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Dyck paths and pattern-avoiding matchings

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Abstract

How many matchings on the vertex set $V = \{1, 2, \ldots, 2n\}$ avoid a given configuration of three edges? Chen, Deng and Du have shown that the number of matchings that avoid three nesting edges is equal to the number of matchings avoiding three pairwise crossing edges. In this paper, we consider other forbidden configurations of size three. We present a bijection between matchings avoiding three crossing edges and matchings avoiding an edge nested below two crossing edges. This bijection uses non-crossing pairs of Dyck paths of length $2n$ as an intermediate step.

Apart from that, we give a bijection that maps matchings avoiding two nested edges crossed by a third edge onto the matchings avoiding all configurations from an infinite family $\mathcal{M}$, which contains the configuration consisting of three crossing edges. We use this bijection to show that for matchings of size $n > 3$, it is easier to avoid three crossing edges than to avoid two nested edges crossed by a third edge.

Our results on pattern-avoiding matchings can be regarded as an extension of previous results on pattern-avoiding permutations.

1. Introduction and basic definitions

The enumeration of pattern-avoiding permutations has received a considerable amount of attention lately (see [9] for a survey). We say that a permutation π of order n *contains* a permutation σ of order k, if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that for every $s, t \in [k]$, $\pi(i_s) < \pi(i_t)$ if and only if $\sigma(s) < \sigma(t)$. One of the central notions in the study of pattern-avoiding permutations is the *Wilf equivalence*: we say that a permutation σ_1 is Wilf-equivalent to a permutation σ_2 if, for every $n \in \mathbb{N}$, the number of permutations

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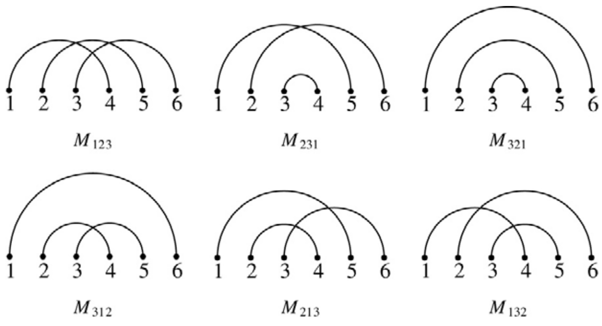

Fig. 1. The six permutational matchings with three edges.

of order n that avoid σ_1 is equal to the number of permutations of order n that avoid σ_2. In this paper, we consider pattern avoidance in matchings. This is a more general concept than pattern avoidance in permutations, since every permutation can be represented by a matching. We will introduce an equivalence relation on matchings that can be regarded as a refinement of the Wilf equivalence, and we determine the classes of this equivalence restricted to matchings representing permutations of order three.

A *matching* of size m is a graph on the vertex set $[2m] = \{1, 2, \ldots, 2m\}$ whose every vertex has degree one. We say that a matching $M = (V, E)$ *contains* a matching $M' = (V', E')$ if there is a monotonic edge-preserving injection from V' to V; in other words, M contains M' if there is a function $f : V' \to V$ such that $u < v$ implies $f(u) < f(v)$ and $\{u, v\} \in E'$ implies $\{f(u), f(v)\} \in E$.

Let M be a matching of size m, and let $e = \{i, j\}$ be an arbitrary edge of M. If $i < j$, we say that i is an *l-vertex* and j is an *r-vertex* of M. Obviously, M has m l-vertices and m r-vertices. Let $e_1 = \{i_1, j_1\}$ and $e_2 = \{i_2, j_2\}$ be two edges of M, with $i_1 < j_1$ and $i_2 < j_2$, and assume that $i_1 < i_2$. We say that the two edges e_1 and e_2 *cross* each other if $i_1 < i_2 < j_1 < j_2$, and we say that e_2 is *nested* below e_1 if $i_1 < i_2 < j_2 < j_1$.

We say that a matching M on the vertex set $[2m]$ is *permutational*, if for every l-vertex i and every r-vertex j we have $i \le m < j$. There is a natural one-to-one correspondence between permutations of order m and permutational matchings of size m: if π is a permutation of m elements, we let M_π denote the permutational matching on the vertex set $[2m]$ whose edge set is the set $\{\{i, m + \pi(i)\},\ i \in [m]\}$. In this paper, we will often represent a permutation π on m elements by the ordered sequence $\pi(1)\pi(2)\cdots\pi(m)$. Thus, for instance, M_{132} refers to the matching on the vertex set $\{1, 2, \ldots, 6\}$, with edge set $\{\{1, 4\}, \{2, 6\}, \{3, 5\}\}$. Fig. 1 depicts most of the matchings relevant for this paper. Note that a permutational matching M_π contains the permutational matching M_σ if and only if π contains σ.

Let $n = 2m$ be an even number. A *Dyck path* of length n is a piecewise linear non-negative walk in the plane, which starts at the point $(0, 0)$, ends at the point $(n, 0)$, and consists of n linear segments ("steps"), of which there are two kinds: an *up-step* connects (x, y) with $(x + 1, y + 1)$, whereas a *down-step* connects (x, y) with $(x + 1, y - 1)$. The non-negativity of the path implies that among the first k steps of the path there are at least $k/2$ up-steps. Let $\mathcal{D}_m$ denote the set of all Dyck paths of length $2m$. It is well known that $|\mathcal{D}_m| = c_m$, where $c_m = \frac{1}{m+1}\binom{2m}{m}$ is the m-th Catalan number (see [8]).

Every Dyck path $D \in \mathcal{D}_m$ can be represented by a *Dyck word* (denoted by $w(D)$), which is a binary word $w \in \{0, 1\}^{2m}$ such that $w_i = 0$ if the i-th step of D is an up-step, and $w_i = 1$ if the i-th step of D is a down-step. It can be easily seen that a word $w \in \{0, 1\}^n$ is a Dyck word of some Dyck path if and only if the following conditions are satisfied:

- The length $n = |w|$ is even.
- The word w has exactly $n/2$ terms equal to 1.
- Every prefix w' of w has at most $|w'|/2$ terms equal to 1.

