)} \circ f_j^{\varepsilon} = g_{\xi(j)}^{\varepsilon} \circ (\psi_j | Y_j \times \varepsilon)$.

The homeomorphism $h: |X| \to |X'|$ naturally induced by $\{\varphi_i, \psi_j\}_{i\in I, j\in J}$ is called an *isomorphism from* X *to* X'.

Definition 3.4. Let $X = (X_k, Y_\ell \times [0,1], f_\ell^\varepsilon)_{k\in K,\ell\in L,\varepsilon=0,1}$ and $X' = (X_i', Y_j' \times [0,1], {f'}_j^\varepsilon)_{i\in I,j\in J,\varepsilon=0,1}$ be orbifold compositions. We say that X' is a *covering of* X if there exists a set of maps $\{\varphi_i, \psi_j\}_{i\in I,j\in J}$ such that, after changing the orientations of $[0,1]$'s if necessary, the following conditions hold:

(1) each φ_i is a covering map (of orbifolds) from X_i' to X_{k_i}, where $k_i \in K$, and each ψ_j is a covering map (of orbifolds) from $Y_j' \times [0,1]$ to $Y_{\ell_j} \times [0,1]$, where $\ell_j \in L$,

(2) for each $j \in J$ and $\varepsilon = 0, 1$, $\varphi_{i(j,\varepsilon)} \circ {f'}_j^\varepsilon = f_{\ell_j}^\varepsilon \circ (\psi_j | Y_j' \times \varepsilon)$,

(3) the continuous map $p : |X'| \to |X|$, which is naturally induced by $\{\varphi_i, \psi_j\}_{i\in I,j\in J}$, is onto and induces the usual covering map from $|X'| - p^{-1}(\Sigma X)$ to $|X| - \Sigma X$.

We call the above map p a *covering map from X' to X*.

Remark 3.5. In the above definition, if each component X_i' is the universal cover of a component X_{k_i}, then for some base point $x_0 \in |X| - \Sigma X$, any path ℓ with the base point x_0 such that $\mathrm{Int}\,\ell \cap \Sigma X = \emptyset$, and any point $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift of ℓ with the base point $\tilde{x}_0$. This holds because the $(f_\ell^\varepsilon)_*$ are monic.

Definition 3.6. Let X be an orbifold composition, $x_0 \in |X| - \Sigma X$ a base point, ℓ a path with the base point x_0 such that $\mathrm{Int}\,\ell \cap \Sigma X = \emptyset$, and $p : \widetilde{X} \to X$ any covering. Fix any point $\tilde{x}_0 \in p^{-1}(x_0)$. Suppose there is a covering $\hat{p} : \widehat{X} \to \widetilde{X}$ such that each component of $\widehat{X}$ is the universal cover of a component of $\widetilde{X}$. Fix any point $\hat{x}_0 \in \hat{p}^{-1}(\tilde{x}_0)$. By Remark 3.5, there exists a unique lift $\hat{\ell}$ to $\widehat{X}$ of ℓ with the base point $\hat{x}_0$. Then we can determine a lift $\tilde{\ell}$ of ℓ uniquely, by putting $\tilde{\ell} = \hat{p} \circ \hat{\ell}$, which is called the *canonical lift of ℓ with the base point $\tilde{x}_0$*.

Definition 3.7. Let X', X be orbifold compositions, and $p : X' \to X$ a covering. We define the *deck transformation group* $\mathrm{Aut}(X', p)$ *of* p by

$$\mathrm{Aut}(X', p) = \{h : X' \to X' \mid h \text{ is an isomorphism such that } p \circ h = p\}.$$

Definition 3.8. Let $\widetilde{X}$, X be orbifold compositions, and $p : \widetilde{X} \to X$ a covering. We say that p is a *universal covering* if for any covering $p' : X' \to X$, there exists a covering $q : \widetilde{X} \to X'$ such that $p = p' \circ q$.

Lemma 3.9. *For any orbifold composition X, there exists a unique universal covering $p : \widetilde{X} \to X$.*

Proof. Put $X_0 = |X| - \Sigma X$. Let H be the normal subgroup of $\pi_1(X_0)$ normally generated by normal loops around ΣX. Then, the Fox completion of the covering of X_0 associated with H can be shown to be the universal cover of X in the sense of orbifold composition.

The uniqueness is derived from the facts that an orbi-covering is an ordinary covering on the nonsingular part and that the ordinary covering associated with the same subgroup is unique. □

We sometimes denote an orbifold composition or a good orbifold X by $(\widetilde{X}, p, |X|)$, where $p:\widetilde{X} \to X$ is the universal covering and $|X|$ is the underlying space of X. A good orbifold is considered as a special case of an orbifold composition.

Proposition 3.10. *Let $\widetilde{X}$, X be orbifold compositions and $p:\widetilde{X} \to X$ a covering. If the restriction of p to each component of $\widetilde{X}$ is universal and $\mathcal{C}(\widetilde{X})$ is a tree, then the covering $p:\widetilde{X} \to X$ is universal.*

Proof. Take any covering $p':X' \to X$. We construct a covering $q:\widetilde{X} \to X'$ as follows: take any point $\tilde{x}_0 \in |\widetilde{X}| - p^{-1}(\Sigma X)$ and fix it. For $\tilde{x} \in |\widetilde{X}|$, take a simple path $\tilde{\ell}_{\tilde{x}}$ with the base point $\tilde{x}_0$ and end point $\tilde{x}$, satisfying the following:

(1) $\tilde{\ell}_{\tilde{x}}(0,1) \subset |\widetilde{X}| - p^{-1}(\Sigma X)$.

(2) $\tilde{\ell}_{\tilde{x}}[0,1]/\approx$ is a simple path in $\mathcal{C}(\widetilde{X})$.

Put $x_0 = p(\tilde{x}_0)$, $\ell_{\tilde{x}} = p \circ \tilde{\ell}_{\tilde{x}}$, and let $x'_0 \in p'^{-1}(x_0)$. Let $\ell'_{\tilde{x}}$ be the canonical lift of $\ell_{\tilde{x}}$ with the base point x'_0. Then a mapping $q:\widetilde{X} \to X'$ is defined by $q(\tilde{x}) = \ell'_{\tilde{x}}(1)$. This map is well-defined, and we can verify that it is a covering and $p = p' \circ q$. □

Definition 3.11. Let $X = (\widetilde{X}, p, |X|)$ be an orbifold composition with the base point $x_0 \in |X| - \Sigma X$. Put

$$\Omega(\widetilde{X}, x_0) = \bigl\{\tilde{\alpha} \mid \tilde{\alpha}:[0,1] \to \widetilde{X} \text{ is a continuous map with } p(\tilde{\alpha}(0)) = p(\tilde{\alpha}(1)) = x_0\bigr\}.$$

For any two elements $\tilde{\alpha}, \tilde{\beta} \in \Omega(\widetilde{X}, x_0)$, $\tilde{\alpha}$ is *equivalent to* $\tilde{\beta}$, denoted by $\tilde{\alpha} \sim \tilde{\beta}$, if there exists an element $\tau \in \operatorname{Aut}(\widetilde{X}, p)$ such that $\tilde{\alpha}(0) = \tau(\tilde{\beta}(0))$ and $\tilde{\alpha}(1) = \tau(\tilde{\beta}(1))$. The relation $\sim$ is an equivalence relation and $\Omega(\widetilde{X}, x_0)/\sim$ is a group with the product defined by

$$[\tilde{\alpha}] \cdot [\tilde{\beta}] = \bigl[\tilde{\alpha} \cdot \rho(\tilde{\beta})\bigr],$$

where $\rho \in \operatorname{Aut}(\widetilde{X}, p)$ is the element such that $\rho(\tilde{\beta}(0)) = \tilde{\alpha}(1)$. The group $\Omega(\widetilde{X}, x_0)/\sim$ is called the *fundamental group of* X and is denoted by $\pi_1^{orb}(X, x_0)$. Note that the fundamental group $\pi_1^{orb}(X, x_0)$ is isomorphic to the deck transformation group $\operatorname{Aut}(\widetilde{X}, p)$. By the symbol σ_A, we mean the element of $\operatorname{Aut}(\widetilde{X}, p)$ which corresponds to $\sigma \in \pi_1^{orb}(X, x_0)$.

Definition 3.12. Let $X = (\widetilde{X}, p, |X|)$ and $Y = (\widetilde{Y}, q, |Y|)$ be orbifold compositions (or orbifolds). By an *orbi-map* $f:X \to Y$, we mean the pair $(\bar{f}, \tilde{f})$ of continuous maps $\bar{f}:|X| \to |Y|$ and $\tilde{f}:\widetilde{X} \to \widetilde{Y}$ satisfying

(i) $\bar{f} \circ p = q \circ \tilde{f}$,

(ii) for each $\sigma \in \operatorname{Aut}(\widetilde{X}, p)$, there exists $\tau \in \operatorname{Aut}(\widetilde{Y}, q)$ such that $\tilde{f} \circ \sigma = \tau \circ \tilde{f}$,

(iii) there exists $x \in |X| - \Sigma X$ such that $\bar{f}(x) \in |Y| - \Sigma Y$.

Definition 3.13. Let $X = (\widetilde{X}, p, |X|)$ and $Y = (\widetilde{Y}, q, |Y|)$ be orbifold compositions, and $f = (\bar{f}, \tilde{f}):X \to Y$ be an orbi-map. By the definition of an orbi-map, there exists a point $x \in |X| - \Sigma X$ such that $\bar{f}(x) \in |Y| - \Sigma Y$. Then the induced homomorphism $f_* : \pi_1^{orb}(X, x) \to \pi_1^{orb}(Y, \bar{f}(x))$ of f is naturally defined by $f_*([\tilde{\alpha}]) = [\tilde{f} \circ \tilde{\alpha}]$.

For an orbi-map and a covering between orbifold compositions, we can define the notions of C-equivalence, orbi-homotopy, and lifting as well as those for an orbi-map and a covering between orbifolds. We derive relations among fundamental groups, coverings, and liftings similar to those for orbifolds. See [21] for the orbifold case.

