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A Generalization of Connor's Inequality to t-Designs with Automorphisms

DONALD L. KREHER

Rochester Institute of Technology,
School of Computer Science and Technology,
Rochester, New York 14623

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In this paper the incidence algebra for t-designs with automorphisms and the fundamental theorem discovered in [4] are exploited to obtain a generalization of Connor's inequality. © 1989 Academic Press, Inc.

1. INTRODUCTION

A *t-design* or *generalized Steiner system* $S(\lambda; t, k, v)$ is a pair $(X, \mathscr{B})$ with a v-set X of points and a family $\mathscr{B}$ of k-subsets of X called blocks, such that each block has k points and any t points are contained in exactly λ blocks. An automorphism of $(X, \mathscr{B})$ is a permutation of X which preserves $\mathscr{B}$. It is well known (see, e.g., [5]) that for $i+j \leqslant t$ the number of blocks of an $S(\lambda; t, k, v)$ design which contains i given points but does not contain any of a set of j other points is

$$b_j^i = \lambda \binom{v-i-j}{k-i} \Big/ \binom{v-t}{k-t}. \tag{1}$$

With this notation we write $b = b_0^0$ for the number of blocks in the design and we write $r = b_1^0 = bk/v$ for the number of blocks containing a given point.

In [1] (also see [2]) W. S. Connor developed a system of inequalities concerning the pairwise intersections $\mu_{ij} = |K_i \cap K_j|$ of m blocks, $K_1, K_2, ..., K_m$ of a $S(\lambda; t, k, v)$ design. The *characteristic matrix* of the m blocks is the m by m matrix

$$C = r(r-\lambda)I - r[\mu_{ij}] + \lambda kJ = r(r-\lambda)I - [r\mu_{ij} - \lambda k], \tag{2}$$

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where $J=[1, 1, ..., 1]^{\mathrm{T}}$ and I is the m by m identity matrix. Connor's theorem is

$$\det(C) \geqslant 0, \tag{3i}$$

$$\det(C)=0 \quad \text{if} \quad m>b-v; \tag{3ii}$$

$$\text{if} \quad m=b-v, \quad \frac{kr(r-\lambda)^{v-1}\det(C)}{r^m(r-\lambda)^m} \quad \text{is a perfect square.} \tag{3iii}$$

Given a $S(\lambda; t, k, v)$ design $(X, \mathscr{B})$ if we define the incidence matrix $W_{i\mathscr{B}}$ to be the $\binom{X}{i}$ by $\mathscr{B}$ matrix given by

$$W_{i\mathscr{B}}[T, K] = \begin{cases} 1, & \text{if } T \subseteq K, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

then parts (3i) and (3ii) of Connor's theorem can be restated as follows: The matrix

$$Q' = r(r-\lambda) - W_{1\mathscr{B}}^{\mathrm{T}} W_{1\mathscr{B}} + \lambda k J \tag{5}$$

is positive semidefinite of rank $\leqslant b-v$, because the matrices C are exactly the principle submatrices of Q'. In [7] R. M. Wilson establishes the following theorem.

THEOREM 1 [7]. *Let P_s denote the matrix of orthogonal projection from the vector space Q^b of b-tuples of rational numbers whose coordinates are indexed by the blocks of an $S(\lambda; t, k, v)$ design with $t \geqslant 2s$ and $v \geqslant k+s$ onto $\mathscr{R}(W_{s\mathscr{B}})$, the row space of the sth incidence matrix $W_{s\mathscr{B}}$. Then*

$$P_s = \sum_{i=0}^{s} (-1)^i \frac{\binom{k-i}{s-i}}{b_s^i} \bar{W}_{i\mathscr{B}}^{\mathrm{T}} W_{i\mathscr{B}}. \tag{6}$$

Equation (5) and thus parts (3i) and (3ii) of Connor's theorem are a consequence of this theorem. In Section 3 we obtain a generalization of Connor's theorem by generalizing Theorem 1.

2. Background and Notation

The tools we use to obtain our generalization were established by Kreher in [4]. The reader who is familiar with this paper may wish to skip this section.