We will use the term Dyck word to refer to any binary word satisfying these conditions. The set of all Dyck words of length $2m$ will be denoted by $\mathcal{D}'_m$.

Let $\mathcal{G}(m)$ denote the set of all matchings on the vertex set $[2m]$. For a matching $M \in \mathcal{G}(m)$, we define the *base* of M (denoted by $b(M)$) to be the binary word $w \in \{0, 1\}^{2m}$ such that $w_i = 0$ if i is an l-vertex of M, and $w_i = 1$ if i is an r-vertex of M. The base $b(M)$ is clearly a Dyck word; conversely, every Dyck word is a base of some matching. If $w_i = 0$ (or $w_i = 1$) we say that i is an *l-vertex* (or an *r-vertex*, respectively) *with respect to the base* w. Let $m \in \mathbb{N}$, let $\mathcal{M}$ be an arbitrary set of matchings, and let $w \in \mathcal{D}'_m$; we define the sets $\mathcal{G}(m, \mathcal{M})$ and $\mathcal{G}(m, w, \mathcal{M})$ as follows:

$$\mathcal{G}(m, \mathcal{M}) = \{M \in \mathcal{G}(m);\ M \text{ avoids all the elements of } \mathcal{M}\}$$
$$\mathcal{G}(m, w, \mathcal{M}) = \{M \in \mathcal{G}(m, \mathcal{M});\ b(M) = w\}.$$

Let $g(m)$, $g(m, \mathcal{M})$ and $g(m, w, \mathcal{M})$ denote the cardinalities of the sets $\mathcal{G}(m)$, $\mathcal{G}(m, \mathcal{M})$ and $\mathcal{G}(m, w, \mathcal{M})$, respectively. The sets $\mathcal{G}(m, w, \mathcal{M})$ form a partition of $\mathcal{G}(m, \mathcal{M})$. In other words, we have

$$\mathcal{G}(m, \mathcal{M}) = \bigcup_w \mathcal{G}(m, w, \mathcal{M}) \qquad \text{and} \qquad g(m, \mathcal{M}) = \sum_w g(m, w, \mathcal{M}),$$

where the union and the sum range over all Dyck words $w \in \mathcal{D}'_m$.

If no confusion can arise, we will write $\mathcal{G}(m, M)$ instead of $\mathcal{G}(m, \{M\})$ and $\mathcal{G}(m, w, M)$ instead of $\mathcal{G}(m, w, \{M\})$.

The aim of this paper is to study the relative cardinalities of the sets $\mathcal{G}(m, F)$, with F being a permutational matching with three edges. For this purpose, we introduce the following notation:

Let $\preccurlyeq$ be the quasiorder relation defined as follows: for two sets $\mathcal{M}$ and $\mathcal{M}'$ of matchings, we write $\mathcal{M} \preccurlyeq \mathcal{M}'$, if for each $m \in \mathbb{N}$ and each $w \in \mathcal{D}'_m$ we have $g(m, w, \mathcal{M}) \leq g(m, w, \mathcal{M}')$. Similarly, we write $\mathcal{M} \cong \mathcal{M}'$ if $\mathcal{M} \preccurlyeq \mathcal{M}'$ and $\mathcal{M} \succcurlyeq \mathcal{M}'$, and we write $\mathcal{M} \prec \mathcal{M}'$ if $\mathcal{M} \preccurlyeq \mathcal{M}'$ and $\mathcal{M} \ncong \mathcal{M}'$. As above, we omit the curly braces when the arguments of these relations are singleton sets.

Note that two permutations π and σ are Wilf equivalent if and only if for every $m \in \mathbb{N}$ the equality $g(m, 0^m 1^m, M_\sigma) = g(m, 0^m 1^m, M_\pi)$ holds, where $0^m 1^m$ is the Dyck word consisting of m consecutive 0-terms followed by m consecutive 1-terms. Thus, if $M_\pi \cong M_\sigma$, then π and σ are Wilf equivalent; however, the converse does not hold in general: it is well known that all the permutations of order three are Wilf equivalent (see [10]), whereas the results of this paper imply that the permutational matchings of size three fall into three $\cong$-classes.

It can be easily checked that $M_{12} \cong M_{21}$; in fact, for every $m \in \mathbb{N}$ and every $w \in \mathcal{D}'(m)$ we have $g(m, w, M_{12}) = g(m, w, M_{21}) = 1$. Recent results by Chen et al. [2] (see also [4]), and subsequent generalization by Chen et al. [3] imply that for every k, the pattern $M_{12\cdots k}$ (consisting of k crossing edges) is $\cong$-equivalent to the pattern $M_{k(k-1)\cdots 1}$ (consisting of k nested edges). In particular, $M_{123} \cong M_{321}$. In this paper, we extend this last equivalence into the following:

Theorem 1. *We have*

$$M_{213} \cong M_{132} \prec M_{123} \cong M_{321} \cong M_{231}.$$

The theorem does not cover the matching M_{312}. In fact, this matching is not equivalent to any other matching: for $w = 0000101111$ we have $41 = g(5, w, M_{123}) < g(5, w, M_{312}) = 42$. We conjecture that $M_{123} \prec M_{312}$.

The proof of Theorem 1 is divided into several independent steps, which are addressed separately in Sections 2 and 3. In Section 2, we prove that $M_{213} \cong M_{132} \prec M_{123}$, and in Section 3, we deal with the matching M_{231}.

2. The forbidden matchings M_{132} and M_{213}

Since the matching M_{132} is the mirror image of the matching M_{213}, it is obvious that $g(m, M_{132})$ is equal to $g(m, M_{213})$ for each $m \in \mathbb{N}$. However, there seems to be no straightforward argument demonstrating the stronger fact that $M_{132} \cong M_{213}$.

For $k \geq 3$, we define $C_k \in \mathcal{G}(k)$ to be the matching with edge set $E(C_k) = \{\{2i-1, 2i+2\};\ 1 \leq i < k\} \cup \{\{2, 2k-1\}\}$. Let $\mathcal{C} = \{C_k;\ k \geq 3\}$.