The next proposition can be proved in a way similar to one in [21, Proposition 2.2].

Proposition 3.14. *Let $X = (\widetilde{X}, p, |X|)$, $Y = (\widetilde{Y}, q, |Y|)$ be orbifold compositions, and $f = (\bar{f}, \tilde{f}) : X \to Y$ an orbi-map. Then for $[\tilde{\alpha}] \in \pi_1^{orb}(X, x)$,*

$$\tilde{f} \circ [\tilde{\alpha}]_A = \big(f_*\big([\tilde{\alpha}]\big)\big)_A \circ \tilde{f}.$$

4. The tree constructions of the universal coverings

4.1. The amalgamation case

Let X be an orbifold composition and $Y \times [0,1]$ one of the edge orbifold components of X. Suppose that $X - Y \times (0,1)$ consists of two disjoint orbifold compositions X^0 and X^1, and attaching orbi-maps from $Y \times \varepsilon$ are mapped into X^ε and denoted by

$$f^\varepsilon : Y \times \varepsilon \to X^\varepsilon, \quad \varepsilon = 0, 1.$$

We construct the universal covering of an orbifold composition X by the "tree construction", and show that the fundamental group $\pi_1^{orb}(X)$ of X is the free product of $\pi_1^{orb}(X^0)$ and $\pi_1^{orb}(X^1)$ with the amalgamated subgroups $f_*^\varepsilon \pi_1^{orb}(Y \times \varepsilon)$, $\varepsilon = 0, 1$.

Let $p^\varepsilon : \widetilde{X}^\varepsilon \to X^\varepsilon$, $\varepsilon = 0, 1$, and $q : \widetilde{Y} \times [0,1] \to Y \times [0,1]$ be the universal coverings. Put $H^\varepsilon = f_*^\varepsilon \pi_1^{orb}(Y \times \varepsilon)$ and $A^\varepsilon =$ (a left coset representative system of $\pi_1^{orb}(X^\varepsilon)$ by H^ε, which includes the identity e), $\varepsilon = 0, 1$. A group G is defined as the free product of $\pi_1^{orb}(X^0)$ and $\pi_1^{orb}(X^1)$ with the amalgamated subgroups H^0 and H^1, under the map $f_*^1 \circ (f_*^0)^{-1}$, denoted by

$$G = \big\langle \pi_1^{orb}(X^0) * \pi_1^{orb}(X^1) \mid H^0 = H^1,\ f_*^1 \circ (f_*^0)^{-1} \big\rangle.$$

And three subsets K, K^0, K^1 of G are defined by

$$K = \big\{e, a_1 a_2 \cdots a_m \mid a_i \neq e,\ a_i \in A^0 \cup A^1,$$
$$a_i,\ a_{i+1} \text{ are not both in } A^0 \text{ or both in } A^1\big\},$$
$$K^0 = \{e, a_1 a_2 \cdots a_m \in K \mid a_m \in A^1\},$$
$$K^1 = \{e, a_1 a_2 \cdots a_m \in K \mid a_m \in A^0\}.$$

For each $k \in K^\varepsilon$, prepare a copy $\widetilde{X}_k^\varepsilon$ of $\widetilde{X}^\varepsilon$, and the identity map $\mathrm{id}_k^\varepsilon : \widetilde{X}_k^\varepsilon \to \widetilde{X}^\varepsilon$. Note that there are $\# A^\varepsilon$ equivalence classes of $\mathrm{Aut}(\widetilde{X}^\varepsilon, p^\varepsilon) \tilde{f}^\varepsilon(\widetilde{Y} \times \varepsilon)$ mod $(H^\varepsilon)_A$, $\varepsilon = 0, 1$. For each $(k, a) \in K^0 \times A^0$, prepare a copy $\widetilde{Y}_{(k,a)} \times [0,1]$ of $\widetilde{Y} \times [0,1]$, and the identity map

$$\mathrm{id}_{(k,a)} : \widetilde{Y}_{(k,a)} \times [0,1] \to \widetilde{Y} \times [0,1].$$

Let $\tilde{f}^{\varepsilon}:\widetilde{Y}\times\varepsilon\to\widetilde{X}^{\varepsilon}$ be structure maps of f^{ε}, $\varepsilon=0,1$. Then we can define structure maps $\tilde{f}^{\varepsilon}_{(k,a)}:\widetilde{Y}_{(k,a)}\times\varepsilon\to\widetilde{X}^{\varepsilon}_{h}$ by

$$\tilde{f}^{\varepsilon}_{(k,a)}=\begin{cases}(\mathrm{id}^0_k)^{-1}\circ a_A\circ\tilde{f}^0\circ\mathrm{id}_{(k,a)}:\widetilde{Y}_{(k,a)}\times 0\to\widetilde{X}^0_k & \text{if } \varepsilon=0,\\ (\mathrm{id}^1_{ka})^{-1}\circ e_A\circ\tilde{f}^1\circ\mathrm{id}_{(k,a)}:\widetilde{Y}_{(k,a)}\times 1\to\widetilde{X}^1_{ka} & \text{if } \varepsilon=1,\ a\neq e,\\ (\mathrm{id}^1_e)^{-1}\circ e_A\circ\tilde{f}^1\circ\mathrm{id}_{(e,e)}:\widetilde{Y}_{(e,e)}\times 1\to\widetilde{X}^1_e & \text{if } \varepsilon=1,\ a=k=e,\\ (\mathrm{id}^1_\ell)^{-1}\circ a'_A\circ\tilde{f}^1\circ\mathrm{id}_{(k,e)}:\widetilde{Y}_{(k,e)}\times 1\to\widetilde{X}^1_\ell & \text{if } \varepsilon=1,\ a=e\neq k,\end{cases}$$

where $k=\ell a',\ \ell\in K^1,\ a'\in A^1$.

Put $\widetilde{X}=(\widetilde{X}^0_k,\widetilde{X}^1_\ell,\widetilde{Y}_{(k,a)}\times[0,1],\tilde{f}^0_{(k,a)},\tilde{f}^1_{(k,a)})_{k\in K^0,\ell\in K^1,a\in A^0}$. Define the projections $p^{\varepsilon}_k:\widetilde{X}^{\varepsilon}_k\to X^{\varepsilon}$and $q_{(h,a)}:\widetilde{Y}_{(h,a)}\times[0,1]\to Y\times[0,1]$ by $p^{\varepsilon}_k=p^{\varepsilon}\circ\mathrm{id}^{\varepsilon}_k$ and $q_{(h,a)}=q\circ\mathrm{id}_{(h,a)}$, $k\in K^{\varepsilon}$, $\varepsilon=0,1$, $(h,a)\in K^0\times A^0$, respectively. Note that p^{ε}_k and $q_{(h,a)}$ are the universal coverings. Furthermore, it is easy to see that $\mathcal{C}(\widetilde{X})$ is a tree. Hence by Proposition 3.10,

$$p=\bigcup_{\substack{k\in K^{\varepsilon},\ \varepsilon=0,1,\\ (h,a)\in K^0\times A^0}}\big(p^{\varepsilon}_k\cup q_{(h,a)}\big):\widetilde{X}\to X$$

is the universal covering.

Lemma 4.1. $\pi_1^{orb}(X,x_0)\cong G$.

Proof. Fix a base point $x_0\in\bar{f}^0(Y\times 0)-\Sigma X$ of X and X^0, and a base point $x_1\in\bar{f}^1(Y\times 1)-\Sigma X$ of X^1. Take a path $\ell:[0,1]\to|Y\times[0,1]|-\Sigma X$ such that $\ell(t)\in|Y\times t|$, $\bar{f}(\ell(0))=x_0$, and $\bar{f}(\ell(1))=x_1$. Fix a base point $\tilde{x}_0\in(p^0_e)^{-1}(x_0)$ of $\widetilde{X}^0_e$. Recall that

$$\mathrm{Aut}(\widetilde{X},p)\cong\pi_1^{orb}(X,x_0)=\Omega(\widetilde{X},x_0)/\sim.$$

Choose $\tilde{\alpha}\in\Omega(\widetilde{X},x_0)$ such that $\tilde{\alpha}(0)=\tilde{x}_0$, $\tilde{\alpha}/\approx$ is a simple path in the associated 1-complex $\mathcal{C}(\widetilde{X})$ of $\widetilde{X}$, and if $\tilde{\alpha}$ goes through $(q_{(k,a)})^{-1}(Y\times[0,1])$, $\tilde{\alpha}$ always uses a lift of ℓ by $q(k,a)$. The restriction of $\tilde{\alpha}$ to each vertex orbifold component is an element of $\pi_1^{orb}(X^0,x_0)$ or $\pi_1^{orb}(X^1,x_1)$. Denote such ordered elements by $g_1,\dots,g_m\in\pi_1^{orb}(X^{\varepsilon},x_0)$, $\varepsilon=0,1$, and define a map $\Phi:\Omega(\widetilde{X},x_0)\to G$ by $\Phi(\tilde{\alpha})=g_1\cdots g_m$.

For each $\tilde{\alpha}\in\Omega(\widetilde{X},x_0)$, there is a path $\tilde{\alpha}'\in\Omega(\widetilde{X},x_0)$ such that $\tilde{\alpha}\sim\tilde{\alpha}'$ and $\Phi(\tilde{\alpha}')=a_1\cdots a_r a h$, where $a_1\cdots a_r\in K^0$, $a\in A^0$ and $h\in H^0$ (possibly, $a=e$ and/or $h=e$). Since $\Phi(\tilde{\alpha})=\Phi(\tilde{\alpha}')$, we obtain the map $\overline{\Phi}:\Omega(\widetilde{X},x_0)/\sim\to G$ defined by $\overline{\Phi}([\tilde{\alpha}])=\Phi(\tilde{\alpha})$.

It is easy to verify that $\overline{\Phi}$ is injective, surjective, and homomorphic. □

4.2. The HNN case

Let X be an orbifold composition and $Y\times[0,1]$ one of the edge orbifold components of X. Suppose that $X-Y\times(0,1)$ is a (connected) orbifold composition X', and the attaching orbi-maps from $Y\times\varepsilon$ are denoted by $f^{\varepsilon}:Y\times\varepsilon\to X'$, $\varepsilon=0,1$. We construct the universal covering of X in a similar manner to the amalgamation case, and show that the fundamental group $\pi_1^{orb}(X)$ of X is the HNN extension of $\pi_1^{orb}(X')$.

