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Multivariate polynomial perturbations of algebraic equations ☆

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ABSTRACT

In this note we study multivariate perturbations of algebraic equations. In general, it is not possible to represent the perturbed solution as a Puiseux-type power series in a connected neighborhood. For the case of two perturbation parameters we provide a sufficient condition that guarantees such a representation. Then, we extend this result to the case of more than two perturbation parameters. We motivate our study by the perturbation analysis of a weighted random walk on the Web Graph. In an instance of the latter the stationary distribution of the weighted random walk, the so-called Weighted PageRank, may depend on two (or more) perturbation parameters in a manner that illustrates our theoretical development.

1. Introduction

We consider a polynomial perturbations of an algebraic equation, namely, an equation of the form

$$0 = P(x, \varepsilon_1, \dots, \varepsilon_m) = \sum_{k=0}^{n} a_k(\varepsilon_1, \dots, \varepsilon_m) x^k, \tag{1}$$

where coefficients $a_k(\varepsilon_1, \dots, \varepsilon_m)$ are in general complex polynomials of perturbation parameters $\varepsilon_1, \dots, \varepsilon_m$. Without loss of generality, we can assume that the perturbation parameters are small and the unperturbed equation $P(x, 0, \dots, 0)$ has a zero solution. Furthermore, as in [3], we also assume that $a_n(0, \dots, 0) \neq 0$.

If the partial derivative $\frac{\partial P}{\partial x}(0, 0, \dots, 0) \neq 0$, by the implicit function theorem the perturbed solution $x(\varepsilon_1, \dots, \varepsilon_m)$ is analytic at the origin. That is, we expand the solution of (1) as the power series

$$x(\varepsilon_1, \dots, \varepsilon_m) = \sum_{k_1, \dots, k_m} c_{k_1, \dots, k_m} \prod_{j=1}^{m} \varepsilon_j^{k_j}, \tag{2}$$

that is convergent in some polydisc centered at the origin and $k_j \in \mathbb{N} \cup \{0\}$.

If the partial derivative $\frac{\partial P}{\partial x}(0, 0, \dots, 0) = 0$, the perturbed solution might not admit the power series representation (2). Firstly, the exponents may no longer remain integers. Furthermore, not every algebraic equation with polynomial coefficients has a solution that admits a unique representation as a Puiseux-type series in some punctured connected neighborhood of the origin. Let us give the following example.

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Example 1. Consider the perturbed polynomial

$$P(x,\boldsymbol{\varepsilon}) = x^2 + 2(\varepsilon_1 - \varepsilon_2)x - 3\left(\varepsilon_1^2 + \varepsilon_2^2\right), \tag{3}$$

with two perturbation parameters ε_1 and ε_2. The partial derivative of P in respect to x at 0 is

$$\frac{\partial P}{\partial x}(x,0,0) = 2x + 2(\varepsilon_1 - \varepsilon_2)|_0 = 0.$$

The two roots of (3) are given by

$$x_{1,2}(\varepsilon) = \varepsilon_2 - \varepsilon_1 \pm \sqrt{4\varepsilon_1^2 - 2\varepsilon_1\varepsilon_2 + 4\varepsilon_2^2}.$$

The solutions $x_{1,2}(\varepsilon)$ are not analytic at $(\varepsilon_1,\varepsilon_2) = (0,0)$. Furthermore, $x_{1,2}(\varepsilon)$ does not have a representation as a unique power series which is convergent in some connected punctured neighborhood of the origin. However, the solutions $x_{1,2}(\varepsilon)$ can be expanded as power series in cones. For instance, if we consider the cone $|\varepsilon_1| < |\varepsilon_2|$, in this cone the solutions admit the following power series expansion

$$x_{1,2}(\varepsilon) = \varepsilon_2 - \varepsilon_1 \pm 2\varepsilon_2\left(1 - \frac{1}{4}\frac{\varepsilon_1}{\varepsilon_2} + \frac{15}{32}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + \frac{15}{128}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^3 + o\left(\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^4\right)\right).$$

In [4] and [5] a general framework for the derivation of the power (possibly, fractional power series) expansions of the perturbed solutions valid in *cones* was provided. In [4] the author studied one algebraic equation with several perturbation parameters and in [5] he generalized results of [4] to the case of a system of algebraic equations.