The goal of this section is to prove that $\mathcal{C} \cong M_{132}$. Since all the elements of $\mathcal{C}$ are symmetric upon mirror reflection, this also proves that $\mathcal{C} \cong M_{213}$ and $M_{132} \cong M_{213}$, see Corollaries 5 and 6 at the end of this section.

Throughout this section, we consider $m \in \mathbb{N}$ and $w \in \mathcal{D}'_m$ to be arbitrary but fixed, and we let $n = 2m$. For the sake of brevity, we write $\mathcal{G}^M$ instead of $\mathcal{G}(m, w, M_{132})$ and $\mathcal{G}^C$ instead of $\mathcal{G}(m, w, \mathcal{C})$. For a matching $G \in \mathcal{G}(m)$ and an arbitrary integer $k \in [n]$, let $G[k]$ denote the subgraph of G induced by the vertices in $[k]$. There are three types of vertices in $G[k]$:

- The r-vertices of G belonging to $[k]$. Clearly, all these vertices have degree one in $G[k]$.
- The l-vertices of G connected to some r-vertex belonging to $[k]$. These have degree one in $G[k]$ as well.
- The l-vertices of G belonging to $[k]$ but not connected to an r-vertex belonging to $[k]$. These are the isolated vertices of $G[k]$, and we will refer to them as the *stubs* of $G[k]$.

Let G be an arbitrary graph from $\mathcal{G}^M$. The sequence

$$G[1],\ G[2],\ G[3], \ldots,\ G[n-1],\ G[n] = G$$

will be called *the construction* of G. It is convenient to view the construction of G as a sequence of steps of an algorithm that produces the matching G by adding one vertex in every step. Two graphs G, G' from $\mathcal{G}^M$ may share an initial part of their construction; however, if $G[k] \neq G'[k]$ for some k, then obviously $G[j] \neq G'[j]$ for every $j \geq k$. It is natural to represent the set of all the constructions of graphs from $\mathcal{G}^M$ by *the construction tree* of $\mathcal{G}^M$ (denoted by $\mathcal{T}^M$), defined by the following properties:

- The construction tree is a rooted tree with n levels, where the root is the only node of level one, and all the leaves appear on level n.
- The nodes of the tree are exactly the elements of the following set:

$$\{G';\ \exists k \in [n], \exists G \in \mathcal{G}^M\colon\ G' = G[k]\}.$$

- The children of a node G' are exactly the elements of the following set:

$$\{G''; \exists k \in [n-1], \exists G \in \mathcal{G}^M\colon\ G' = G[k], G'' = G[k+1]\}.$$

It follows that the level of every node G' of the tree $\mathcal{T}^M$ is equal to the number of vertices of G'. Also, the leaves of $\mathcal{T}^M$ are exactly the elements of $\mathcal{G}^M$, and the nodes of the path from the root to a leaf G form the construction of G.

The construction tree of $\mathcal{G}^C$, denoted by $\mathcal{T}^C$, is defined in complete analogy with the tree $\mathcal{T}^M$. Our goal is to prove that the two trees are isomorphic, hence they have the same number of leaves, i.e., $|\mathcal{G}^M| = |\mathcal{G}^C|$.

We say that a graph G' on the vertex set $[k]$ is *consistent* with w, if $G' = G[k]$ for some matching $G \in \mathcal{G}(m)$ with base w.

Lemma 2.

1. *A graph G' is a node of $\mathcal{T}^M$ if and only if G' satisfies these three conditions:*
   (a) *G' is consistent with w.*
   (b) *G' avoids M_{132}.*
   (c) *G' does not contain a sequence of five vertices $x_1 < x_2 < \cdots < x_5$ such that x_2 is a stub, while $\{x_1, x_4\}$ and $\{x_3, x_5\}$ are edges of G'.*
2. *A graph H' is a node of $\mathcal{T}^C$ if and only if H' satisfies these three conditions:*
   (a) *H' is consistent with w.*
   (b) *H' avoids $\mathcal{C}$.*
   (c) *For every $p \geq 3$, H' does not contain an induced subgraph isomorphic to $C_p[2p-1]$ (by an order preserving isomorphism). In other words, for every $p \geq 3$, H' does not contain a sequence of $2p-1$ vertices $x_1 < x_2 < \cdots < x_{2p-1}$, where $2p-3$ is a stub, and the remaining $2p-2$ vertices induce the edges $\{\{x_{2i-1}, x_{2i+2}\}, 1 \leq i \leq p-2\} \cup \{\{x_2, x_{2p-1}\}\}$.*

Proof. We first prove the first part of the lemma. Let G' be a node of $\mathcal{T}^M$. Clearly, G' satisfies conditions (a) and (b) of the first part of the lemma. Assume that G' fails to satisfy condition (c). Choose $G \in \mathcal{G}^M$ such that $G' = G[k]$ for some $k \in [n]$. Let x_6 denote the r-vertex of G connected to x_2. Then $x_6 > k$, because x_2 was a stub of $G' = G[k]$, which implies that $x_6 > x_5$ and the six vertices $x_1 < \cdots < x_6$ induce a subgraph isomorphic to M_{132}, which is forbidden. This shows that the conditions (a)–(c) are necessary.

To prove the converse, assume that G' satisfies the three conditions, and let $V(G') = [k]$. We will extend G' into a graph G with base w, by adding the vertices $k+1, k+2, \ldots, n$ one by one, and each time that we add a new r-vertex i, we connect i with the smallest stub of the graph constructed in the previous steps. We claim that this algorithm yields a graph $G \in \mathcal{G}^M$. For contradiction, assume that this is not the case, and that there are six vertices $x_1 < x_2 < \cdots < x_6$ inducing a copy of M_{132}. By condition (b), we know that these six vertices are not all contained in G', which means that $x_6 > k$. Also, by condition (c), we know that $x_5 > k$. In the step of the above construction when we added the r-vertex x_5, both x_2 and x_3 were stubs. Since x_5 should have been connected to the smallest available stub, it could not have been connected to x_3, which contradicts the assumption that $x_1, \ldots, x_6$ induce a copy of M_{132}. Thus $G \in \mathcal{G}^M$, as claimed.