In addition to the definitions found in the introduction, we mention a few more definitions and notational conventions. An $S(\lambda; t, k, v)$ design is also known as a $t-(v, k, \lambda)$ design and when $\lambda = 1$ it is the familiar Steiner system $S(t, k, v)$.

If X is a finite set and F a field, then an *X-vector* U over F is a function $U: X \to F$, and we write $U = (U[x]: x \in X)$. The set of all X-vectors over F is F^X. Similarly, given finite sets A and B an A by B *matrix* M over F is a function $M: A \times B \to F$ and we write $M = (M[a, b]: a \in A \text{ and } b \in B)$. The set of all X by X matrices over a field F is denoted by $\mathrm{Mat}_F(X)$. Multiplication of matrices is the usual matrix product. That is, if M is A by B and N is B by C then MN is the A by C matrix whose $[a, c]$-entry is

$$(MN)[a, c] = \sum_{b \in B} M[a, b] N[b, c].$$

The vector space over F spanned by the rows of M is the row space, $\mathscr{R}(M)$; the column space $\mathscr{C}(M)$ is defined similarly.

For notation definitions and theorems on permutation groups the reader is directed to the book by Wielandt [6]. Here we introduce some notation and concepts relevant to the present paper. If X is a set, then $\mathrm{Sym}(X)$ denotes the symmetric group on X. A group G is said to act on a set X if there is a function $X \times G \to X$ (usually denoted by $(x, g) \to x^g$) such that for all $g, h \in G$ and $x \in X$,

$$x^1 = x \qquad \text{and} \qquad x^{(gh)} = (x^g)^h.$$

Such a function is said to be a group action of G on X and is denoted by $G \mid X$. Thus, if $G \mid X$ is a group action, then G may be thought of as being mapped homomorphically onto a subgroup of $\mathrm{Sym}(X)$ and x^g is the image of $x \in X$ under $g \in G$. If $x \in X$, the *stabilizer* in G of x is the subgroup $G_x = \{g \in G: x^g = x\}$ and the *orbit* of x under G is $x^G = \{x^g: g \in G\}$. We note that $|G| = |x^G| \cdot |G_x|$. A group action $G \mid X$ induces a natural action on the *power set* $P(X)$, on the collection $\binom{X}{t}$ of t-subsets of X and on $\mathrm{Mat}_F(X)$. If $S \subseteq X$ and $g \in G$, then we define S^g by: $S^g = \{s^g: s \in S\}$; if F is a field and $M \in \mathrm{Mat}_F(P(X))$, then M^g is defined by $M^g[S, T] = M[S^g, T^g]$. The set of all orbits of group action $G \mid \Omega$ is denoted by Ω/G.

If $M \in \mathrm{Mat}_F(P(X))$ has the property that $M^g = M$ for all $g \in G$, then M is said to be *G-invariant*. The set of all G-invariant matrices is denoted by $\mathrm{Alg}(G \mid X)$. That is,

$$\mathrm{Alg}(G \mid X) = \{M \in \mathrm{Mat}_F(P(X)): M^g = M \text{ for all } g \in G\}.$$

It is easy to verify that $\mathrm{Alg}(G \mid X)$ is an algebra over the rationals Q.

If $G \mid X$ is a group action, define the $P(X)/G$ by $P(X)/G$ matrices A, B, and D as follows: For each $\Delta, \Gamma \in P(X)/G$,

$$A[\Delta, \Gamma] = |\{K \in \Gamma: K \supseteq T_0\}|,$$

$$B[\Delta, \Gamma] = |\{T \in \Delta: T \subseteq K_0\}|,$$

$$D[\Delta, \Gamma] = \begin{cases} |\Delta| & \text{if } \Delta = \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

where $T_0 \in \Delta$ and $K_0 \in \Gamma$ are any fixed representatives.

To emphasize the dependence of A, B, and D on the group action $G \mid X$ we write $A(G \mid X)$, $B(G \mid X)$, and $D(G \mid X)$ for A, B, and D, respectively. Now, because $T \subseteq K$ implies $T^g \subseteq K^g$ for an g in G and subsets T and K of X, it is easy to establish that $A(1 \mid X) = B(1 \mid X) \in \mathrm{Alg}(G \mid X)$. See, for example, [4].