The goal of the present work is to identify a sufficiently general situation when the area of convergence is represented by a *connected* neighborhood. In particular, we show that for two perturbation parameters that case can be identified by the condition that the partial derivative of the discriminant with respect to any one of the perturbation parameters does not vanish at the origin. We also identify a situation when we can deal with more than two perturbation parameters.

Our approach is based on the Newton polygon method. The application of the Newton polygon to the perturbed equations with one perturbation parameter is explained in [3,9,10]. In particular, the authors of [3] combine the Newton polygon with the Newton-like iteration method to efficiently compute the Puiseux series expansions of the perturbed solutions.

The structure of the paper is as follows: In Section 2 we present our main result which provides a sufficient condition for the existence of the Puiseux-type series of the perturbed solution convergent in a connected neighborhood of the origin. We also show how our results can be applied in the case of more than two perturbation parameters. This is a generalisation of the results from [1,2] where the case of one perturbation parameter was analysed. Then, in Section 3 we provide an illustrative example of the perturbation analysis of a weighted random walk on the Web graph. The random walk stationary distribution, Weighted PageRank, is used as a popularity measure for Web pages.

2. Results

Let $P(x,\boldsymbol{\varepsilon}) = \sum_{k=0}^{n} a_k(\varepsilon_1,\varepsilon_2)x^k$ be a polynomial of degree $n \geqslant 2$ in a variable x with polynomial coefficients in $\boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2)$ such that $a_n(0,0) \neq 0$. Consider the corresponding algebraic equation

$$P(x,\boldsymbol{\varepsilon}) = 0 \tag{4}$$

in a neighbourhood $\mathbf{U}$ of $\boldsymbol{\varepsilon} = (0,0)$ in $\mathbb{C}^2$. Let $x^{(i)}(\varepsilon_1,\varepsilon_2)$, $i = 1,\dots,n$, be (possibly multiple) roots of the polynomial $P(x,\boldsymbol{\varepsilon})$ with respect to x. Then, the descriminant of $P(x,\boldsymbol{\varepsilon})$ is defined by

$$D(\varepsilon_1,\varepsilon_2) = a_n(\varepsilon_1,\varepsilon_2)\prod_{i<j}\left(x^{(i)}(\varepsilon_1,\varepsilon_2) - x^{(j)}(\varepsilon_1,\varepsilon_2)\right).$$

If the discriminant $D(0,0) \neq 0$, Eq. (4) has n distinct solutions, which are analytic in the coefficients of P, and, so, in $\boldsymbol{\varepsilon}$. Consequently, the partial derivative of P with respect to x at any unperturbed root is not equal to zero and the perturbed solution admits the power series representation (2). Here we are predominantly interested in the case $D(0,0) = 0$. As a regularity condition in this case we assume that

$$\frac{\partial D}{\partial \varepsilon_1}(0,0) \neq 0. \tag{5}$$

By a regular algebraic function in variable ε_i, $i = 1,2$, we will mean a Puiseux series expansion

$$\sum_{k=0}^{\infty} r_k \varepsilon_i^{\frac{k}{d_i}},$$

for some integer $d_i \geqslant 1$ with non-negative powers of ε_i. Such a series is presumed to converge in any one-dimensional disc V_i such that $0 \notin V_i$ and $0 \times V_i \subset \mathbf{U}$.

Theorem 1. *Let the regularity condition* (5) *hold. Then, for a polynomial equation* (4) *any solution admits a form of a Puiseux series in* ε_1,

$$x(\boldsymbol{\varepsilon}) = \sum_{k=0}^{\infty} c_k(\varepsilon_2)\varepsilon_1^{\frac{k}{d_1}}, \tag{6}$$

where d_1 *is some positive integer and coefficients* $c_k(\varepsilon_2)$ *that are algebraic functions in* ε_2.