The proof of the second part of the lemma follows along the same lines. To see that the conditions (a)–(c) of the second part are sufficient, note that every graph satisfying these conditions can be extended into a graph $H \in \mathcal{G}^C$ by adding new vertices one by one, and connecting every new r-vertex to the biggest stub available when the r-vertex is added. We omit the details. $\square$

Let G' be a node of $\mathcal{T}^M$. We define a binary relation $\sim$ on the set of stubs of G' by the following rule: $u \sim v$ if and only if either $u = v$ or there is an edge $\{x, y\} \in E(G')$ such that $x < u < y$ and $x < v < y$.

Let H' be a node of $\mathcal{T}^C$. We define a binary relation $\approx$ on the set of stubs of H' by the following rule: $u \approx v$ if and only if either $u = v$ or H' contains a sequence of edges $e_1, e_2, \ldots, e_p$, where $p \geq 1$, $e_i = \{x_i, y_i\}$, the edge e_i crosses the edge e_{i+1} for each $i < p$,

Fig. 2. The relation $\approx$. Here $u \approx v$, assuming u and v are stubs.

and at the same time $x_1 < u < y_1$ and $x_p < v < y_p$ (see Fig. 2; note that we may assume, without loss of generality, that the edge e_i does not cross any other edge of the sequence except for e_{i-1} and e_{i+1}, and that no two edges of the sequence are nested: indeed, a minimal sequence $(e_i)_{i=1}^{p}$ witnessing $u \approx v$ clearly has these properties). We remark that the relation $\approx$ has an intuitive geometric interpretation: assume that the vertices of H' are represented by points on a horizontal line, ordered left-to-right according to the natural order, and assume that every edge of H' is represented by a half-circle connecting the corresponding endpoints. Then $u \approx v$ if and only if every vertical line separating u from v intersects at least one edge of H'.

Using condition (c) of the first part of Lemma 2, it can be easily verified that for every node G' of the tree $\mathcal{T}^M$, the relation $\sim$ is an equivalence relation on the set of stubs of G'. Let $\langle x\rangle_\sim^{G'}$ denote the block of $\sim$ containing the stub x. Clearly, the blocks of $\sim$ are contiguous with respect to the ordering $<$ of the stubs of G'; i.e., if $x < y < z$ are three stubs of G', then $x \sim z$ implies $x \sim y \sim z$.

Similarly, $\approx$ is an equivalence relation on the set of stubs of a node H' of $\mathcal{T}^C$ (notice that, contrary to the case of $\sim$, the fact that $\approx$ is an equivalence relation does not rely on the particular properties of the nodes of $\mathcal{T}^C$ described in Lemma 2). The block of $\approx$ containing x will be denoted by $\langle x\rangle_\approx^{H'}$. These blocks are contiguous with respect to the ordering $<$ as well.

Lemma 3.

1. *Let G' be a node of level $k < n$ in the tree $\mathcal{T}^M$. The following holds:*
   (a) *Let G'' be an arbitrary child of G' in the tree $\mathcal{T}^M$. This implies that $V(G'') = [k+1]$. If the vertex $k+1$ is an l-vertex with respect to w, then G'' is the only child of G', and $k+1$ is a stub in G''. In this case, $\langle x\rangle_\sim^{G'} = \langle x\rangle_\sim^{G''}$ for every stub x of G', and $\langle k+1\rangle_\sim^{G''} = \{k+1\}$. On the other hand, if $k+1$ is an r-vertex, then in the graph G'' the vertex $k+1$ is connected to a vertex x satisfying $x = \min \langle x\rangle_\sim^{G'}$. In this case, we have $\langle y\rangle_\sim^{G'} = \langle y\rangle_\sim^{G''}$ whenever $y < x$, and all the stubs $z > x$ of G'' form a single $\sim$-block in G''.*
   (b) *If $k+1$ is an r-vertex, then for every stub x satisfying $x = \min \langle x\rangle_\sim^{G'}$, G' has a child G'' which contains the edge $\{x, k+1\}$. This implies, together with part* (a)*, that if $k+1$ is an r-vertex, then the number of children of G' in $\mathcal{T}^M$ is equal to the number of its $\sim$-blocks.*
2. *Let H' be a node of level $k < n$ in the tree $\mathcal{T}^C$. The following holds:*
   (a) *Let H'' be an arbitrary child of H' in the tree $\mathcal{T}^C$. This implies that $V(H'') = [k+1]$. If the vertex $k+1$ is an l-vertex with respect to w, then H'' is the only child of H', and $k+1$ is a stub in H''. In this case, $\langle x\rangle_\approx^{H'} = \langle x\rangle_\approx^{H''}$ for every stub x of H', and $\langle k+1\rangle_\approx^{H''} = \{k+1\}$. On the other hand, if $k+1$ is an r-vertex, then in the graph H'' the vertex $k+1$ is connected to a vertex x satisfying $x = \max \langle x\rangle_\approx^{H'}$. In this case, we have $\langle y\rangle_\approx^{H'} = \langle y\rangle_\approx^{H''}$ whenever $y < x$ and $y \notin \langle x\rangle_\approx^{H'}$, and all the other stubs of H'' form a single $\approx$-block in H''.*
   (b) *If $k+1$ is an r-vertex, then for every stub x satisfying $x = \max \langle x\rangle_\approx^{H'}$, H' has a child H'' which contains the edge $\{x, k+1\}$. This implies, together with part* (a)*, that if $k+1$ is an r-vertex, then the number of children of H' in $\mathcal{T}^C$ is equal to the number of its $\approx$-blocks.*

Proof. We first prove part 1(a). The case when $k + 1$ is an l-vertex follows directly from the definition of $\sim$, so let us assume that $k + 1$ is an r-vertex, and let x be the vertex connected to $k + 1$ in G''. Assume, for contradiction, that $x \neq \min \langle x \rangle_{\sim}^{G'}$, and choose $y \in \langle x \rangle_{\sim}^{G'}$ such that $y < x$. Since $y \sim x$, G' must contain an edge $e = \{u, v\}$, with $u < y < x < v$. Then the five vertices $u, y, x, v, k + 1$ form in G'' a configuration that was forbidden by Lemma 2, part 1(c). This shows that $x = \min \langle x \rangle_{\sim}^{G'}$. The edge $\{x, k+1\}$ guarantees that all the stubs larger than x are $\sim$-equivalent in G'', whereas the equivalence classes of the stubs smaller than x are unaffected by this edge. This concludes the proof of part 1(a).