Let $p : \widetilde{X}' \to X'$, and $q : \widetilde{Y} \times [0,1] \to Y \times [0,1]$ be the universal coverings. Put $H^\varepsilon = f_*^\varepsilon \pi_1^{orb}(Y \times \varepsilon)$ and $A^\varepsilon =$ (a left coset representative system of $\pi_1^{orb}(X')$ by H^ε, which includes the identity e), $\varepsilon = 0, 1$. A group G is defined as the HNN extension of $\pi_1^{orb}(X')$ relative to H^0, H^1 and $f_*^1 \circ (f_*^0)^{-1}$, denoted by

$$G = \langle \pi_1^{orb}(X'), t \mid t^{-1} H^0 t = H^1, f_*^1 \circ (f_*^0)^{-1} \rangle.$$

And a subset K of G is defined by

$$K = \{ e, a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \cdots a_m t^{\varepsilon_m} \mid a_i \neq e,\ a_i \in A^0 \cup A^1, \\ \text{if } a_i \in A^\varepsilon,\ \text{then } \varepsilon_i = (-1)^\varepsilon,\ \varepsilon = 0, 1 \}.$$

For each $k \in K$, prepare a copy $\widetilde{X}'_k$ of $\widetilde{X}'$, and the identity map $\mathrm{id}_k : \widetilde{X}'_k \to \widetilde{X}'$. Note that there are $\# A^\varepsilon$ equivalent classes of $\mathrm{Aut}(\widetilde{X}', p) \tilde{f}^\varepsilon(\widetilde{Y} \times \varepsilon)$ mod $(H^\varepsilon)_A$, $\varepsilon = 0, 1$. And for each $(k, a) \in K \times A^0$, prepare a copy $\widetilde{Y}_{(k,a)}$ of $\widetilde{Y}$, and the identity map $\mathrm{id}_{(k,a)} : \widetilde{Y}_{(k,a)} \times [0,1] \to \widetilde{Y} \times [0,1]$. Let $\tilde{f}^\varepsilon : \widetilde{Y} \times \varepsilon \to \widetilde{X}'$ be structure maps, $\varepsilon = 0, 1$. Then we can define structure maps $\tilde{f}^\varepsilon_{(k,a)} : \widetilde{Y}_{(k,a)} \times \varepsilon \to \widetilde{X}'_h$ by

$$\tilde{f}^\varepsilon_{(k,a)} = \begin{cases} (\mathrm{id}_k)^{-1} \circ a_A \circ \tilde{f}^0 \circ \mathrm{id}_{(k,a)} : \widetilde{Y}_{(k,a)} \times 0 \to \widetilde{X}'_k & \text{if } \varepsilon = 0, \\ (\mathrm{id}_{kat})^{-1} \circ e_A \circ \tilde{f}^1 \circ \mathrm{id}_{(k,a)} : \widetilde{Y}_{(k,a)} \times 1 \to \widetilde{X}'_{kat} & \text{if } \varepsilon = 1,\ a \neq e, \\ (\mathrm{id}_{kt})^{-1} \circ e_A \circ \tilde{f}^1 \circ \mathrm{id}_{(k,e)} : \widetilde{Y}_{(k,e)} \times 1 \to \widetilde{X}'_{kt} & \text{if } \varepsilon = \varepsilon' = 1,\ a = e, \\ (\mathrm{id}_\ell)^{-1} \circ a'_A \circ \tilde{f}^1 \circ \mathrm{id}_{(k,e)} : \widetilde{Y}_{(k,e)} \times 1 \to \widetilde{X}'_\ell & \text{if } \varepsilon = -\varepsilon' = 1,\ a = e, \end{cases}$$

where $k = \ell a' t^{\varepsilon'}$, $\ell \in K^1$.

Put $\widetilde{X} = (\widetilde{X}'_k, \widetilde{Y}_{(k,a)} \times [0,1], \tilde{f}^0_{(k,a)}, \tilde{f}^1_{(k,a)})_{k \in K, a \in A^0}$. Define the projections $p_k : \widetilde{X}'_k \to X'$ and $q_{(k,a)} : \widetilde{Y}_{(k,a)} \times [0,1] \to Y \times [0,1]$ by $p_k = p \circ \mathrm{id}_k$ and $q_{(k,a)} = q \circ \mathrm{id}_{(k,a)}$, $k \in K$, $\varepsilon = 0, 1$, respectively. As in the amalgamation case, we can see that $\bigcup (p_k \cup q_{(k,a)}) : \widetilde{X} \to X$ is the universal covering and obtain the following lemma.

Lemma 4.2. $\pi_1^{orb}(X, x_0) \cong G$.

5. Extensions and constructions of orbi-maps

Definition 5.1. Let X be an orbifold composition. Define

$$O_1(X) = \{ f : \partial D \to X \mid D \text{ is a discal 2-orbifold, } f \text{ is an orbi-map} \},$$

$$O_2(X) = \{ f : S \to X \mid S \text{ is a spherical 2-orbifold, } f \text{ is an orbi-map} \},$$

$$O_3(X) = \{ f : \mathcal{D}B \to X \mid \mathcal{D}B \text{ is the double of a ballic 3-orbifold } B, \\ f \text{ is an orbi-map} \}.$$

We call $f : \partial D \to X \in O_1(X)$ *trivial* if there exists an orbi-map $g : D \to X$ such that $g|\partial D = f$, and call $O_1(X)$ *trivial* if any element of $O_1(X)$ is trivial. We call $f : S \to X \in O_2(X)$ *trivial* if there exists an orbi-map $g : c * S \to X$ such that $g|S = f$, where $c * S$ is the cone on S, and call $O_2(X)$ *trivial* if any element of $O_2(X)$ is trivial. We define the trivialities of $O_3(X)$ similarly.

Note that if $O_i(X)$ is trivial, then any covering $\widetilde{X}$ of X inherits the triviality.

Proposition 5.2. *Let F be a compact 2-orbifold and X be an orbifold composition. If $O_1(X)$ is trivial, then for any homomorphism $\varphi:\pi_1^{orb}(F,y)\to\pi_1^{orb}(X,x)$, there exists an orbi-map $f:(F,y)\to(X,x)$ such that $f_*=\varphi$.*

Proof. Let $F_0=F-\operatorname{Int}U(\Sigma F)$, where $U(\Sigma F)$ is the small regular neighborhood of ΣF. We construct an orbi-map from F_0 to X associated with φ. Since $O_1(X)$ is trivial, This orbi-map is extendable to the desired orbi-map. □

The following Propositions 5.3 and 5.4 are proved similarly.

Proposition 5.3. *Let M be a compact 3-orbifold and X an orbifold composition such that $O_1(X)$ and $O_2(X)$ are trivial. Then for any homomorphism $\varphi:\pi_1^{orb}(M,x)\to\pi_1^{orb}(X,y)$, there exists an orbi-map $f:(M,x)\to(X,y)$ such that $f_*=\varphi$.*

Proposition 5.4. *Let M be a 3-orbifold and X be an orbifold composition such that $O_3(X)$ is trivial. If $f,g:M\to X$ are C-equivalent orbi-maps, then f and g are orbi-homotopic.*

The following Lemmas 5.5–5.7 give sufficient conditions which enable us to extend certain orbi-maps.

Lemma 5.5. *Let X be an orbifold composition, D a discal 2-orbifold, and $f:\partial D\to X$ an orbi-map. If $\operatorname{Fix}([f]_A)\neq\emptyset$, then f is extendable to an orbi-map from D to X.*

Proof. Let $q:D^2\to D$ be the universal covering. Choose a point $x\in\operatorname{Fix}([f]_A)$. We can construct the structure map of the desired orbi-map by mapping the cone point of D^2 to x and performing the skeletonwise and equivariant extension. □

Let S be a spherical 2-orbifold and $q:\widetilde{S}\to S$ the universal covering. Let τ be an element of $\pi_1^{orb}(S)$ and x_τ the point of ΣS such that $[\ell]^k=\tau$, where ℓ is the normal loop around x_τ and k is an integer. By the symbol $\mu(\ell)$, we mean the local normal loop around x_τ such that $\ell=m^{-1}\cdot\mu(\ell)\cdot m$, where m is a path. Let $\tilde{x}_\tau$ be the point of $q^{-1}(\Sigma S)$ such that the lift of $\mu(\ell)$ following the lift of m^{-1} is a path around $\tilde{x}_\tau$.

Lemma 5.6. *Let X be an orbifold composition, S a spherical 2-orbifold, and $f:S\to X$ an orbi-map. Suppose that there is a point $\tilde{d}\in\operatorname{Fix}(f_*\pi_1^{orb}(S))_A$, and for any $\tau\in\pi_1^{orb}(S)$ there is an interval ℓ_σ including $\tilde{d}$ and $\tilde{f}(\tilde{x}_\tau)$ which is fixed by σ_A, where $\sigma=f_*(\tau)$. If π_2 of the universal cover $\widetilde{X}$ of X is 0, then f is extendable to an orbi-map from the cone on S to X.*

Proof. Let $q:\widetilde{S}\to S$ be the universal covering, $\tilde{f}:\widetilde{S}\to\widetilde{X}$ the structure map of f, and $B=c*S$ be the cone on S, where c is the cone point of B. Let $\bar{q}:\widetilde{B}\to B$ be the universal covering and $\tilde{c}=\bar{q}^{-1}(c)$; i.e., $\widetilde{B}=\tilde{c}*\widetilde{S}$ and $\bar{q}(t\tilde{x}+(1-t)\tilde{c})=tq(\tilde{x})+(1-t)c$, $\tilde{x}\in\widetilde{S}$.