The fundamental theorem discovered by Kreher in [4] is

THEOREM 2. *There is an epimorphism* $\tau: \mathrm{Alg}(G \mid X) \to \mathrm{Mat}_Q(P(X))$ *which has the properties*

(i) $\tau: A(1 \mid X) \to A(G \mid X)$

(ii) $\tau: B^T(1 \mid X) \to B^T(G \mid X)$.

We will use this theorem to generalize Wilson's and consequently Connor's theorem to designs with a given automorphism group.

3. Generalization of Connor's Inequality

Before giving the generalization we introduce some useful notation. If $M \in \mathrm{Mat}_Q(P(X)/G)$, then M_{tk} denotes the $\binom{X}{t}/G$ by $\binom{X}{k}/G$ submatrix of M corresponding to the rows and columns labeled by $\binom{X}{t}/G$ and $\binom{X}{k}/G$, respectively. Using this notation with $A = A(G \mid X)$ we may state the observation of Kramer and Mesner, see [3]:

> Given integers $0 \leqslant t \leqslant k \leqslant v$, a v-set X and $G \leqslant \mathrm{Sym}(X)$ there exists an $S(\lambda; t, k, v)$ design $(X, \mathscr{B})$ with G as an automorphism group if and only if there is a nonnegative integer solution U to the system:
>
> $$A_{tk} U = \lambda J, \qquad \text{where} \quad J = [1, 1, 1, ..., 1]^{\mathrm{T}}.$$
>
> Furthermore, the corresponding design is simple if $U[\Gamma] \in \{0, 1\}$ for each orbit Γ of k-subsets.

In 1982, Wilson [7] obtained some very useful identities among these matrices when the group is trivial. His W_{tk} matrix is our $A_{tk}(1 \mid X)$, where

1 represents the identity group on X. Restating the fundamental theorem of [4] with this notation we have

THEOREM 2′. *There is an epimorphism* $\tau\colon \mathrm{Alg}(G) \to \mathrm{Mat}_Q(P(X)/G)$ *with the properties*

(i) $\tau\colon W_{tk} \to A_{tk}$;

(ii) $\tau\colon W_{tk}^{\mathrm{T}} \to B_{tk}^{\mathrm{T}}$.

Finally, for convenience, if $(X, \mathscr{B})$ is an $S(\lambda; t, k, v)$ preserved by $G \leqslant \mathrm{Sym}(X)$, we denote by $\bar{A}_{t\mathscr{B}}$ and $\bar{B}_{t\mathscr{B}}$ the submatrices of A_{tk} and B_{tk} whose columns are indexed by $\mathscr{B}/G$. Similarly, we define $\bar{W}_{t\mathscr{B}}$ as that submatrix of W_{tk} with columns corresponding to $\mathscr{B}$. We are now in a position to generalized Connor's inequalities.

We first present some elementary relations among the A_{tk}'s and introduce a new family of matrices $\bar{A}_{tk}$. The matrix $\bar{A}_{tk}$ denotes the $\binom{X}{t}/G$ by $\binom{X}{k}/G$ matrix whose $[\varDelta, \varGamma]$-entry is

$$\bar{A}_{tk}[\varDelta, \varGamma] = |\{T \in \varDelta\colon T \cap K_0 = \varnothing\}|,$$

where $K_0 \in \varGamma$ is any fixed representative. Similarly, we define $\bar{B}_{tk}$ and $\bar{W}_{tk}$ as follows:

$$\bar{B}_{tk}[\varDelta, \varGamma] = |\{K \in \varGamma\colon K \cap T_0 = \varnothing\}|$$

$$\bar{W}_{tk}[T, K] = \begin{cases} 1 & \text{if } T \cap K = \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, $\bar{A}_{t\mathscr{B}}$, $\bar{B}_{t\mathscr{B}}$, and $\bar{W}_{t\mathscr{B}}$ are defined for a $S(\lambda; t, k, v)$ design $(X, \mathscr{B})$ in a similar fashion.