To prove part 1(b), it is sufficient to show that after choosing a vertex x such that $x = \min \langle x \rangle_{\sim}^{G'}$ and adding the edge $\{x, k + 1\}$ to G', the resulting graph G'' satisfies the three conditions of the first part of Lemma 2. Condition 1(a) of Lemma 2 is satisfied automatically. If G'' fails to satisfy condition 1(b), then G' fails to satisfy one of the conditions 1(b) and 1(c) of that lemma, which is impossible. Similarly, if G'' fails to satisfy condition 1(c), then either G' fails to satisfy this condition as well, or G' contains a stub y with $y < x$ and $y \sim x$, contradicting our choice of x.

The proof of the second part of this lemma follows along the same lines as the proof of the first part, and we omit it. □

We are now ready to state and prove the main theorem of this section.

Theorem 4. *The trees* $\mathcal{T}^M$ *and* $\mathcal{T}^C$ *are isomorphic.*

Proof. Our aim is to construct a mapping ϕ with the following properties:

- The mapping ϕ maps the nodes of $\mathcal{T}^M$ to the nodes of $\mathcal{T}^C$, preserving their level.
- If G' is a child of G in $\mathcal{T}^M$, then $\phi(G')$ is a child $\phi(G)$ in $\mathcal{T}^C$. Furthermore, if G_1 and G_2 are two distinct children of a node G in $\mathcal{T}^M$, then $\phi(G_1)$ and $\phi(G_2)$ are two distinct children of $\phi(G)$ in $\mathcal{T}^C$.
- Let G be an arbitrary node of $\mathcal{T}^M$, and let $H = \phi(G)$. Let $\langle x_1 \rangle_{\sim}^{G}, \langle x_2 \rangle_{\sim}^{G}, \ldots, \langle x_s \rangle_{\sim}^{G}$ be the sequence of all the distinct blocks of $\sim$ in G, uniquely determined by the condition $x_1 < x_2 < \cdots < x_s$. Similarly, let $\langle y_1 \rangle_{\approx}^{H}, \langle y_2 \rangle_{\approx}^{H}, \ldots, \langle y_t \rangle_{\approx}^{H}$ be the sequence of all the distinct blocks of $\approx$ in H, uniquely determined by the condition $y_1 < y_2 < \cdots < y_t$. Then $s = t$ and $|\langle x_i \rangle_{\sim}^{G}| = |\langle y_i \rangle_{\approx}^{H}|$ for each $i \in [s]$.

These conditions guarantee that ϕ is an isomorphism, because, thanks to Lemma 3, we know that the number of children of each node of $\mathcal{T}^M$ (or $\mathcal{T}^C$) at level k is either equal to one if $k+1$ is an l-vertex or equal to the number of blocks of its $\sim$ relation (or $\approx$ relation, respectively) if $k+1$ is an r-vertex.

The mapping ϕ is defined recursively for nodes of increasing level. The root of $\mathcal{T}^M$ is mapped to the root of $\mathcal{T}^C$. Assume that the mapping ϕ has been determined for all the nodes of $\mathcal{T}^M$ of level at most k, for some $k \in [n - 1]$, and that it does not violate the properties stated above. Let G be a node of level k, let $H = \phi(G)$. If $k + 1$ is an l-vertex, then G has a unique child G' and H has a unique child H'. In this case, define $\phi(G') = H'$. Let us now assume that $k + 1$ is an r-vertex. Let $\langle x_1 \rangle_{\sim}^{G}, \langle x_2 \rangle_{\sim}^{G}, \ldots, \langle x_s \rangle_{\sim}^{G}$ be the sequence of all the distinct blocks of $\sim$ on G, with $x_1 < x_2 < \cdots < x_s$. We may assume, without loss of generality, that $x_i = \min \langle x_i \rangle_{\sim}^{G}$ for $i \in [s]$. By assumption, $\approx$ has s blocks on H. Let $\langle y_1 \rangle_{\approx}^{H}, \langle y_2 \rangle_{\approx}^{H}, \ldots, \langle y_s \rangle_{\approx}^{H}$ be the sequence of these blocks, where $y_1 < y_2 < \cdots < y_s$ and $y_i = \max \langle y_i \rangle_{\approx}^{H}$ for every $i \in [s]$. By Lemma 3, the nodes G and H have s children in $\mathcal{T}^M$ and $\mathcal{T}^C$. Let G_i be the graph obtained from G by addition of the edge $\{x_i, k + 1\}$, let H_i be the graph obtained from H by addition of the edge $\{y_i, k + 1\}$, for $i \in [s]$. By Lemma 3, the graphs $\{G_i;\ i \in [s]\}$ (or $\{H_i;\ i \in [s]\}$) are exactly the children of G (or H, respectively). We define $\phi(G_i) = H_i$. The $\sim$-blocks of G_i are exactly the sets $\langle x_1\rangle^G_\sim, \langle x_2\rangle^G_\sim, \ldots, \langle x_{i-1}\rangle^G_\sim$ and $\left(\bigcup_{j\geq i}\langle x_j\rangle^G_\sim\right) \setminus \{x_i\}$, while the $\approx$-blocks of H_i are exactly the sets $\langle y_1\rangle^H_\approx, \langle y_2\rangle^H_\approx, \ldots, \langle y_{i-1}\rangle^H_\approx$ and $\left(\bigcup_{j\geq i}\langle y_j\rangle^H_\approx\right)\setminus\{y_i\}$. This implies that the corresponding blocks of G_i and H_i have the same number and the same size, as required (note that if $i = s$ and $\langle x_s\rangle^G_\sim = \{x_s\}$, then the last block in the above list of $\sim$-blocks of G_i is empty; however, this happens if and only if the last entry in the list of $\approx$-blocks of H_i is empty as well, so it does not violate the required properties of ϕ).