We can construct the structure map of the desired orbi-map by mapping $\tilde{c}$ to $\tilde{d}$, $\tilde{c} * \tilde{x}_\tau$ into ℓ_σ, and performing the skeletonwise and equivariant extension. □

Lemma 5.7. *Let X be an orbifold composition, B a ballic 3-orbifold, and $f : \mathcal{D}B \to X$ an orbi-map. Suppose that there is a point $\tilde{d} \in \mathrm{Fix}(f_* \pi_1^{orb}(\partial B))_A$, and for $\tau \in \pi_1^{orb}(\partial B)$ there is an interval ℓ_σ including $\tilde{d}$ and $\tilde{f}(\tilde{x}_\tau)$ which is fixed by σ_A, where $\sigma = f_*(\tau)$. If π_2 and π_3 of the universal cover $\widetilde{X}$ of X is 0, then f is extendable to an orbi-map from the cone on $\mathcal{D}B$ to X.*

Proof. The proof is similar to that of Lemma 5.6. □

Lemma 5.8. *Let M be an irreducible 3-orbifold. Let $p : \widehat{M} \to M$ be the universal covering and $\sigma \in \mathrm{Aut}(\widehat{M}, p)$ be an orientation preserving element of finite order. Suppose that $\widehat{M}$ is noncompact. Then*:

(i) $\mathrm{Fix}(\sigma) \neq \emptyset$ *and is homeomorphic to an interval* (*i.e., homeomorphic to either* $[0, 1]$, $[0, 1)$, *or* $(0, 1)$),

(ii) *if M is orientable, then $O_1(M)$ is trivial.*

Proof. Note first that (ii) follows from (i) and Lemma 5.5, so we need only prove (i).

(i) Let n be the order of σ and G be the subgroup of $\mathrm{Aut}(\widehat{M}, p)$ generated by σ. Let $\widetilde{M}$ be the orbifold $\widehat{M}/G$ and $q : \widehat{M} \to \widetilde{M}$ be the universal covering.

First we claim that there is no trivalent point in $\Sigma\widetilde{M}$. Otherwise, there is a noncyclic spherical 2-orbifold S in $\widetilde{M}$. By the Orbifold Loop Theorem 1.1, $i_* : \pi_1^{orb}(S) \to \pi_1^{orb}(\widetilde{M})$ is monic. This contradicts the fact that $\pi_1^{orb}(\widetilde{M}) = G \cong \mathbb{Z}_n$.

Furthermore, since σ is orientation preserving, $\Sigma\widetilde{M}$ has neither isolated points nor mirror boundaries. Hence, each component of $\Sigma\widetilde{M}$ is either an interval or a simple closed curve properly embedded in $|\widetilde{M}|$, and so is each component of $\mathrm{Fix}(\sigma)$ in $\widehat{M}$.

By the lifting of irreducibility [24, 6.13], $\widehat{M}$ is irreducible. Since $\widehat{M}$ is noncompact, $\widehat{M}$ is a homology 0-disc. (See [2].) By [2, Theorem 5.2], in case n is prime, $\mathrm{Fix}(\sigma)$ is a homology 0-disc. Then, $\mathrm{Fix}(\sigma)$ is not empty and is an interval.

Consider the case $n = pr$, p is prime and $r > 1$. Since σ^r has prime order, $\mathrm{Fix}(\sigma^r)$ is an interval. Hence, from the fact that $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\sigma^r)$, $\mathrm{Fix}(\sigma)$ is either an interval or empty set. To complete the proof, we have only to show that $\mathrm{Fix}(\sigma) \neq \emptyset$. Suppose $\mathrm{Fix}(\sigma) = \emptyset$. Let R be the subgroup of G generated by σ^r. Let $\overline{M}$ be the orbifold $\widehat{M}/R$, $t : \widehat{M} \to \overline{M}$ be the universal covering, and $\bar{t} : \overline{M} \to \widetilde{M}$ be the covering with $q = \bar{t} \circ t$. Note that $\bar{t}$ is a regular covering since R is a normal subgroup of G. Let L be the interval $\mathrm{Fix}(\sigma^r)$ and $\bar{L} = \Sigma\overline{M}$. Note that $t|L$ is a homeomorphism from L to $\bar{L}$ and $t^{-1}(\bar{L}) = L$. For any $\tau \in \mathrm{Aut}(\overline{M}, \bar{t})$, $\tau(\bar{L}) = \tau(\Sigma\overline{M}) = \Sigma\overline{M} = \bar{L}$.

We claim that τ acts on $\bar{L}$ preserving the orientation. Otherwise, since τ preserves the orientation of $\overline{M}$, $\Sigma\widetilde{M}$ must have a trivalent point of the dihedral type. Contradiction.

Combining this fact and the finiteness of the order of τ, we conclude that τ acts trivially on $\bar{L}$. That is, $\bar{L} = \mathrm{Fix}(\mathrm{Aut}(\overline{M}, \bar{t}))$. Hence, $\bar{t}|\bar{L}$ is a homeomorphism from $\bar{L}$ to $\widetilde{L}$, where $\widetilde{L} = \bar{t}(\bar{L})$. Moreover, since $\bar{t}$ is regular, $\bar{t}^{-1}(\widetilde{L}) = \bar{L}$. Thus, $q|L = (\bar{t}|\bar{L}) \circ (t|L)$

and $q^{-1}(\widetilde{L}) = t^{-1}(\bar{t}^{-1}(\widetilde{L})) = t^{-1}(\bar{L}) = L$. This implies that, for any $\omega \in \mathrm{Aut}(\widehat{M}, q)$, $L = \mathrm{Fix}(\omega)$. Contradiction. □

Lemma 5.9. *Let M be an irreducible 3-orbifold, and $p : \widehat{M} \to M$ the universal covering. Let G be any subgroup of $\mathrm{Aut}(\widehat{M}, p)$, which is isomorphic to the orbifold fundamental group of a spherical 2-orbifold S such that all elements of G preserve the orientation of $\widehat{M}$. Suppose that $\widehat{M}$ is noncompact. Then*:

(i) $\mathrm{Fix}(G) \neq \emptyset$,

(ii) *if M is orientable, then the $O_i(M)$'s are trivial, $i = 1, 2, 3$.*

Proof. Note first that (ii) follows from (i), Lemmas 5.5–5.8, so we need only prove (i).

In case $G \cong \mathbb{Z}_n$, this lemma reduces to Lemma 5.8. So we may assume that G is a triangle group. Let $\widetilde{M}$ be the orbifold $\widehat{M}/G$ and $q : \widehat{M} \to \widetilde{M}$ be the universal covering. Since $\pi_1^{orb}(S) \cong \pi_1^{orb}(\widetilde{M})$, we can construct an orbi-map $f : S \to \widetilde{M}$ such that f_* is an isomorphism by using Proposition 5.2 and Lemma 5.8. From the compactness of S, there is a compact 3-suborbifold N of $\widetilde{M}$ such that $f(S) \subset \mathrm{Int}\, N$.

Put $\mathcal{N} = \{(N, f) \mid f$ is an orbi-map from S to $\widetilde{M}$ such that $f_* : \pi_1^{orb}(S) \to \pi_1^{orb}(\widetilde{M})$ is an isomorphism, and N is a compact 3-suborbifold of $\widetilde{M}$ such that $f(S) \subset \mathrm{Int}\, N\}$. Then $\mathcal{N} \neq \emptyset$. We define the *complexity* c of an element (N, f) of $\mathcal{N}$ as follows:

> Let L be the maximum of the orders of the local groups of $\Sigma^{(1)}\widetilde{M}$ and s be the minimal number of the Euler numbers of all components of ∂N. Choose numbers $r \in \mathbb{Z}$ and $m \in \{0, 1, 2, \ldots, L-1\}$ satisfying $-r + (m-1)/L < s \leqslant -r + m/L$. Let $n_{-r+i+j/L}$ be the numbers of the components of ∂N whose Euler numbers are more than $-r + i + (j-1)/L$ and not more than $-r + i + j/L$. Define $c(N, f) = (n_{-r+m/L}, n_{-r+(m+1)/L}, \ldots, n_{-r+1}, n_{-r+1+1/L}, \ldots, n_2)$ and order $c(\mathcal{N})$ lexicographically.

Since $c(\mathcal{N}) \geqslant (0, \ldots, 0)$ and has discrete values, there is an element $(N_0, f_0) \in \mathcal{N}$ which attains the minimal value of $c(\mathcal{N})$.

Claim. *Each component of ∂N_0 is a spherical 2-orbifold.*

Otherwise, we can find an element $(N_1, f_1) \in \mathcal{N}$ such that $c(N_1, f_1) < c(N_0, f_0)$ as follows: Let $S_1, \ldots, S_k$ be a maximal system of incompressible spherical 2-suborbifolds of N and $B_1, \ldots, B_k$ be the ballic 3-suborbifolds of $\widetilde{M}$ such that $\partial B_i = S_i$. Put $\overline{N}_0 = N_0 \cup B_1 \cup \cdots \cup B_k$. Note that $(\overline{N}_0, f_0) \in \mathcal{N}$. From the minimality of $c(N_0, f_0)$, there is a nonspherical component F of $\partial \overline{N}_0$. Since $\pi_1^{orb}(\widetilde{M})$ is finite, F is never incompressible in $\widetilde{M}$. Let D be a compressing discal 2-orbifold with respect to F. Using the innermost arguments, we can replace the pair (F, D), if necessary, by one satisfying $D \cap \partial \overline{N}_0 = \partial D$. Hence it follows that either $\mathrm{Int}(D) \subset \widetilde{M} - \overline{N}_0$ or $\mathrm{Int}(D) \subset \mathrm{Int}(\overline{N}_0)$.

In case $\mathrm{Int}(D) \subset \widetilde{M} - \overline{N}_0$; let N_1 be the orbifold derived from $\overline{N}_0$ by attaching $D \times I$ as a 2-handle. Put $f_1 = f_0$. Then, $(N_1, f_1) \in \mathcal{N}$.