PROPOSITION 3. *Given an* $S(\lambda; t, k, v)$ *design* $(X, \mathscr{B})$ *and integers* $0 \leqslant i \leqslant j \leqslant k$, *then*

$$A_{ij} A_{j\mathscr{B}} = \binom{k-i}{j-i} A_{i\mathscr{B}} \tag{7i}$$

$$A_{ij} \bar{A}_{j\mathscr{B}} = \binom{v-k-i}{j-i} \bar{A}_{i\mathscr{B}} \tag{7ii}$$

$$\bar{A}_{j\mathscr{B}} = \sum_{i=0}^{j} (-1)^i B_{ij}^{\mathrm{T}} A_{i\mathscr{B}} \tag{7iii}$$

$$A_{j\mathscr{B}} = \sum_{i=0}^{j} (-1)^i B_{ij}^{\mathrm{T}} \bar{A}_{i\mathscr{B}} \tag{7iv}$$

$$\bar{A}_{j\mathscr{B}} B_{i\mathscr{B}}^{T} = b_i^j B_{ij}^{\mathrm{T}} \tag{7v}$$

$$A_{j\mathscr{B}} \bar{B}_{i\mathscr{B}}^{T} = b_j^i \bar{B}_{ij}^{\mathrm{T}}, \tag{7vi}$$

where $i+j \leqslant t$ in (7v) *and* (7vi). *Furthermore, these equations hold when the A's and B's are interchanged.*

Proof. Equation (7i) is just a special case of Proposition 9 in [4] and the proof of (7ii) is similar. For (7iii) consider the $[J, K]$-entry of $R = \sum_{i=0}^{j} (-1)^i W_{ij}^T W_{i\mathscr{B}}$, $J \in \binom{X}{j}$ and $K \in \mathscr{B}$, which is

$$R[J, K] = \sum_{i=0}^{j} (-1)^i \binom{\mu}{i} = \begin{cases} 1 & \text{if } \mu = 0 \\ 0 & \text{if } \mu \neq 0 \end{cases} = \bar{W}_{j\mathscr{B}}[J, K],$$

where $\mu = |J \cap K|$. Hence applying Theorem 2 yields the required result. For Eq. (7iv) again we appeal to Theorem 2 by examining the $[J, K]$-entry of $\sum_{i=0}^{j} (-1)^i W_{ij}^T \bar{W}_{i\mathscr{B}}$, $J \in \binom{X}{j}$ and $K \in \mathscr{B}$; it is

$$\sum_{i=0}^{j} (-1)^i \binom{j-\mu}{i} = \begin{cases} 1 & \text{if } \mu = j \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } J \subseteq K, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu = |J \cap K|$. Thus this entry is just $W_{j\mathscr{B}}[J, K]$. For (7v) we show $\bar{W}_{j\mathscr{B}} W_{i\mathscr{B}}^T = b_i^j \bar{W}_{ij}^T$ and apply the fundamental theorem. If $J \in \binom{X}{j}$, and $I \in \binom{X}{i}$, then the $[J, I]$-entry of the left-hand side is

$$|\{K \in \mathscr{B}: J \cap K \neq \varnothing \text{ and } I \subseteq K\}| = \left.\begin{cases} b_i^j & \text{if } |I \cap J| = 0 \\ 0 & \text{otherwise} \end{cases}\right\} = b_i^j \bar{W}_{ij}^T[J, I].$$

The proof of (7vi) is similar. Finally, to complete the proof we can interchange A's with B's by applying Proposition 4 of [4]. ∎

We should remark that Wilson in 1984, see [7], states this proposition for the trivial automorphism group. Indeed, in the same paper Wilson also shows that for an $S(\lambda; t, k, v)$ design $(X, \mathscr{B})$ with $s \leqslant t/2$ and $s \leqslant k \leqslant v-s$, the matrix $P_s(1 \mid X)$ corresponding to the orthogonal projection from $Q^{\mathscr{B}}$ onto $\mathscr{R}(W_{s\mathscr{B}})$ is given by

$$P_s(1 \mid X) = \sum_{i=0}^{s} (-1)^i \binom{k-i}{s-i} (b_s^i)^{-1} \bar{W}_{i\mathscr{B}}^T W_{i\mathscr{B}}.$$

That is, $P_s(1 \mid X)$ has the following properties:

(i) $W_{s\mathscr{B}} P_s(1 \mid X) = W_{s\mathscr{B}}$;

(ii) if $U \cdot W_{s\mathscr{B}}^T = 0$, then $U \cdot P_s(1 \mid X) = 0$;

(iii) $(P_s(1 \mid X))^2 = P_s(1 \mid X)$.