Remark 1. In the above $\varepsilon_1^{\frac{1}{d_1}}$ is a multi-valued function with d_1 branches corresponding to $\varepsilon_1^{\frac{1}{d_1}} e^{2m\pi i}$; $m = 0, 1, \ldots, d_1 - 1$. These branches correspond to the different roots of the polynomial (4).

Proof. We use the Newton Polygon technique with respect to the variable ε_1, following the algorithm outlined in [3]. Detailed description of the Newton polygon process can be found in [9,10]. Let

$$a_k(\varepsilon_1, \varepsilon_2) = \sum_{j=0}^{m_k} a_{k,j}(\varepsilon_2)\varepsilon_1^j. \tag{7}$$

Without loss of generality, see [3], we can assume that $a_n(0,0) \neq 0$. This assumption excludes negative powers of ε_1 in (6). For non-zero $a_{k,j}$ we plot the points (k, j) in a quadrant of the two-dimensional lattice coordinates (ξ, η), with $\xi, \eta \in \mathbb{N}_0$ where ξ corresponds to the degrees of x and η corresponds to the degree of ε_1 in $a_k(\varepsilon_1, \varepsilon_2)$. We join those points with a convex polygon arc Γ so that each of its vertices is one of the plotted points and no other plotted point lies below any line extending any segment of Γ. We choose any segment of the polygon arc Γ. It is described by $\eta + \gamma_1\xi = \beta_1$ of the arc Γ with rational $\gamma_1 > 0$. By the continuity argument γ_1 and β_1 can be selected universal for ε_2. Let g_1 denote the set of indices k for which vertices corresponding to $a_{k,j(k)}$ lie on the chosen segment. Then, we need to solve the polynomial equation

$$\sum_{k \in g_1} a_{k,j(k)}(\varepsilon_2)c^k = 0. \tag{8}$$

By Puiseux theorem [8,10], $c_1(\varepsilon_2)$ in (6) is a root of (8) and hence an algebraic function that admits a Puiseux series representation in ε_2. We note, that since the power in ε_2 of the highest degree term in $c_1(\varepsilon_2)$ is uniformly bounded by the total degree of the original polynomial P, the negative power in ε_2 of any such Puiseux series expansion is also uniformly bounded. If $c_1(\varepsilon_2)$ is a simple root, then the Newton polygon process stops. If $c_1(\varepsilon_2)$ is a multiple root, we need to perform one more iteration. As the smallest degree of ε_1 in D is equal to 1, Theorem 6.2 in [3] guarantees that the Newton Polygon process terminates at most in two steps. Suppose that $c_1(\varepsilon_2)$ is a multiple root. Then, we make a change of variables $P_1(x, \boldsymbol{\varepsilon}) \leftarrow \varepsilon_1^{-\beta_1} P(\varepsilon_1^{\gamma_1}(x + c_1(\varepsilon_2)), \varepsilon)$ and we construct the Newton polygon for $P_1(x, \boldsymbol{\varepsilon})$. We note that the coefficients of $P_1(x, \boldsymbol{\varepsilon})$, in general, are algebraic in ε_2 and polynomial in ε_1. Its second iteration provides the eventual value of the exponent denominator d_1 in (6) which is the smallest common denominator of γ_1 and γ_2. By Lemma 6.1 and Theorem 6.2 from [3], the substitutions $\varepsilon_1^{-\beta_2} P_1(\varepsilon_1^{\gamma_2} x, \boldsymbol{\varepsilon})$ for P_1 and then $P_1(x, \varepsilon_1^{d_1}, \varepsilon_2)$ for the updated P_1 ensure that the resulting polynomial has only simple roots. These roots are, therefore, analytic functions in the coefficients of the polynomial, which in the original perturbation parameters are algebraic functions in ε_1 with algebraic coefficients in ε_2. □

Remark 2. By choosing different segments of the Newton polygon one can obtain series expansions for all the roots of the polynomial equation (4).