In case $\mathrm{Int}(D) \subset \mathrm{Int}(\overline{N}_0)$; let N' be the orbifold derived from $\overline{N}$ by cutting open along D. First, we consider the case that N' consists of two components N_1 and N_2. Then,

$\pi_1^{orb}(\overline{N}_0)$ is the free product of $\pi_1^{orb}(N_1)$ and $\pi_1^{orb}(N_2)$ with the amalgamated subgroup $\pi_1^{orb}(D)$ under the maps naturally induced by inclusions. Since $(f_0)_*\pi_1^{orb}(S)$ is a finite subgroup of $\pi_1^{orb}(\overline{N}_0)$, by [14, Lemma 6.8(1)], $(f_0)_*\pi_1^{orb}(S)$ is conjugate to a subgroup of either $\pi_1^{orb}(N_1)$ or $\pi_1^{orb}(N_2)$. Hence, we may assume that there is an element g of $\pi_1^{orb}(\overline{N}_0)$ such that

$$g\big((f_0)_*\pi_1^{orb}(S)\big)g^{-1} < \pi_1^{orb}(N_1).$$

Let φ be a homomorphism from $\pi_1^{orb}(S)$ to $\pi_1^{orb}(N_1)$ defined by $\varphi(\sigma) = g(f_*(\sigma))g^{-1}$ for $\sigma \in \pi_1^{orb}(S)$. From the construction, $\overline{N}_0$ is irreducible. Hence, by Proposition 1.5, N_1 is irreducible. Let $p_1 : \widehat{N}_1 \to N_1$ be the universal covering, and σ be any element of $\mathrm{Aut}(\widehat{N}_1, p_1)$ of finite order. In case $\#\pi_1^{orb}(N_1) = \infty$, by Lemma 5.8, σ has a fixed point in $\widehat{N}_1$. In case $\#\pi_1^{orb}(N_1) < \infty$, each component of ∂N_1 must be a spherical 2-orbifold. Since N_1 is irreducible, N_1 is a ballic 3-orbifold. Then, σ has a fixed point in $\widehat{N}_1$. Hence, by Proposition 5.2 and Lemma 5.5, we can construct an orbi-map $f_1 : S \to N_1$ such that

$$(f_1)_* = \varphi : \pi_1^{orb}(S) \to \pi_1^{orb}(N_1).$$

Since $\varphi : \pi_1^{orb}(S) \to \pi_1^{orb}(\widetilde{M})$ is an isomorphism, so is $(f_1)_* : \pi_1^{orb}(S) \to \pi_1^{orb}(\widetilde{M})$. Thus, we have $(N_1, f_1) \in \mathcal{N}$.

In case N' is connected, $\pi_1^{orb}(\overline{N}_0)$ is an HNN group. Then, by using [14, Lemma 6.8(2)], we construct $(N_1, f_1) \in \mathcal{N}$, similarly.

In any case, it is clear that $c(N_1, f_1) < c(N_0, f_0)$, which yields the claim.

Let $S_1, \ldots, S_k$ be the incompressible spherical 2-orbifold components of ∂N_0, and $B_1, \ldots, B_k$ be the ballic 3-suborbifolds of $\widetilde{M}$ such that $\partial B_i = S_i$. At least one of the B_i's includes N_0. Otherwise, it follows that $\mathrm{Int}\, B_i \cap \mathrm{Int}\, N_0 = \emptyset$ for all i. Then, $N_0 \cup B_1 \cup \cdots \cup B_k$ is a closed 3-suborbifold of $\widetilde{M}$; i.e., $N_0 \cup B_1 \cup \cdots \cup B_k = \widetilde{M}$. This contradicts the noncompactness of $\widetilde{M}$. Thus, we may assume that $B_1 \supset N_0$. Hence, $f(S) \subset B_1$. On the other hand, since $f_* : \pi_1^{orb}(S) \to \pi_1^{orb}(\widetilde{M})$ is an isomorphism, $f_* : \pi_1^{orb}(S) \to \pi_1^{orb}(B_1)$ is monic. Furthermore, since $\pi_1^{orb}(B_1) \to \pi_1^{orb}(\widetilde{M})$ is also monic, $\pi_1^{orb}(B_1)$ is isomorphic to $\pi_1^{orb}(\widetilde{M})$ ($\cong \pi_1^{orb}(S)$). Then, ΣB_1 is the same type as Σ (the cone on S). Let $\widehat{B}_1$ be a component of $q^{-1}(B_1)$. Since $q|\widehat{B}_1 : \widehat{B}_1 \to B_1$ is $\#\pi_1^{orb}(\partial B_1)$-sheeted orbi-covering and $\#\pi_1^{orb}(\partial B_1) = \#G < \infty$, $q^{-1}(B_1) = \widehat{B}_1$. That is, $\widehat{B}_1$ is invariant under G. Hence, for any $\sigma \in G$, σ fixes a line segment including $q^{-1}(v)$, where v is the trivalent point of ΣB_1. □

Proposition 5.10. *Let $X = (X^\varepsilon, Y \times [0,1], f^\varepsilon)_{\varepsilon=0,1}$ be an orbifold composition, where each X^ε is an orientable, irreducible* 3*-orbifold, and Y is an orientable* 2*-orbifold. If the universal coverings of X^ε and Y are all noncompact, then $O_i(X)$ is trivial, $i = 1, 2, 3$.*

Proof. Let $p : \widetilde{X} \to X$ be the universal covering. From the uniqueness of the universal covering Lemma 3.9, we may assume that $\widetilde{X}$ is the orbifold composition constructed as illustrated in Section 4.

Claim. *Let G be any subgroup of* $\mathrm{Aut}(\widetilde{X}, p)$*, which is isomorphic to the fundamental group of a spherical* 2*-orbifold. Then there is a vertex or edge orbifold $\widetilde{Z}$ of $\widetilde{X}$ such that $G(\widetilde{Z}) = \widetilde{Z}$.*

Considering the associated 1-complex of $\widetilde{X}$, the claim is derived from Lemma 2.2. Then the triviality of $O_1(X)$ follows from Lemmas 5.8, 5.5 and the claim. Note that the edge orbifold is a good orientable and irreducible 3-orbifold.

Take any element $f \in O_2(X)$, $f: S \to X$. Let $q: \widetilde{S} \to S$ be the universal covering and $\tilde{f}: \widetilde{S} \to \widetilde{X}$ the structure map of f. Let $B = c * S$ be the cone on S and c the cone point of B. Let $\bar{q}: \widetilde{B} = \tilde{c} * \widetilde{S} \to B$ be the universal covering, $\tilde{c} = \bar{q}^{-1}(c)$ and $\bar{q}(t\tilde{x} + (1-t)\tilde{c}) = tq(\tilde{x}) + (1-t)c$, $\tilde{x} \in \widetilde{S}$.

By the claim and Lemma 5.9, $(f_* \pi_1^{orb}(S))_A$ has a fixed point, say $\tilde{d}$, in a vertex or edge orbifold $\widetilde{Z}$ of $\widetilde{X}$.

Choose any $\tau \in \pi_1^{orb}(S)$. Let $\tilde{x}_\tau$ be the point defined in the paragraph preceding Lemma 5.6. We put $\sigma = f_*(\tau)$. Since σ_A fixes a vertex or edge orbifold $\widetilde{Z}_\sigma$ of $\widetilde{X}$, it follows that σ_A fixes an interval in $\widetilde{Z}_\sigma$ by using Lemma 5.8. Note that if $\widetilde{Z}'$ is any edge orbifold fixed by σ_A, then the fixed set interval is a fiber of $\widetilde{Z}'$. Hence, by Proposition 2.1, we can find an interval connecting $\tilde{f}(\tilde{x}_\tau)$ and $\tilde{d}$ which is fixed by σ_A. Note that $\pi_2(\widetilde{X}) = 0$ from the construction of $\widetilde{X}$. Then the triviality of $O_2(X)$ follows from Lemma 5.6.

All that remains to be shown is the triviality of $O_3(X)$, which is derived from the facts $\pi_3(\widetilde{X}) = 0$ and Lemma 5.7. □

Let X be an orbifold composition and F be a core of an edge orbifold $Y \times [0,1]$ of X. When we consider each connected component (or its closure) of $|X| - |F|$, it naturally admits an orbifold composition structure by restricting the structure of X. We denote it by $X - F$, etc. In this situation, a component of type $Y \times [\varepsilon, \frac{1}{2}]$ (respectively $Y \times [\varepsilon, \frac{1}{2})$), $\varepsilon = 0, 1$, appears, and is called a closed (respectively open) half-edge orbifold of the orbifold composition. Iterating this process, we can consider an orbifold composition with several half-edge orbifolds. Concerning the new types of orbifold compositions described above, the same arguments and statements hold as those in Sections 3–5.

6. More on orbifold compositions

Let X be an orbifold composition. An orbifold Y belongs to the set δX if Y satisfies one of the following conditions:

(i) Y is a boundary component of a vertex orbifold of X such that Y is disjoint from any images of attaching maps of X.

(ii) Y is the core of a closed half-edge of X such that $\partial Y = \emptyset$.

Theorem 6.1 (Transversality theorem). *Let M be a compact and orientable 3-orbifold, and X a 3-orbifold composition with trivial $O_i(X)$'s, $i = 2, 3$. Suppose that there is an edge orbifold whose core is an orientable and nonspherical 2-orbifold F such that $O_i(X - F)$ is trivial, $i = 2, 3$. Then, for any orbi-map $f: M \to X$, there is an orbi-map $g: M \to X$ such that*

(i) *g is orbi-homotopic to f,*

(ii) *each component of $g^{-1}(F)$ is a compact, properly embedded, 2-sided, incompressible 2-suborbifold in M, and*

(iii) *for properly chosen product neighborhoods* $F \times [-1,1]$ *of* $F = F \times 0$ *in* X, *and* $g^{-1}(F) \times [-1,1]$ *of* $g^{-1}(F) = g^{-1}(F) \times 0$ *in* M, $\bar{g}$ *maps each fiber* $x \times |[-1,1]|$ *homeomorphically to the fiber* $\bar{g}(x) \times |[-1,1]|$ *for each* $x \in |g^{-1}(F)|$, *where* $\bar{g} : |M| \to |X|$ *is the underlying map of* g.