Hence, in view of Theorem 2, the fundamental theorem, when there is a group $G \leqslant \mathrm{Sym}(X)$ preserving $\mathscr{B}$, we write

$$P_s(G \mid X) = \sum_{i=0}^{s} (-1)^i \binom{k-i}{s-i} (b_s^i)^{-1} \bar{B}_{i\mathscr{B}}^T A_{i\mathscr{B}}.$$

Then $P_s = P_s(G \mid X)$ has the following properties:

(i) $A_{s\mathscr{B}} P_s = A_{s\mathscr{B}}$;

(ii) if $U \cdot B^{\mathrm{T}}_{s\mathscr{B}} = 0$, then $U \cdot P_s = 0$;

(iii) $P_s^2 = P_s$.

Properties (i) and (iii) follow, of course, from the fundamental theorem. To verify property (ii), however, we make liberal use of Proposition 3. Equations (7i) and (7ii) show that

$$\mathscr{R}(B_{0\mathscr{B}}) \subseteq \mathscr{R}(B_{1\mathscr{B}}) \subseteq \cdots \subseteq \mathscr{R}(B_{k\mathscr{B}})$$

and

$$\mathscr{R}(\bar{B}_{0\mathscr{B}}) \subseteq \mathscr{R}(\bar{B}_{1\mathscr{B}}) \subseteq \cdots \subseteq \mathscr{R}(\bar{B}_{k\mathscr{B}}),$$

respectively. Therefore, by (7iii) and (7iv) we have for each $j = 0, 1, 2, ..., k$,

$$\mathscr{R}(\bar{B}_{j\mathscr{B}}) \subseteq \bigcup_{i=0}^{j} \mathscr{R}(B_{i\mathscr{B}}) \subseteq \mathscr{R}(B_{j\mathscr{B}})$$

and

$$\mathscr{R}(B_{j\mathscr{B}}) \subseteq \bigcup_{i=0}^{j} \mathscr{R}(\bar{B}_{i\mathscr{B}}) \subseteq \mathscr{R}(\bar{B}_{j\mathscr{B}}),$$

respectively. Thus, $\mathscr{R}(B_{j\mathscr{B}}) = \mathscr{R}(\bar{B}_{j\mathscr{B}})$ for all $j = 0, 1, 2, ..., s$. Whence, it follows that $P_s(G \mid X)$ annihilates the row space of $B^{\mathrm{T}}_{s\mathscr{B}}$ and so property (ii) follows.

We now give the central theorem of this paper from which we will derive generalizations of Connor's inequalities.

THEOREM 4. *Let* $(X, \mathscr{B})$ *be an* $S(\lambda; t, k, v)$ *design preserved by* $G \leqslant \mathrm{Sym}(X)$ *with* $s \leqslant t/2$ *and* $s \leqslant k \leqslant v - s$. *Then,* $D^{1/2} P_s D^{-1/2}$ *is the matrix corresponding to the orthogonal projection from* $Q^{\mathscr{B}/G}$ *onto* $\mathscr{R}(A_{s\mathscr{B}} D^{-1/2})$.