Remark 3. We note that the condition (5) allows us to represent the perturbed solution as Puiseux-type series in the variables ε_2 and D.

We note that the regularity condition (5) is essential as the example in the introduction clearly demonstrates.
Let us illustrate the construction of the Puiseux-type series of the perturbed solution by the following example.

Example 2. Consider the perturbed polynomial

$$P(x, \boldsymbol{\varepsilon}) = x^2 + 2(\varepsilon_1 - \varepsilon_2)x + \left(\varepsilon_1 + \varepsilon_2^2\right).$$

Its discriminant is

$$D(\varepsilon_1, \varepsilon_2) = (\varepsilon_1 - \varepsilon_2)^2 - \left(\varepsilon_1 + \varepsilon_2^2\right) = \varepsilon_1^2 - \varepsilon_1 - 2\varepsilon_1\varepsilon_2.$$

Clearly, $D(0,0) = 0$ and

$$\frac{\partial D}{\partial \varepsilon_1}(0,0) = -1 \neq 0.$$

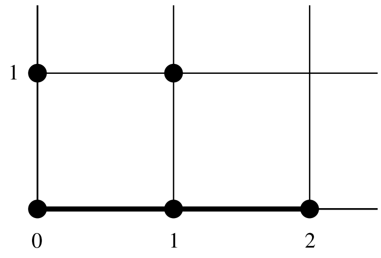

Fig. 1. Newton diagram for Example 2; 1st step.

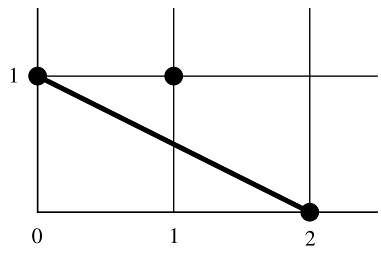

Fig. 2. Newton diagram for Example 2; 2nd step.

Thus, the condition of Theorem 1 is satisfied. Here,

$$a_0(\varepsilon_1,\varepsilon_2)=\varepsilon_1+\varepsilon_2^2,\qquad a_1(\varepsilon_1,\varepsilon_2)=2\varepsilon_1-2\varepsilon_2,\qquad a_2(\varepsilon_1,\varepsilon_2)=1.$$

Thus,

$$a_{0,0}(\varepsilon_2)=\varepsilon_2^2,\qquad a_{0,1}(\varepsilon_2)=1,\qquad a_{1,0}(\varepsilon_2)=-2\varepsilon_2,\qquad a_{1,1}(\varepsilon_2)=2,\qquad a_{2,0}(\varepsilon_2)=1.$$

Fix ε_2 and plot an associated Newton polygon as in Fig. 1.

The Newton polygon consists of one horizontal line. This means that the Puiseux series of any perturbed solution starts with a term depending only on ε_2. To find this term we set $\varepsilon_1=0$. The polynomial equation $P(x,\boldsymbol{\varepsilon})=0$ reduces to

$$x^2-2\varepsilon_2x+\varepsilon_2^2=0,$$

which has as a multiple root $x=\varepsilon_2$. Next, we make a change of variable $x=\varepsilon_2+x_1$ and transform $P(x,\boldsymbol{\varepsilon})=0$ into

$$(\varepsilon_2+x_1)^2+2(\varepsilon_1-\varepsilon_2)(\varepsilon_2+x_1)+\left(\varepsilon_1+\varepsilon_2^2\right)=0,$$

which simplifies to

$$\tilde{P}(x_1,\varepsilon_1,\varepsilon_2)=x_1^2+2\varepsilon_1x_1+2\varepsilon_1\varepsilon_2+\varepsilon_1=0.$$

For the above equation, as before, we construct the Newton polygon (see Fig. 2).