This concludes the proof of the theorem. $\square$

Corollary 5. $M_{132} \cong \mathcal{C}$.

Proof. Since $g(m, w, M_{132})$ is equal to the number of leaves of the tree $\mathcal{T}^M$, and $g(m, w, \mathcal{C})$ is equal to the number of leaves of the tree $\mathcal{T}^C$, this is a direct consequence of Theorem 4. $\square$

Corollary 6. $M_{132} \cong M_{213}$.

Proof. Let $\overline{w}$ denote the Dyck word defined by the relation $\overline{w}_i = 0$ if and only if $w_{n-i+1} = 1$. By inverting the linear order of the vertices of a matching M with base w, we obtain a matching $\overline{M}$ with base $\overline{w}$. Since every matching $C_k \in \mathcal{C}$ satisfies $\overline{C_k} = C_k$, we know that a matching M avoids $\mathcal{C}$ if and only if $\overline{M}$ avoids $\mathcal{C}$, and hence $g(m, w, \mathcal{C}) = g(m, \overline{w}, \mathcal{C})$. Note that $\overline{M_{213}} = M_{132}$. This gives

$$g(m, w, M_{132}) = g(m, w, \mathcal{C}) = g(m, \overline{w}, \mathcal{C}) = g(m, \overline{w}, M_{132}) = g(m, w, M_{213}),$$

as claimed. $\square$

Corollary 7. $M_{123} \succ M_{132}$.

Proof. Notice that $M_{123} = C_3 \in \mathcal{C}$, and all the other graphs in $\mathcal{C}$ avoid M_{123}. This implies that $\mathcal{G}(m, w, \mathcal{C}) \subseteq \mathcal{G}(m, w, M_{123})$, and for every $m \geq 4$ there is a $w \in \mathcal{D}'_m$ for which this is a proper inclusion, because C_m clearly belongs to $\mathcal{G}(m, M_{123}) \setminus \mathcal{G}(m, \mathcal{C})$. The claim follows, as a consequence of Corollary 5. $\square$

3. The matching M_{231} and non-crossing pairs of Dyck paths

In this section, we prove that $M_{231} \cong M_{321}$. We first introduce some notation: recall that $\mathcal{D}(m)$ denotes the set of all Dyck paths of length $2m$. For two Dyck paths P_1 and P_2 of length $2m$, we say that (P_1, P_2) is a *non-crossing pair* if P_2 never reaches above P_1. Let $\mathcal{D}^2_m$ denote the set of all the non-crossing pairs of Dyck paths of length $2m$ and, for a Dyck word w of length $2m$, let $\mathcal{D}^2_m(w)$ be the set of all the pairs $(P_1, P_2) \in \mathcal{D}^2_m$ whose first component P_1 is the path represented by the Dyck word w.

Recently, Chen et al. [2] have proved that $M_{123} \cong M_{321}$ by a bijective construction involving Dyck paths. Their proof in fact shows that the cardinality of the set $D^2_m(w)$ is equal to the number of matchings with base w avoiding M_{123}, and at the same time equal to the number of matchings with base w avoiding M_{321}. In our notation, this corresponds to the following claim:

$$\forall m \in \mathbb{N}\ \forall w \in \mathcal{D}'_m\ g(m, w, M_{123}) = |D^2_m(w)| = g(m, w, M_{321}). \tag{1}$$

Another proof of (1), using a more general approach, has been obtained by Chen et al. [3].

In this section, we extend the equalities (1) to the matching M_{231} by proving $|D^2_m(w)| = g(m, w, M_{231})$. This shows that $M_{231} \cong M_{321} \cong M_{123}$.

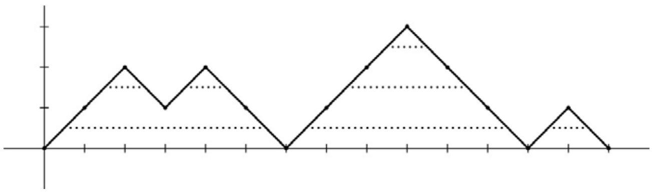

Fig. 3. An example of a Dyck path. The dotted segments represent the tunnels.

We remark that the number of non-crossing pairs of Dyck paths of length $2m$ (and hence the number of M-avoiding matchings of size m, where M is any of M_{123}, M_{321} or M_{231}) is equal to $c_{m+2}c_m - c_{m+1}^2$, where c_m is the m-th Catalan number (see [8]). The sequence $(c_{m+2}c_m - c_{m+1}^2;\ m \in \mathbb{N})$ is listed as the entry A005700 in the On-Line Encyclopedia of Integer Sequences [11]. It is noteworthy, that Bonichon [1] has shown a completely different combinatorial interpretation of this sequence, in terms of realizers of plane triangulations.

Let us fix $m \in \mathbb{N}$ and $w \in \mathcal{D}'_m$. Let M be a matching with base w. Let $1 = x_1 < \cdots < x_m$ denote the sequence of all the l-vertices with respect to w, and $y_1 < \cdots < y_m = 2m$ be the sequence of all the r-vertices with respect to w. Let y_k be the neighbour of x_1 in M. An edge $\{x_i, y_j\}$ of M is called *short* if $y_j < y_k$, and it is called *long* if $y_j > y_k$. Let $E_{\mathrm{S}}(M)$ and $E_{\mathrm{L}}(M)$ denote the set of the short edges and long edges, respectively, so that we have $E(M) = E_{\mathrm{S}}(M) \cup E_{\mathrm{L}}(M) \cup \{\{1, y_k\}\}$. An l-vertex x_i is called *short* (or *long*) if it is incident with a short edge (or a long edge, respectively).