Proof. Let $U \in \mathscr{R}(A_{s\mathscr{B}} D^{-1/2})$, then $0 = U \cdot D^{-1/2} A^{\mathrm{T}}_{s\mathscr{B}} = U \cdot D^{-1/2} \times (D^{-1} B_{s\mathscr{B}} D)^{\mathrm{T}} = U \cdot D^{-1/2} D B^{\mathrm{T}}_{s\mathscr{B}} D^{-1} = U \cdot D^{1/2} B^{\mathrm{T}}_{s\mathscr{B}} D^{-1}$, whence $U \cdot D^{1/2} B^{\mathrm{T}}_{s\mathscr{B}} = 0$. Thus, by property (ii), $U \cdot D^{1/2} P_s = 0$. Hence, $(D^{1/2} P_s D^{-1/2})^{\mathrm{T}}$ annihilates $U \in \mathscr{R}(A_{s\mathscr{B}} D^{-1/2})$. Now we need to show that $D^{1/2} P_s D^{-1/2}$ fixes the vectors in $U \in \mathscr{R}(A_{s\mathscr{B}} D^{-1/2})$.

The matrix $D^{1/2} P_s D^{-1/2}$ fixes vectors in $U \in \mathscr{R}(A_{s\mathscr{B}} D^{-1/2})$ if and only if $(A_{s\mathscr{B}} D^{-1/2})(D^{1/2} P_s D^{-1/2}) = A_{s\mathscr{B}} D^{-1/2}$. From property (i) we have

$$A_{s\mathscr{B}} D^{-1/2} = A_{s\mathscr{B}} D^{1/2} P_s D^{-1/2} = A_{s\mathscr{B}} D^{-1/2} D^{1/2} P_s D^{-1/2},$$

which concludes the proof. ▮

We give an obvious but important corollary.

COROLLARY 5. *For an* $S(\lambda; t, k, v)$ *design* $(X, \mathscr{B})$ *preserved by* $G \leqslant \mathrm{Sym}(X)$ *with* $s \leqslant t/2$ *and* $s \leqslant k \leqslant v-s$, *the matrix* $D^{1/2} P_s D^{-1/2}$ *is symmetric, idempotent, positive semidefinite, and has rank* $|\binom{X}{s}/G|$.

Proof. This is always the case for a matrix corresponding to the orthogonal projection onto a subspace. Also note that the projection has rank equal to the dimension of the subspace, in this case the dimension of $\mathscr{R}(A_{s\mathscr{B}} D^{-1/2})$. By the proof of Theorem 6 in [4] we have

$$\dim(\mathscr{R}(A_{s\mathscr{B}})) = \mathrm{rank}(A_{s\mathscr{B}}) \geqslant \mathrm{rank}(A_{sk}) = |\tbinom{X}{s}/G|.$$

This, however, is the number of rows of $A_{s\mathscr{B}}$, whence, $\dim(\mathscr{R}(A_{s\mathscr{B}} D^{-1/2})) = \mathrm{rank}(A_{s\mathscr{B}}) = |\binom{X}{s}/G|$. ▮

If Δ and Γ are orbits of blocks of an $S(\lambda; t, k, v)$ design $(X, \mathscr{B})$ preserved by $G \leqslant \mathrm{Sym}(X)$, we define $\mu_i = \mu_i(\Delta, \Gamma)$ to be the number of blocks in Γ which intersect an orbit representative, say $K_0 \in \Delta$, in exactly i points. That is,

$$\mu_i = |\{K \in \Gamma \colon |K \cap K_0| = i\}|.$$

Writing $\mu = [\mu_0, \mu_1, ..., \mu_k]$, we then have

LEMMA 6. (i) $B^{T}_{i\mathscr{B}} A_{i\mathscr{B}}[\Delta, \Gamma] = \sum_{j=0}^{k} \mu_j \binom{j}{i}$;

(ii) $\bar{B}^{\mathrm{T}}_{i\mathscr{B}} A_{i\mathscr{B}}[\Delta, \Gamma] = \sum_{j=0}^{k} \mu_j \binom{k-j}{i}$.

Proof. To prove (i) we construct a graph on $\Delta \cup \Gamma \cup \binom{X}{i}$ as follows:

(a) If $K \in \Delta$ and $I \in \binom{X}{i}$ then K is adjacent to I just when K contains I;

(b) if $I \in \binom{X}{i}$ and $K' \in \Gamma$ then I is adjacent to K' just when $I \subseteq K'$.

There are no other edges.