From the Newton polygon we conclude $\gamma=1/2$ and $\beta=1$ and the defining equation, consisting of zero degree terms of $\varepsilon_1^{-1}\tilde{P}(\varepsilon_1^{1/2}c,\varepsilon_1,\varepsilon_2)=0$ with respect to ε_1, is

$$c^2+(1+2\varepsilon_2)=0,$$

which has two simple non-zero roots

$$c_{1,2}=\pm\sqrt{-1-2\varepsilon_2}.$$

Thus, the Puiseux-type series expansion for the perturbed solution with two first terms is given by

$$x_{1,2}(\boldsymbol{\varepsilon})=\varepsilon_2\pm\sqrt{-1-2\varepsilon_2}\,\varepsilon_1^{1/2}+\cdots.$$

We note that the above series in ε_1 and ε_2 has infinitely many terms. However, an equivalent series expansion of $x_{1,2}(x,\boldsymbol{\varepsilon})$ in D and ε_2 has a benefit of having only a finite number of Puiseux terms.

Sacrificing the possibility of expanding the coefficients c_k in (6) as a Puiseux series, we can generalize Theorem 1 to the case of more than two perturbation parameters. For the polynomial equation (1) let D be the discriminant of P with respect to the variable x.

Theorem 2. *Let the following regularity condition*[1] *hold*

$$\frac{\partial D}{\partial\varepsilon_1}(0,0,\ldots,0)\neq0. \tag{9}$$

Then, for a polynomial equation (1) *any solution admits the form of a Puiseux series in* ε_1:

$$x(\boldsymbol{\varepsilon})=\sum_{k=0}^{\infty}c_k(\varepsilon_2,\ldots,\varepsilon_m)\varepsilon_1^{\frac{k}{d_1}}, \tag{10}$$

for some positive integer d_1 *and coefficients* $c_k(\varepsilon_2,\ldots,\varepsilon_m)$ *that are algebraic functions in* $\varepsilon_2,\ldots,\varepsilon_m$.

[1] Of course, the regularity condition (9) can be replaced by $\frac{\partial D}{\partial\varepsilon_j}(0,0,\ldots,0)\neq0$ for at least one $j=1,\ldots,m$.

Proof. This follows the lines of the proof of Theorem 1 with replacing ε_2 by $\boldsymbol{\varepsilon}' = (\varepsilon_2, \dots, \varepsilon_m)$. The only difference appears when we determine coefficients $c_k(\varepsilon')$ as algebraic functions of ε' being solutions of some polynomial equations. Apart of the univariate case $\varepsilon' = \varepsilon_2$, such algebraic functions do not necessarily possess Puiseux series representation in punctured neighbourhoods of the origin.[2] □

3. Perturbation analysis of Weighted PageRank

Our study is partly motivated by the perturbation analysis of Weighted PageRank. PageRank is one of the principal measures used by Google to rank relevant answers to a user's query [7]. Google precomputes and updates offline the ranking of all pages on the Web and then uses it online only for those pages that are relevant to the query. Denote by N the total number of pages on the Web and define the $N \times N$ matrix W as follows:

$$w_{ij} = \begin{cases} 1/d_i, & \text{if page } i \text{ links to } j, \\ 1/N, & \text{if page } i \text{ does not have outgoing links}, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

for $i, j = 1, \dots, N$, where d_i is the number of outgoing links from page i. The matrix W corresponds to the one-step transition of the random walk on the graph where the nodes are Web pages and edges are hyper-links between the nodes. The PageRank π is defined as a stationary distribution of the random walk on the Web graph with random restart with probability α at the probability distribution row-vector v defining user preference. The transition matrix of such a random walk is given by

$$T = (1-\alpha)W + \alpha e v. \tag{12}$$

Here we use the symbol e for a column vector of ones having by default an appropriate dimension. In the standard PageRank, there is no preference among the outgoing links. However, one could give preference based on the link type. For instance, one could assign less weight to navigation links (e.g., links within one organization) and more weight to recommendation links (e.g., links which refer to reputable sources of relevant information). This leads to a more general definition of Weighted PageRank [6]. A distribution of weight among navigation and recommendation links can be controlled by a parameter ε_1. For instance, ε_1 can represent the total weight attributed to navigation links from a node and $1-\varepsilon_1$ the total weight attributed to the recommendation links from that node. Thus, the transition matrix for the Weighted PageRank (12) takes the form