Proof. Let G be any component of $f^{-1}(F)$. Let U_G and U'_G be sufficiently small compact neighborhoods of G such that $f(U_G) \subset F \times [-\frac{1}{2}, \frac{1}{2}]$, $\mathrm{Int}(U_G) \supset U'_G$, and ∂U_G and $\partial U'_G$ are parallel in U_G. By Proposition 5.10 and [21, 5.4], we may assume that $f|U'_G$ is an orbi-map. Triangulate $F \times [-\frac{1}{2}, \frac{1}{2}]$ as a product. By modifying $f|U'_G$ to a simplicial orbi-map, we have that G is a compact, properly embedded, and 2-sided 2-suborbifold in U'_G. Note that this modification can be performed by an orbi-homotopy which fixes $M - \mathrm{Int}(U_G)$. Iterating the modifications, we may assume that each component of $f^{-1}(F)$ is a compact, properly embedded, and 2-sided 2-suborbifold in M. The remainder of the proof is similar to [21, 5.5]. □

Theorem 6.2 (I-bundle theorem). *Let* M *be a compact, orientable and irreducible* 3-*orbifold with boundary, and* X *be a* 3-*orbifold composition. Let* $f : (M, \partial M) \to (X, \delta X)$ *be an orbi-map such that* f_* *is monic. Suppose there is a path* $\alpha : (I, \partial I) \to (|M| - \Sigma M, |\partial M|)$, *incompressible components* B_0, B_1 *of* ∂M, *and a component* C *of* δX *which satisfy the following*:

(i) $\alpha(0) \neq \alpha(1)$.

(ii) $\bar{f}(\alpha(0)) = \bar{f}(\alpha(1)) \in |\delta X| - \Sigma X$.

(iii) $[\tilde{f} \circ \hat{\alpha}] = 1$ *in* $\pi_1^{orb}(X)$, *where* $\hat{\alpha}$ *is a lift of* α *to the universal cover* $\widetilde{M}$ *of* M *and* $f = (\bar{f}, \tilde{f})$.

(iv) B_i (*respectively* C) *includes* $\alpha(i)$ (*respectively* $\bar{f}(\alpha(0))$), $\mathrm{Ker}(\pi_1^{orb}(C) \to \pi_1^{orb}(X)) = 1$, *and* $(f|B_i) : B_i \to C$ *is a covering*, $i = 0, 1$ (*possibly* $B_0 = B_1$).

Then M *is an I-bundle over a closed* 2-*orbifold.*

Proof. Let $\eta_0 : \pi_1^{orb}(B_0, x_0) \to \pi_1^{orb}(M, x_0)$ be the homomorphism induced by the inclusion orbi-map $B_0 \to M$ and $p : (\widetilde{M}, \tilde{x}_0) \to (M, x_0)$ be the covering associated with $\eta_0 \pi_1^{orb}(B_0, x_0)$. By an argument parallel to [23, 4.1 and 4.2], we can show that $\widetilde{M}$ is compact. Hence, $p : (\widetilde{M}, \tilde{x}_0) \to (M, x_0)$ is a finite covering. Therefore,

$$\left|\pi_1^{orb}(M, x_0); \eta_0 \pi_1^{orb}(B_0, x_0)\right| < \infty.$$

From [21, 6.3], M is an I-bundle over a closed 2-orbifold. □

Theorem 6.3 (Retraction theorem). *Let* M *be an orientable* 3-*orbifold which is orbi-isomorphic to an I-bundle over a closed* 2-*orbifold* F. *Let* X *be a* 3-*orbifold composition with trivial* $O_i(X)$*'s*, $i = 2, 3$. *Let* $f : (M, \partial M) \to (X, \delta X)$ *be an orbi-map such that* $f|\partial M$ *is not an orbi-embedding and such that, for each component* B *of* ∂M, *there is a component* C *of* δX *with* $f(B) \subset C$ *and* $(f|B) : B \to C$ *an orbi-covering.*

If there is a point $x \in |F| - \Sigma F$ *such that* $f|(\varphi^{-1}(x))$ *is orbi-homotopic* (6.3.1) *to a path in* C *rel.* $\{x\} \times \partial I$, *where* $\varphi : M \to F$ *is a fibration, then there is an orbi-homotopy* $f_t : M \to X$ *such that* $f_0 = f$, $f_1(M) \subset \delta X$, *and* $f_t|\partial M = f|\partial M$.

Proof. Let $s_1, \dots, s_k$ be a system of simple closed curves on $|F| - \Sigma F$ such that $s_i \cap s_j = x$ if $i \neq j$, and cutting F open along $s_1, \dots, s_k$ derives discal orbifolds $D_1, \dots, D_r$. We construct the desired orbi-map $H : M \times J \to X$, $J = [0,1]$ as follows: First, $H|\{\varphi^{-1}(x) \times J\}$ is defined by the orbi-homotopy (6.3.1). Then we can define $H|\{\varphi^{-1}(s_i) \times J\}$ and $H|\{\varphi^{-1}(D_i) \times J\}$ by using the triviality of $O_2(X)$ and $O_3(X)$, respectively. See [23, 4.3] for details. □

Remark 6.4. In Theorem 6.3, if $f_* : \pi_1^{orb}(M) \to \pi_1^{orb}(X)$ is an isomorphism and C is orientable, then condition (6.3.1) holds. Furthermore, M is orbi-isomorphic to the product I-bundle over B_0, and B_0 is orbi-isomorphic to C.

Proof. The proof follows by an argument parallel to [23, 4.6]. □

Theorem 6.5 (Amalgamation theorem). *Let A_i, $i = 1, 2$, be groups which contain subgroups H_i, $i = 1, 2$. Suppose there is an isomorphism $\varphi : H_1 \to H_2$. Let A_i', $i = 1, 2$, be subgroups of A_i containing H_i. If the natural homomorphism $\phi : \langle A_1' * A_2' \mid H_1 = H_2, \varphi\rangle \to \langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ is an isomorphism, then $A_i = A_i'$, $i = 1, 2$.*

Proof. See [3, Proposition 2.5]. □

Theorem 6.6 (HNN theorem). *Let A be a group which contains subgroups H_i, $i = 1, 2$. Suppose there is an isomorphism $\varphi : H_1 \to H_2$. Let A' be a subgroup of A, containing H_i, $i = 1, 2$. If the natural homomorphism $\phi : \langle A', t' \mid t'^{-1} H_1 t' = H_2, \varphi\rangle \to \langle A, t \mid t^{-1} H_1 t = H_2, \varphi\rangle$ is an isomorphism, then $A = A'$.*

Proof. Let H be the subgroup of A which is generated by H_1 and H_2. Let $G = \langle H, s \mid s^{-1} H_1 s = H_2, \varphi\rangle$. From the remark preceding Lemma 2 on p. 238 of [14],

$$\langle A, t \mid t^{-1} H_1 t = H_2, \varphi\rangle = \langle A, G \mid H = \varphi(H), \varphi\rangle$$

and

$$\langle A', t' \mid t'^{-1} H_1 t' = H_2, \varphi\rangle = \langle A', G \mid H = \varphi(H), \varphi\rangle.$$

Then, by Theorem 6.5, we can derive the conclusion. □

7. Main Theorem

In this section, we assume that all free products with amalgamations are nontrivial.

Definition 7.1. Let M be a 3-orbifold with trivial $O_1(M)$. Let S be a closed, orientable, nonspherical 2-orbifold. Suppose $\pi_1^{orb}(M) = \langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and there is an isomorphism $\psi : \pi_1^{orb}(S) \to H_1$. Let $p_i : X_i \to M$ be the orbi-covering associated with A_i, $i = 1, 2$. Note that $O_1(X_i)$ is trivial, $i = 1, 2$. Put $\widetilde{H}_i = p_{i*}^{-1}(H_i)$, $i = 1, 2$. Note that $(p_{1*}|\widetilde{H}_1)^{-1} \circ \psi$ (respectively $(p_{2*}|\widetilde{H}_2)^{-1} \circ \varphi \circ \psi$) is an isomorphism from $\pi_1^{orb}(S)$ to

$\widetilde{H}_1$ (respectively $\widetilde{H}_2$). By Proposition 5.2, we can construct orbi-maps $h_1 : S \to X_1$ and $h_2 : S \to X_2$ such that $h_{1*} = (p_{1*}|\widetilde{H}_1)^{-1} \circ \psi$ and $h_{2*} = (p_{2*}|\widetilde{H}_2)^{-1} \circ \varphi \circ \psi$. We call the orbifold composition $X = (X_1, X_2, S \times [0,1], h_1, h_2)$ the *orbifold composition associated with* $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$. We also define the *orbifold composition associated with* $\langle A, t \mid t^{-1}H_1t = H_2, \varphi\rangle$ similarly.

From Lemma 4.1 (respectively Lemma 4.2), it holds that

$$\pi_1^{orb}(X) = \left\langle \pi_1^{orb}(X_1) * \pi_1^{orb}(X_2) \mid h_{1*}\pi_1^{orb}(S) = h_{2*}\pi_1^{orb}(S), h_{2*} \circ h_{1*}^{-1} \right\rangle$$

(respectively $\langle \pi_1^{orb}(X'), t \mid t^{-1}h_{1*}\pi_1^{orb}(S)t = h_{2*}\pi_1^{orb}(S), h_{2*} \circ h_{1*}^{-1}\rangle$). Furthermore, we have the following proposition.

Proposition 7.2. *Let M be a* 3*-orbifold with $O_1(M)$ trivial. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose $\pi_1^{orb}(M) = \langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ (respectively $\langle A, t \mid t^{-1}H_1t = H_2, \varphi\rangle$) and there is an isomorphism $\psi : \pi_1^{orb}(S) \to H_1$. Let X be the orbifold composition associated with $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ (respectively $\langle A, t \mid t^{-1}H_1t = H_2, \varphi\rangle$). Then there is an isomorphism $\Psi : \pi_1^{orb}(X) \to \pi_1^{orb}(M)$ such that*

(i) $\Psi(\pi_1^{orb}(X_i)) = A_i$, $i = 1, 2$ *(respectively $\Psi(\pi_1^{orb}(X')) = A$),*

(ii) $\Psi(\widetilde{H}_i) = H_i$, $i = 1, 2$ *(note that $h_{i*}\pi_1^{orb}(S) = \widetilde{H}_i$),*

(iii) $\Psi \circ (h_{2*} \circ h_{1*}^{-1}) = \varphi \circ \Psi$.