We also partition $\binom{X}{i}$ into its G-orbits $[\Omega_1, \Omega_2, ..., \Omega_n]$, where $n = |\binom{X}{i}/G|$. Let $K_0 \in \Delta$. The number of paths of length 2 from K_0 into Γ are counted in two ways. First, K_0 is adjacent to $B^{\mathrm{T}}_{i\mathscr{B}}[\Delta, \Omega_j]$ i-subsets in Ω_j for each $j = 1, 2, ..., n$. Also, if $I \in \Omega_j$, then I is adjacent to $A_{i\mathscr{B}}[\Omega_j, \Gamma]$ blocks in Γ. Thus, the number of paths of length 2 from K_0 into Γ is

$$\sum_{j=1}^{n} B^{\mathrm{T}}_{i\mathscr{B}}[\Delta, \Omega_j] A_{i\mathscr{B}}[\Omega_j, \Gamma] = B^{\mathrm{T}}_{i\mathscr{B}} A_{i\mathscr{B}}[\Delta, \Gamma].$$

On the other hand, the number of paths of length 2 from K_0 to a particular K' in Γ is $|\{I \in \binom{X}{i} \colon K_0 \supseteq I$ and $I \subseteq B\}| = \binom{m}{i}$, where $m = |K_0 \cap K'|$.

Therefore, summing over all $K' \in \Gamma$ gives the desired result. The proof of (ii) can be done in the same fashion. ∎

It follows from Lemma 6 that the $[\Delta, \Gamma]$-entry of $P_s(G \mid X)$ is $f_s(\mu(\Delta, \Gamma))$, where

$$f_s(Y) = \sum_{i=0}^{s} (-1)^i \binom{k-i}{s-i} (b_s^i)^{-1} \sum_{j=0}^{k} Y[j] \binom{k-j}{i}.$$

Interchanging the order of summation we see that

$$f_s(Y) = \sum_{j=0}^{k} \beta_{sj} Y[j],$$

where

$$\beta_{sj} = \sum_{i=0}^{s} (-1)^i \binom{k-i}{s-i} (b_s^i)^{-1} \binom{k-j}{i}.$$

Thus, if $\beta_s = [\beta_{s0}, \beta_{s1}, ..., \beta_{sk}]$, then $f_s(Y) = \beta_s Y$. Hence, the $[\Delta, \Gamma]$-entry of $P_s(G \mid X)$ is just $\beta_s \cdot \mu(\Delta, \Gamma)$. It is important to observe that β_s is independent of the choice of group action. Therefore it may be computed a priori to choosing a group.

COROLLARY 7. *Let $\Delta_1, \Delta_2, ..., \Delta_m$ be some set of orbits of blocks of an $S(\lambda; t, k, v)$ design $(X, \mathscr{B})$ preserved by $G \leqslant \mathrm{Sym}(X)$ with integers $s \leqslant t/2$ and $k+s \leqslant v$. Then the m by m matrix*

$$I - [\beta_s \cdot \mu(\Delta_i, \Delta_j)]$$

has nonnegative determinate and is singular if $m > |\mathscr{B}/G| - |\binom{X}{s}/G|$.

Proof. Let $M = [\beta_s \cdot (\Delta_i, \Delta_j)]$. Then $I - D^{1/2} M D^{-1/2}$ is a principal submatrix of $I - D^{1/2} P_s D^{-1/2}$ which by Corollary 5, is positive semi-definite of rank equal to $|\mathscr{B}/G| - |\binom{X}{s}/G|$. Hence, $\mathrm{rank}(I - M) = \mathrm{rank}(I - D^{1/2} M D^{-1/2}) \leqslant |\mathscr{B}/G| - |\binom{X}{s}/G|$. ∎

When G is trivial and $t = 2$, Corollary 7 is Connor's inequality [1]. Also note that when $m = 1$, Corollary 7 says that if the orbit Δ is to be used in the construction of an $S(\lambda; t, k, v)$ with $s \leqslant t/2$, $k+s \leqslant v$, and automorphism group G, then Δ must satisfy

$$1 - \beta_s \cdot \mu(\Delta, \Delta) \geqslant 0.$$

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