$$T = (1-\alpha)W(\varepsilon_1) + \alpha e v, \tag{13}$$

where

$$W(\varepsilon_1) = \varepsilon_1 W_1 + (1-\varepsilon_1)W_2,$$

and where the matrix W_1 corresponds to the transitions along the navigation links and W_2 corresponds to the transitions along the recommendation links. The most significant particular cases are when α is close to 0 and when it is close to 1. A connection between this application and the theory developed in the previous section will arise when we define the second perturbation parameter ε_2 to be either α or $1-\alpha$ in (13). When $\alpha \approx 0$ the process follows closely a random walk on the original Web graph and when $\alpha \approx 1$, the preference vector v and the number of incoming links to a node dominate the contribution to the value of PageRank. Thus, when $\varepsilon_2 = \alpha$ we analyze the case when α approaches 0, and when $\varepsilon_2 = 1-\alpha$ we analyze the case when α approaches 1.

We are going to derive asymptotic expansions for the PageRank $\pi(\varepsilon_1, \varepsilon_2)$ and also for the second eigenvalue of the transition matrix (13) which indicates the rate of convergence to the steady state. Let us demonstrate the range of possibilities of asymptotic expansions with the following examples.

Example 3. Define matrix W and the preference vector v as

$$W = \begin{bmatrix} 1-\varepsilon_1 & \varepsilon_1 \\ \varepsilon_1 & 1-\varepsilon_1 \end{bmatrix}, \qquad v = [1 \quad 0].$$

First, we study the case $\alpha \approx 0$. Then, by (13) the transition matrix $T(\varepsilon_1, \varepsilon_2)$ takes the form

$$T = \begin{bmatrix} (1-\varepsilon_1)(1-\varepsilon_2) + \varepsilon_2 & \varepsilon_1(1-\varepsilon_2) \\ \varepsilon_1(1-\varepsilon_2) + \varepsilon_2 & (1-\varepsilon_1)(1-\varepsilon_2) \end{bmatrix}.$$

[2] In general, as demonstrated in Example 3 below, an algebraic function can only be expanded as a convergent series in some sector emanating from the origin.

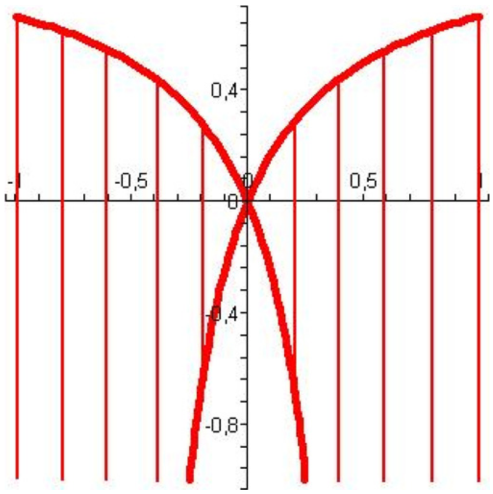

Fig. 3. The $(\varepsilon_1, \varepsilon_2)$-convergence region when $\alpha = \varepsilon_2 \approx 0$ in Example 3.

PageRank π is a solution of the following linear system for $\pi = [\pi_1\ \pi_2]$,

$$\pi T = \pi, \qquad \pi e = 1.$$

Equivalently, $x = \pi_1$ is a solution of the equation

$$P(x, \varepsilon_1, \varepsilon_2) = \varepsilon_1(1-\varepsilon_2)x + (1-\varepsilon_1)(1-\varepsilon_2)(1-x) - (1-x) = 0 \tag{14}$$

and $\pi_2 = 1 - x$. Solving equation (14), we obtain

$$\pi = \left[\frac{\varepsilon_1 - \varepsilon_1\varepsilon_2 + \varepsilon_2}{2\varepsilon_1 + \varepsilon_2 - 2\varepsilon_1\varepsilon_2}, \frac{\varepsilon_1 - \varepsilon_1\varepsilon_2}{2\varepsilon_1 + \varepsilon_2 - 2\varepsilon_1\varepsilon_2}\right].$$