Proof. Let $a_1, \ldots, a_m$ (respectively $b_1, \ldots, b_n$) be a generating system of $\pi_1^{orb}(X_1)$ (respectively $\pi_1^{orb}(X_2)$). We can construct the desired isomorphism Ψ by defining $\Psi(a_i) = p_{1*}(a_i)$ and $\Psi(b_j) = p_{2*}(b_j)$. □

Definition 7.3. Let M be a 3-orbifold, and S be a closed, orientable, and nonspherical 2-orbifold. We say that S *algebraically splits* $\pi_1^{orb}(M)$ *as an amalgamated free product* if $\pi_1^{orb}(M)$ is expressed as a free product with an amalgamation, $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$, and there is an isomorphism $\Psi : H_1 \to \pi_1^{orb}(S)$.

We say that the splitting above *respects the peripheral structure* of M if for each component G of ∂M, some conjugate of $\eta_*\pi_1^{orb}(G)$ is contained in either A_1 or A_2, where η is the inclusion orbi-map $G \to M$.

Proposition 7.4. *Let M be a compact, orientable, and irreducible* 3*-orbifold. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose S algebraically splits $\pi_1^{orb}(M)$ as an amalgamated free product $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Let X be the orbifold composition associated with $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$. Then there is an orbi-map $f : M \to X$ such that f_* is an isomorphism and $f(\partial M) \cap (S \times (0,1)) = \emptyset$.*

Proof. Since $\pi_1^{orb}(M)$ has the form $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$, $\pi_1^{orb}(M)$ is infinite and the universal cover of M is noncompact. Then $O_1(M)$ is trivial using Lemma 5.8. By Proposition 7.2, there is an isomorphism $\Psi : \pi_1^{orb}(M) \to \pi_1^{orb}(X)$ such that $\Psi(A_i) = \pi_1^{orb}(X_i)$, $\Psi(H_i) = \widetilde{H}_i$, $i = 1, 2$, and $\Psi \circ \varphi = (h_{2*} \circ h_{1*}^{-1}) \circ \Psi$. By Proposition 5.10, $O_1(X)$

and $O_2(X)$ are trivial. Hence, by Proposition 5.3, there is an orbi-map $f' : M \to X$ which induces the isomorphism Ψ. Then all we have to do is show that if F is a component of ∂M, there is an orbi-homotopy $H : F \times [0,1] \to X$ such that $H|(F \times 0) = f'|F$ and $H|(F \times 1)$ is an orbi-map into either X_1 or X_2. We construct this orbi-homotopy in a piecewise fashion. Define $H|(F \times 0) = f'|F$. Choose a triangulation $K_{|F|}$ of $|F|$ so that for each 2-simplex $e \in K_{|F|}$, $\partial e \cap \Sigma F = \emptyset$ and $(\mathrm{Int}\, e) \cap \Sigma F =$ (at most one point).

Let F_1 be the subspace of $F \times [0,1]$ whose underlying space is $|K_{|F|}^{(1)}| \times |[0,1]|$. From the hypothesis that the splitting respects the peripheral structure, some conjugation of $\Psi(\eta_* \pi_1^{orb}(F))$ is contained in either $\pi_1^{orb}(X_1)$ or $\pi_1^{orb}(X_2)$. Hence, we can extend $H|(F \times 0)$ to $(F \times 0) \cup F_1$ such that $H(K_{|F|}^{(1)} \times 1)$ is included in either X_1 or X_2. Note that

$$\mathrm{Ker}\left(\pi_1^{orb}(X_j) \to \pi_1^{orb}(X)\right) = 1$$

by the definition of an orbifold composition. So we can extend $H|\{(F \times 0) \cup F_1\}$ to $(F \times 0) \cup F_1 \cup (F \times 1)$ such that $H(F \times 1)$ is included in either X_1 or X_2. Since $O_2(X)$ is trivial, we can extend $H|\{(F \times 0) \cup (F \times 1) \cup F_1\}$ to $F \times [0,1]$ □

Definition 7.5. Let F be a closed, properly embedded, 2-sided, incompressible, and separating 2-suborbifold in M. Let M_1, M_2 be the orbifolds derived from M by cutting open along F and $\eta_i : F \to M_i$, $i = 1, 2$, be the inclusion orbi-maps. Note that $\pi_1^{orb}(M)$ is expressed as the amalgamated free product $\langle \pi_1^{orb}(M_1) * \pi_1^{orb}(M_2) \mid \eta_{1*}\pi_1^{orb}(F) = \eta_{2*}\pi_1^{orb}(F), \eta_{2*} \circ \eta_{1*}^{-1} \rangle$. We say that F *geometrically realizes the algebraic splitting* $\langle A_1 * A_2 \mid H_1 = H_2, \varphi \rangle$ of $\pi_1^{orb}(M)$ if there is an isomorphism $\Psi : \pi_1^{orb}(M) \to \pi_1^{orb}(M)$ such that

(i) $\Psi(\pi_1^{orb}(M_i)) = A_i$, $i = 1, 2$,
(ii) $\Psi(\eta_{i*}\pi_1^{orb}(F \times i)) = H_i$, $i = 1, 2$, and
(iii) $\Psi \circ (\eta_{2*} \circ \eta_{1*}^{-1}) = \varphi \circ \Psi$.

Theorem 7.6. *Let M be a compact, orientable, and irreducible* 3*-orbifold. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose S algebraically splits $\pi_1^{orb}(M)$ as an amalgamated free product $\langle A_1 * A_2 \mid H_1 = H_2, \varphi \rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

Proof. Let $X = (X_1, X_2, S \times [0,1], h_1, h_2)$ be an orbifold composition associated with $\langle A_1 * A_2 \mid H_1 = H_2, \varphi \rangle$. By Proposition 7.4, we can construct an orbi-map $f : M \to X$ such that f_* is an isomorphism and $f(\partial M) \cap (S \times (0,1)) = \emptyset$.

Note that, by Proposition 5.10, $O_i(X)$ is trivial, $i = 1, 2, 3$. Since $O_i(X_j)$ is trivial, $i = 1, 2, 3$, $j = 1, 2$, $O_i(X - S \times \frac{1}{2})$ is trivial, $i = 1, 2, 3$. From Theorem 6.1, we may assume that each component of $f^{-1}(S \times \frac{1}{2})$ is a compact, properly embedded, 2-sided, incompressible 2-suborbifold in M, and f is transverse between product neighborhoods of $f^{-1}(S \times \frac{1}{2})$ and of $S \times \frac{1}{2}$. Let $F_1, \ldots, F_k$ be the components of $f^{-1}(S \times \frac{1}{2})$. Since $f^{-1}(S \times \frac{1}{2}) \cap \partial M = \emptyset$, each F_i is closed, $i = 1, \ldots, k$. By [21, 7.2] and [23, 3.2], we may assume that $f|F_i : F_i \to S \times \frac{1}{2}$, $i = 1, \ldots, k$, is an orbi-covering.

Claim 1. $k = 1$. (*By modifying f through an orbi-homotopy.*)

Suppose $k \geqslant 2$. Let $M_1, \ldots, M_\ell$ be the components derived from M by cutting open along $F_1, \ldots, F_k$. From the surjectivity of f_*, there is a path $\beta : (I, \partial I) \to (|M| - \Sigma M, f^{-1}(S \times \frac{1}{2}))$ such that $\beta(0) \neq \beta(1)$, $\bar{f}(\beta(0)) = \bar{f}(\beta(1))$, and $[\tilde{f} \circ \hat{\beta}] = 1$ in $\pi_1^{orb}(X)$, where $\hat{\beta}$ is a lift of β to the universal cover of M and $f = (\bar{f}, \tilde{f})$. This path β is called a *binding tie* and can be expressed as the form $\beta = \alpha_1 \cdots \alpha_m$ such that $\mathrm{Int}\,\alpha_i \cap f^{-1}(S \times \frac{1}{2}) = \emptyset$, $\tilde{f} \circ \hat{\alpha}_i$ represents an element of either $\pi_1^{orb}(X_1)$ or $\pi_1^{orb}(X_2)$ and $[\tilde{f} \circ \hat{\alpha}_j]$, $[\tilde{f} \circ \hat{\alpha}_{j+1}]$ are not both in $\pi_1^{orb}(X_1)$ or both in $\pi_1^{orb}(X_2)$, where $\hat{\alpha}_i$ is a lift of α_i to the universal cover of M. We may assume that the number m is minimal. Then we claim $m = 1$. Suppose $m \geqslant 2$. Since $[\tilde{f} \circ \hat{\alpha}_1] \cdots [\tilde{f} \circ \hat{\alpha}_m] = 1$ in $\pi_1^{orb}(X)$, $[\tilde{f} \circ \hat{\alpha}_i] \in \pi_1^{orb}(S \times \frac{1}{2})$ for some i, $i = 1, \ldots, m$, by [14, Theorem 2.6]. Let ℓ be a loop in $S \times \frac{1}{2} - \Sigma(S \times \frac{1}{2})$ such that $[\ell] = [\tilde{f} \circ \hat{\alpha}_i]$ in $\pi_1^{orb}(S \times \frac{1}{2})$. Let γ be a lift of ℓ^{-1} by the orbi-covering $f|F_{j_i}$ with initial point $\alpha_i(1)$, where F_{j_i} is the component of $f^{-1}(S \times \frac{1}{2})$ including $\alpha_i(1)$. In case $\gamma(1) \neq \alpha_i(0)$, put $\delta = \alpha_i \cdot \gamma$. Otherwise, put $\delta = \alpha_1 \cdots \alpha_{i-1} \cdot \gamma^{-1} \cdot \alpha_{i+1} \cdots \alpha_m$. In any case, by modifying δ along the product structure of the regular neighborhood of F_{j_i}, we have another binding tie, i.e., a path $\delta' : (I, \partial I) \to (|M| - \Sigma M, f^{-1}(S \times \frac{1}{2}))$ such that $\delta'(0) \neq \delta'(1)$, $\bar{f}(\delta'(0)) = \bar{f}(\delta'(1))$, and $[\tilde{f} \circ \hat{\delta}'] = 1$ in $\pi_1^{orb}(X)$, where $\hat{\delta}'$ is a lift of δ' to the universal cover of M. Since δ' intersects with $f^{-1}(S \times \frac{1}{2})$ in fewer points than β, this contradicts the minimality of m. Hence $m = 1$. Then, β is included in one of the components $M_1, \ldots, M_\ell$.