Lemma 8. *Let M be a matching with base w. M avoids M_{231} if and only if M satisfies the following three conditions:*

- *The subgraph of M induced by the short edges avoids M_{231}.*
- *The subgraph of M induced by the long edges avoids M_{231}.*
- *Every short l-vertex precedes all the long l-vertices.*

Proof. The first two conditions are clearly necessary. The third condition is necessary as well, for if M contained an edge $\{x_s, y_s\} \in E_S$ and an edge $\{x_l, y_l\} \in E_L$ with $x_s > x_l$, then the six vertices $1 < x_l < x_s < y_s < y_k < y_l$ would induce a copy of M_{231}.

To see that the three conditions are sufficient, assume for contradiction that a graph M satisfies these conditions but contains the forbidden configuration induced by some vertices $x_a < x_b < x_c < y_d < y_e < y_f$. We first note that $y_f > y_k$: indeed, it is impossible to have $y_f = y_k$, because y_f is not connected to the leftmost vertex, and the inequality $y_f < y_k$ would imply that all the three edges of the forbidden configuration are short, which is ruled out by the first condition of the lemma. Thus, the edge $\{x_b, y_f\}$ is long, and hence $\{x_c, y_d\}$ is long as well, by the third condition. This implies that $y_d > y_k$, hence $y_e > y_k$ as well, and all the three edges of the configuration are long, contradicting the second condition of the lemma. $\square$

To construct the required bijection between $\mathcal{G}(m, w, M_{231})$ and $\mathcal{D}^2_m(w)$, we will use the intuitive notion of a "tunnel" in a Dyck path, which has been employed in bijective constructions involving permutations in, e.g., [5,6] or [7]. Let P be a Dyck path. A *tunnel* in P is a horizontal segment t whose left endpoint is the center of an up-step of P, its right endpoint is the center of a down-step of P, and no other point of t belongs to P (see Fig. 3). A path $P \in \mathcal{D}_m$ has exactly m tunnels. An up-step u and a down-step d of P are called *partners* if P has a tunnel connecting u and d. Let $u_1(P), \ldots, u_m(P)$ denote the up-steps of P and $d_1(P), \ldots, d_m(P)$ denote the down-steps of P, in the left-to-right order in which they appear on P.

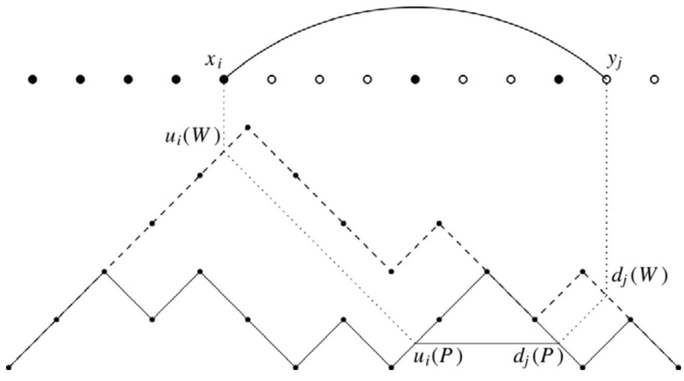

Fig. 4. The correspondence between a pair of Dyck paths (W, P) and a matching $M(W, P)$. A tunnel between $u_i(P)$ and $d_j(P)$ corresponds to an edge $\{x_i, y_j\}$. The filled dots above the pair of paths represent the l-vertices of the matching, the empty dots represent the r-vertices.

Let $W \in \mathcal{D}_m$ be the Dyck path represented by the Dyck word w, let $(W, P) \in \mathcal{D}_m^2(w)$ be a non-crossing pair of Dyck paths. Let $M(W, P)$ be the unique matching with base w satisfying the condition that $\{x_i, y_j\}$ is an edge of M if and only if $u_i(P)$ is the partner of $d_j(P)$. To see that this definition is valid, we need to check that if $u_i(P)$ is partnered to $d_j(P)$ in a path P not exceeding W, then $x_i < y_j$ in the matchings with base w. This is indeed the case, because the horizontal coordinate of $u_i(W)$ (which determines the position of x_i in the matching) does not exceed the horizontal coordinate of $u_i(P)$, while the horizontal coordinate of $d_j(P)$ does not exceed the horizontal coordinate of $d_j(W)$ (note that a half-line starting in the center of $u_i(P)$ directed north-west intersects W in the center of $u_i(W)$; similarly, a half-line starting in the center of $d_j(P)$ directed north-east hits the center of $d_j(W)$). See Fig. 4.

Lemma 9. *If $(W, P) \in \mathcal{D}_m^2(w)$, then $M(W, P)$ avoids M_{231}.*

Proof. Choose an arbitrary $(W, P) \in \mathcal{D}_m^2(w)$ and assume, for contradiction, that there are six vertices $x_a < x_b < x_c < y_d < y_e < y_f$ in $M(W, P)$ which induce the forbidden configuration. Let t_{cd}, t_{ae} and t_{bf} be the tunnels corresponding to the three edges $x_c y_d$, $x_a y_e$ and $x_b y_f$, respectively. Note that the projection of t_{cd} onto some horizontal line h is a subset of the projections of t_{ae} and t_{bf} onto h. Thus, the three tunnels lie on different horizontal lines and there is a vertical line intersecting all of them.

Since $a < b$, the tunnel t_{ae} must lie below t_{bf}, otherwise the subpath of P between $u_a(P)$ and $u_b(P)$ would intersect t_{ae}. On the other hand, $e < f$ implies that t_{ae} lies above t_{bf}, a contradiction. □

The aim of the next lemma is to show that the mapping $P \mapsto M(W, P)$ can be inverted.