The component π_1 can be rewritten in the form

$$\pi_1 = \frac{1}{2}\left(1 + \left(1 + 2\frac{\varepsilon_1}{\varepsilon_2} - 2\varepsilon_1\right)^{-1}\right).$$

The latter expression cannot be expanded as a Laurent series in either one of the perturbation parameters in a punctured neighbourhood of the origin. However, it admits a Laurent series representation for any pair $(\varepsilon_1, \varepsilon_2)$ in the region described by

$$-\frac{1}{2} < \frac{\varepsilon_1}{\varepsilon_2} - \varepsilon_1 < \frac{1}{2},$$

which is depicted as the marked area in Fig. 3. The same is true for the second component π_2.

Let us now study the case $\alpha \approx 1$. Then, the transition matrix $T(\varepsilon_1, \varepsilon_2)$ takes the form

$$T = \begin{bmatrix} (1-\varepsilon_1)(1-\varepsilon_2) + \varepsilon_2 & \varepsilon_1(1-\varepsilon_2) \\ \varepsilon_1(1-\varepsilon_2) + \varepsilon_2 & (1-\varepsilon_1)(1-\varepsilon_2) \end{bmatrix}.$$

Solving a system of two linear equations, we obtain the following expression for the PageRank:

$$\pi = \left[\frac{1 - \varepsilon_2 + \varepsilon_1\varepsilon_2}{1 - \varepsilon_2 + 2\varepsilon_1\varepsilon_2}, \frac{\varepsilon_1\varepsilon_2}{1 - \varepsilon_2 + 2\varepsilon_1\varepsilon_2}\right],$$

which can be expanded as a Taylor series in a neighbourhood of the origin.

The second eigenvalue of T in this example can be easily calculated. It is given by

$$\lambda_2 = \operatorname{tr}(T) - \lambda_1 = \operatorname{tr}(T) - 1 = 1 - 2\varepsilon_1 - \varepsilon_2 + 2\varepsilon_1\varepsilon_2,$$

which is an analytic function.

In the above example we have encountered Laurent series expansion for the PageRank distribution and the second eigenvalue of the associated transition matrix. The next example shows that a Puiseux series expansion for the second eigenvalue of T may arise.

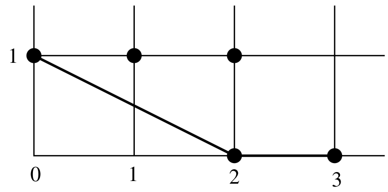

Fig. 4. Newton diagram for Example 4.

Example 4. Define the matrix W and the preference vector v as

$$W = \begin{bmatrix} 1-\varepsilon_1 & \varepsilon_1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad v = [1 \quad 0 \quad 0].$$

Consider $\alpha \approx 0$ in (13). Then, with $\varepsilon_2 = \alpha$ and the transition matrix $T(\varepsilon_1, \varepsilon_2)$ becomes

$$T = \begin{bmatrix} (1-\varepsilon_1)(1-\varepsilon_2)+\varepsilon_2 & (1-\varepsilon_2)\varepsilon_1 & 0 \\ \varepsilon_2 & 0 & 1-\varepsilon_2 \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of T is given by

$$Q = x^3 + (-1+\varepsilon_1-\varepsilon_1\varepsilon_2)x^2 + \left(-\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_2^2\right)x - \varepsilon_1 + 2\varepsilon_1\varepsilon_2 - \varepsilon_1\varepsilon_2^2.$$

If $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$, then Q reduces to

$$Q = x^3 - x^2 = x^2(x-1).$$

Thus, we conclude that the second and the third eigenvalues branch at zero.