Suppose M_1 is such a component. We may assume that $f(M_1) \subset X_1$. Hence, by Theorems 6.2, 6.3, and Remark 6.4, we can modify $f|M_1$ through an orbi-homotopy rel. ∂M_1 to an orbi-map $f_1 : M_1 \to X_1$ which satisfies $f_1(M_1) \subset S \times \frac{1}{2}$. Hence we can remove one or two of $F_1, \ldots, F_k$. Repeating this process, if $k \geqslant 2$, we can finally assume $k = 0, 1$. If $k = 0$, $f_* \pi_1^{orb}(M) < A_1$ or A_2. This contradicts the fact that f_* is an isomorphism and the decomposition of $\pi_1^{orb}(M)$ is nontrivial. Thus, $k = 1$.

Claim 2. $f|F_1 : F_1 \to S$ *is an orbi-isomorphism.*

Otherwise, we can remove F_1 by using an argument similar to the proof of Claim 1.

Claim 3. F_1 *is separating.*

Otherwise, there is a loop α in M, which intersects F_1 transversely in a single point. By Claim 2 and Theorem 6.1, $f \circ \alpha$ intersects S transversely in a single point. This contradicts the fact that S is separating.

Let M_1, M_2 be the components derived from M by cutting open along F. Note that $f(M_i) \subset X_i$ and $(f|M_i)_* : \pi_1^{orb}(M_i) \to \pi_1^{orb}(X_i)$, $i = 1, 2$, are monics. By Claim 2,

$f_*\pi_1^{orb}(F)=\pi_1^{orb}(S)$. Since $f_*\eta_{i*}=h_{i*}f_*$, $f_*\eta_{i*}\pi_1^{orb}(F)=h_{i*}\pi_1^{orb}(S)$, $i=1,2$. Hence, all maps in the following commutative diagram are isomorphisms.

$$\begin{array}{ccc}
\eta_{1*}\pi_1^{orb}(F) & \xrightarrow[f_*|\eta_{1*}\pi_1^{orb}(F)]{} & h_{1*}\pi_1^{orb}(S) \\
\uparrow{\scriptstyle \eta_{1*}} & & \uparrow{\scriptstyle h_{1*}} \\
\pi_1^{orb}(F) & & \pi_1^{orb}(S) \\
\downarrow{\scriptstyle \eta_{2*}} & & \downarrow{\scriptstyle h_{2*}} \\
\eta_{2*}\pi_1^{orb}(F) & \xrightarrow[f_*|\eta_{2*}\pi_1^{orb}(F)]{} & h_{2*}\pi_1^{orb}(S)
\end{array}$$

Thus, $f_*\circ(\eta_{2*}\circ\eta_{1*}^{-1})=(h_{2*}\circ h_{1*}^{-1})\circ f_*$. Note that

$$f_*:\langle\pi_1^{orb}(M_1)*\pi_1^{orb}(M_2)\mid\eta_{1*}\pi_1^{orb}(F)=\eta_{2*}\pi_1^{orb}(F),\eta_{2*}\circ\eta_{1*}^{-1}\rangle$$
$$\to\langle\pi_1^{orb}(X_1)*\pi_1^{orb}(X_2)\mid h_{1*}\pi_1^{orb}(S)=h_{2*}\pi_1^{orb}(S),h_{2*}\circ h_{1*}^{-1}\rangle$$

is an isomorphism. By Theorem 6.5, $(f|M_i)_*:\pi_1^{orb}(M_i)\to\pi_1^{orb}(X_i)$, $i=1,2$, are isomorphisms. Then the composite of f_* and Ψ given in Proposition 7.2 gives the desired isomorphism. □

Definition 7.7. Let M be a 3-orbifold. Let S be a closed, orientable, and nonspherical 2-orbifold. We say that S *algebraically splits* $\pi_1^{orb}(M)$ *as an HNN extension* if $\pi_1^{orb}(M)$ is expressed as an HNN extension, $\langle A,t\mid t^{-1}H_1t=H_2,\varphi\rangle$, and there is an isomorphism $\Psi:H_1\to\pi_1^{orb}(S)$.

We say that the splitting above *respects the peripheral structure* of M if for each component G of ∂M, some conjugate of $\eta_*\pi_1^{orb}(G)$ is contained in A, where η is the inclusion orbi-map $G\to M$.

Proposition 7.8. *Let M be a compact, orientable, and irreducible* 3*-orbifold. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose S algebraically splits $\pi_1^{orb}(M)$ as an HNN extension $\langle A,t\mid t^{-1}H_1t=H_2,\varphi\rangle$ and this splitting respects the peripheral structure of M. Let X be the orbifold composition associated with $\langle A,t\mid t^{-1}H_1t=H_2,\varphi\rangle$. Then there is an orbi-map $f:M\to X$ such that f_* is an isomorphism and $f(\partial M)\cap(S\times(0,1))=\emptyset$.*

Proof. Similarly to Proposition 7.4. □

Definition 7.9. Let F be a closed, properly embedded, 2-sided, incompressible, and nonseparating 2-suborbifold in M. Let M' be the orbifold derived from M by cutting open along F and $\eta_i : F \times i \to M'$, $i = 0, 1$, be the inclusion orbi-maps. Note that $\pi_1^{orb}(M)$ is expressed as the HNN extension

$$\left\langle \pi_1^{orb}(M'), t \mid t^{-1}\eta_{0*}\pi_1^{orb}(F \times 0)\, t = \eta_{1*}\pi_1^{orb}(F \times 1), \eta_{1*} \circ \eta_{0*}^{-1} \right\rangle.$$

We say that F *geometrically realizes the algebraic splitting* $\langle A, t \mid t^{-1}H_1 t = H_2, \varphi\rangle$ *of* $\pi_1^{orb}(M)$ if there is an isomorphism $\Psi : \pi_1^{orb}(M) \to \pi_1^{orb}(M)$ such that

(1) $\Psi(\pi_1^{orb}(M')) = A$,
(2) $\Psi(\eta_i \pi_1^{orb}(F \times i)) = H_i$, $i = 0, 1$, and
(3) $\Psi(\eta_{1*}\eta_{0*}^{-1}) = \varphi \circ \Psi$.

Theorem 7.10. *Let M be a compact, orientable, and irreducible* 3*-orbifold. Let S be a closed, orientable, and nonspherical* 2*-orbifold. Suppose S algebraically splits $\pi_1^{orb}(M)$ as an HNN extension $\langle A, t \mid t^{-1}H_1 t = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

Proof. Let $X = (X, S \times [0, 1], h_1, h_2)$ be an orbifold composition associated with $\langle A, t \mid t^{-1}H_1 t = H_2, \varphi\rangle$. By Proposition 7.8, we can construct an orbi-map $f : M \to X$ such that f_* is an isomorphism and $f(\partial M) \cap (S \times (0, 1)) = \emptyset$. Note that, by Proposition 5.10, $O_1(X)$, $O_2(X)$, and $O_3(X)$ are trivial. As in the proof of Theorem 7.6 (using the normal form of the HNN group), we can modify f through an orbi-homotopy so that $f^{-1}(S)$ consists of one, and only one, component F which is a closed, properly embedded, 2-sided, and incompressible 2-suborbifold in M.

Claim 1. *There is a loop in $|M| - \Sigma M$ whose algebraic intersection number with F is one.*

Since S is nonseparating in X, there is a loop β in $|X| - \Sigma X$ which intersects S transversely in a single point. Since f_* is an isomorphism, there is a loop α in $|M| - \Sigma M$ such that $f_*[\alpha] = [\beta]$ in $\pi_1^{orb}(X)$. We may assume that α intersects F transversely. Since $[f \circ \alpha] = [\beta]$ in $\pi_1^{orb}(X)$, there is an orbi-map $h : S^1 \times [0, 1] \to X$ such that $h|(S^1 \times 0) = f \circ \alpha$ and $h|(S^1 \times 1) = \beta$. Hence $\bar{h}$ is a map from $S^1 \times [0, 1]$ to $|X|$ such that $\bar{h}|(S^1 \times 0) = \bar{f} \circ \alpha$ and $\bar{h}|(S^1 \times 1) = \beta$. Therefore, the algebraic intersection number of $\bar{f} \circ \alpha$ and S is one. Since f is an orbi-isomorphism between $F \times [0, 1]$ and $S \times [0, 1]$, the algebraic intersection number of α and F is also one.

Claim 2. *There is a loop in $|M| - \Sigma M$ whose geometric intersection number with F is one.* (*Thus F is nonseparating.*)

From Claim 1, there is a loop α_1 in $|M| - \Sigma M$ which intersects F transversely, and whose algebraic intersection number with F is one. Let $p_1, \ldots, p_{2m+1}$, $m \geqslant 0$, be all points of $\alpha_1 \cap F$. Suppose $m \geqslant 1$. Then we may assume that the algebraic intersection number of α_1 and F at p_1 is $+1$ and at p_2 is -1. Hence we can find a loop α_2 in $|M| - \Sigma M$ which intersects F transversely and $\alpha_2 \cap F = \alpha_1 \cap F - \{p_1, p_2\}$. Repeating this process, we can find a desired loop.

Let M' be the component derived from M by cutting open along F. Note that $f(M') \subset X'$ and $(f|M')_* : \pi_1^{orb}(M') \to \pi_1^{orb}(X')$ is monic. The remainder of the proof is similar to the proof of Theorem 7.6 except for using Theorem 6.6 instead of Theorem 6.5. □

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