Lemma 10. *For every $M \in \mathcal{G}(m, w, M_{231})$ there is a unique Dyck path P such that $(W, P) \in \mathcal{D}_m^2(w)$ and $M = M(W, P)$.*

Proof. We proceed by induction on m. The case $m = 1$ is clear, so let us assume that $m > 1$ and that the lemma holds for every $m' < m$ and every $w' \in \mathcal{D}'_{m'}$. Let us choose an arbitrary $w \in \mathcal{D}'_m$, and an arbitrary $M \in \mathcal{G}(m, w, M_{231})$, and define k such that $\{1, y_k\}$ is an edge of M. Let M_S be the matching from $\mathcal{G}(k-1)$ that is isomorphic to the subgraph of M induced by the short edges, let $M_L \in \mathcal{G}(m-k)$ be isomorphic to the subgraph induced by the long ones, let w_S and w_L be the respective bases of M_S and M_L, and let W_S and W_L be the Dyck paths corresponding to w_S

and w_{L}. By induction, we know that $M_{\mathrm{S}} = M(W_{\mathrm{S}}, P_{\mathrm{S}})$ and $M_{\mathrm{L}} = M(W_{\mathrm{L}}, P_{\mathrm{L}})$ for some Dyck paths P_{S} and P_{L}, where P_{S} does not exceed W_{S}, and P_{L} does not exceed W_{L}. Let w_{X} be the Dyck word $0w_{\mathrm{S}}1w_{\mathrm{L}}$, and let W_{X} be the corresponding Dyck path. Note that W_{X} does not exceed W: assume that W has t up-steps occurring before the k-th down-step; then W_{X} is obtained from W by omitting the $t-k$ up-steps $u_{k+1}(W), u_{k+2}(W), \ldots, u_t(W)$, and inserting $t-k$ new up-steps directly after the k-th down-step.

Let P be the Dyck path obtained by concatenating the following pieces:

- An up-step from $(0, 0)$ to $(1, 1)$.
- A shifted copy of P_{S} from $(1, 1)$ to $(2k-1, 1)$.
- A down-step from $(2k-1, 1)$ to $(2k, 0)$.
- A shifted copy of P_{L} from $(2k, 0)$ to $(2m, 0)$.

Since P clearly does not exceed W_{X}, it does not exceed W either. Let us check that $M = M(W, P)$:

- The base of M is equal to the base of $M(W, P)$. Thus, to see that M is equal to $M(W, P)$, it suffices to check that M_{S} and M_{L} are isomorphic to the matchings induced by the short edges of $M(W, P)$ and the long edges of $M(W, P)$, respectively.
- The up-step $u_1(P)$ is clearly partnered to the down-step $d_k(P)$ (which connects $(2k-1, 1)$ to $(2k, 0)$). Thus, $M(W, P)$ contains the edge $\{x_1, y_k\}$. It follows that $M(W, P)$ has $k-1$ short edges, incident to the l-vertices $x_2, \ldots, x_k$ and r-vertices $y_1, \ldots, y_{k-1}$.
- The $k-1$ up-steps $u_2(P), u_3(P), \ldots, u_k(P)$ as well as the $k-1$ down-steps $d_1(P), d_2(P), \ldots, d_{k-1}(P)$ all belong to the shifted copy of P_{S}. Since shifting does not affect the partnership relations, we see that the short edges of $M(W, P)$ form a matching isomorphic to $M_{\mathrm{S}} = M(W_{\mathrm{S}}, P_{\mathrm{S}})$.
- Similarly, the up-steps $u_{k+1}(P), u_{k+2}(P), \ldots, u_m(P)$ are partnered to the down-steps $d_{k+1}(P), d_{k+2}(P), \ldots, d_m(P)$ according to the tunnels of P_{L}. The corresponding long edges form a matching isomorphic to M_{L}.

It follows that $M = M(W, P)$.

We now show that P is determined uniquely: assume that $M = M(W, Q)$ for some $Q \in \mathcal{D}_m$. Since $\{1, y_k\} \in E(M)$, the path Q must contain a down-step from $(2k-1, 1)$ to $(2k, 0)$, and this down-step must be the first down-step of Q to reach the line $y = 0$. This shows that the subpath of Q between $(1, 1)$ and $(2k-1, 1)$ is a shifted copy of some Dyck path $Q_S \in \mathcal{D}_{k-1}$. The tunnels of this path must define a matching isomorphic to $M_{\mathrm{S}} = (W_{\mathrm{S}}, P_{\mathrm{S}})$. By induction, we know that P_{S} is determined uniquely, hence $P_{\mathrm{S}} = Q_{\mathrm{S}}$. By the same argument, we see that the subpath of Q from $(2k, 0)$ to $(2m, 0)$ is a shifted copy of P_{L}. This shows that $P = Q$, and P is unique, as claimed. $\square$

We are now ready to prove the following theorem:

Theorem 11. *For each $m \in \mathbb{N}$ and for each $w \in \mathcal{D}'_m$, $g(m, w, M_{231})$ is equal to $|\mathcal{D}^2_m(w)|$.*

Proof. Putting together Lemmas 9 and 10, we infer that the function that maps a pair $(W, P) \in \mathcal{D}_m(w)$ to the matching $M(W, P)$ is a bijection between $\mathcal{D}^2_m(w)$ and $\mathcal{G}(m, w, M_{231})$. This gives the required result. $\square$

Corollary 12. $M_{231} \cong M_{321}$.

Proof. This a direct consequence of Theorem 11 and the results of Chen et al. [2]. $\square$

4. Conclusion and open problems

We have introduced an equivalence relation $\cong$ on the set of permutational matchings, and we have determined how this equivalence partitions the set of permutational matchings of order 3. However, many natural questions remain unanswered:

- Is it true that $M_{123} \prec M_{312}$? We conjecture that the answer is yes.
- Into how many blocks does the equivalence relation $\cong$ partition the set $\mathcal{G}(m)$? Is it possible to characterize the minimal and the maximal elements of $(\mathcal{G}(m), \preccurlyeq)$?
- Chen et al. [3] have shown that the matching of k pairwise nested edges and the matching of k pairwise crossing edges are $\cong$-equivalent. What other examples of pairs of arbitrarily large $\cong$-equivalent (or just $\preccurlyeq$-comparable) matchings are there?

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