Next, we calculate the discriminant D of the characteristic polynomial, which equals

$$\begin{aligned} D = {} & -4\varepsilon_1 - 15\varepsilon_1^2 + 8\varepsilon_2\varepsilon_1 - 4\varepsilon_2^2\varepsilon_1 + 54\varepsilon_2\varepsilon_1^2 - 12\varepsilon_1^3 - 71\varepsilon_2^2\varepsilon_1^2 \\ & + 66\varepsilon_2\varepsilon_1^3 + 4\varepsilon_1^4 + 40\varepsilon_2^3\varepsilon_1^2 - 146\varepsilon_2^2\varepsilon_1^3 - 20\varepsilon_1^4\varepsilon_2 + 166\varepsilon_2^3\varepsilon_1^3 \\ & + 41\varepsilon_2^2\varepsilon_1^4 - 8\varepsilon_2^4\varepsilon_1^2 - 102\varepsilon_2^4\varepsilon_1^3 - 44\varepsilon_2^3\varepsilon_1^4 + 26\varepsilon_2^4\varepsilon_1^4 \\ & + 32\varepsilon_2^5\varepsilon_1^3 - 8\varepsilon_2^5\varepsilon_1^4 - 4\varepsilon_2^6\varepsilon_1^3 + \varepsilon_1^4\varepsilon_2^6. \end{aligned}$$

Clearly, we have

$$D(0,0) = 0, \qquad \frac{\partial D}{\partial \varepsilon_1}(0,0) = -4, \qquad \frac{\partial D}{\partial \varepsilon_2}(0,0) = 0.$$

According to Theorem 1, there exist a Puiseux series of the form given by (6). From the Newton diagram associated with polynomial Q (see Fig. 4) we conclude that $d_1 = 1/2$. Substituting $x(\varepsilon_1, \varepsilon_2) = c_1(\varepsilon_2)\varepsilon_1^{1/2} + c_2(\varepsilon_2)\varepsilon_1 \ldots$ into the equation $Q = 0$ and collecting coefficients at the same power of ε_1, we obtain the following equations for c_1 and c_2:

$$c_1^2 = -(1-\varepsilon_2)^2, \qquad c_2 = \frac{1}{2}\left(-\varepsilon_2+\varepsilon_2^2\right) + \frac{1}{2}c_1^2.$$

Consequently, we obtain

$$c_1 = \pm \mathrm{i}(1-\varepsilon_2),$$

and

$$c_2 = \frac{1}{2}(-1+\varepsilon_2).$$

It can be easily verified that

$$\lambda_1 = 1, \qquad \lambda_{2,3} = \frac{1}{2}\left(\varepsilon_1 \pm \sqrt{\varepsilon_1^2 - 4\varepsilon_1}\right)(-1+\varepsilon_2)$$

are the three roots of $Q(x, \varepsilon_1, \varepsilon_2) = 0$. One can further verify that the roots $\lambda_{2,3}$ have the first two terms $\pm\mathrm{i}(1-\varepsilon_2)\varepsilon_1^{1/2}$ and $\frac{1}{2}(-1+\varepsilon_2)\varepsilon_1$ in its Puiseux series expansion with respect to ε_1.

We note that according to the proof of Theorem 1 one can have at most two iterations of the Newton polygon procedure. However, in this case we have at most three distinct eigenvalues and we are certain that the branching terminates after the first substitution.

4. Conclusions

We analysed perturbed algebraic equations whose coefficients are polynomials in several perturbation parameters. In general, it is not possible to represent the perturbed solution as a Puiseux-type power series in a connected neighborhood. For the case of two perturbation parameters we provide a sufficient condition that distinguishes one perturbation parameter and guarantees such a representation. Then, we extend this result to the case of more than two perturbation parameters at the expense of sacrificing the possibility of Puiseux series expansions for the coefficients of the Puiseux series in the distinguished perturbation parameter. Our approach is based on Newton diagram technique. We motivate our theoretical development by the perturbation analysis of the Weighted PageRank, a frequently used measure for Web pages popularity. It is seen that – even in the artificially simple two and three state related Markov chains – the concept of the Weighted PageRank exhibits the full range of possible cases of asymptotic expansions for several perturbation parameters. Our Example 4 shows that the mixing time could be large for random walks with nearly periodic behaviour.

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