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Simple (γ, δ) Algebras Are Associative*

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A (γ, δ) algebra over a field F is a nonassociative algebra satisfying an identity of the form, $(a, b, c) + \gamma(b, a, c) + \delta(c, a, b) = 0$, for fixed $\gamma, \delta \in F$, and $\gamma^2 - \delta^2 + \delta = 1$. We assume that F is of characteristic $\neq 2, \neq 3$; however, we do not assume that the algebra is finite-dimensional over F. We show that any simple (γ, δ) algebra is associative with the possible exception of the cases $(\pm 1, 0)$ and $(1, 1)$. The approach used in this paper is to represent the identities by matrices by way of the group algebra representation. This enables us to manipulate identities by the well-known techniques of matrix theory.

INTRODUCTION

(γ, δ) algebras were first introduced by Albert in [1]. They form a residue class in his classification of almost alternative algebras. They are originally defined as nonassociative algebras over a field F which satisfy an identity of the form

$$(a, b, c) + \gamma(b, a, c) + \delta(c, a, b) = 0 \tag{1}$$

for fixed γ and δ in F such that $\gamma^2 - \delta^2 + \delta = 1$. It is not clear whether Albert intended third-power associativity to hold and/or a corresponding right-hand version of Eq. (1): $-(b, c, a) + \gamma(b, a, c) + (\delta - 1)(c, a, b) = 0$. Later authors assumed these extra conditions. In this paper, however, we shall assume (γ, δ) means only that Eq. (1) needs to be satisfied.

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Albert showed [1, Theorem 9] that any nil finite-dimensional (γ, δ) algebra is nilpotent; the cases $(-1, 1)$ and $(1, 0)$ were left open as possible exceptions. His arguments require finite-dimensionality. Later authors have studied special cases. One of these cases is $(-1, 0)$, $(1, 1)$ [5, 6, 9]. The other case, $(-1, 1)$, is of considerable interest [4, 10, 14].

Until 1975, no results had been published on the general (γ, δ) algebras since Albert's original paper in 1949 except for the results in [6, 8]. In [8], Kokoris used the idempotent decomposition to study simple finite-dimensional (γ, δ) algebras $(\delta \neq 0, 1)$. In [6, Theorem 12], it is shown that simple, not necessarily finite-dimensional (γ, δ) algebras $(\gamma \neq \pm 1)$ which are not associative will have no proper left or right ideals. In 1975, Kleinfeld and Kleinfeld [7] showed that (γ, δ) division rings are associative, and Nikitin [11] showed that simple finite-dimensional (γ, δ) algebras are associative.

In this paper we show that simple (γ, δ) algebras are associative; the cases $(\pm 1, 0)$ and $(1, 1)$ may be exceptions. If we assume Eq. (1) and third-power associativity as well, then simple $(1, 0)$ algebras are associative.

We actually prove the stronger result that in any (γ, δ) algebra R, excluding cases $(\pm 1, 0)$ and $(1, 1)$, the associators (R, R, R) form a locally nilpotent ideal. The result stated in the title follows since a simple algebra cannot be locally nilpotent.

Our approach is to study identities by matrix techniques. Our work here is of interest in itself since this approach can be used very generally.

Notation

We assume that R represents a nonassociative algebra over a field F of characteristic $\neq 2$, $\neq 3$. We do *not* assume that R is a finite-dimensional algebra over F. The letters (γ, δ) refer to Eq. (1), which R is assumed to satisfy. The associator (a, b, c) and commutator $[a, b]$ are defined by $(a, b, c) = (ab)\, c - a(bc)$; $[a, b] = ab - ba$. In formulas whose arguments are elements of R, we often let one or more arguments be subsets of R. By this we mean the subspace spanned by the elements generated by the formula as the arguments are chosen from the indicated subsets. Thus, $(R, R, R) =$ the span of $\{(a, b, c) \mid a, b, c \in R\}$; $(x, x, R) =$ the span of $\{(x, x, a) \mid a \in R\}$. When an algebra R is under discussion, we let $U = \{u \in R \mid [u, R] = 0\}$. We always use the letter U for this set, and the letter u always means an element of this set. If S is any subset of R, by $\langle S \rangle$ we mean the ideal of R generated by S.

When an algebra R is under discussion, the letters C and A represent fixed sets. $C =$ the ideal generated by $[[R, R], R]$, and $A =$ the ideal generated by (R, R, R).

If n is a positive integer, then $R^2 = RR$ and $R^n = \sum_{i+j=n} R^i R^j$. We say R is nilpotent if $R^n = 0$ for some n. We say R is locally nilpotent if every finitely generated subalgebra of R is nilpotent.

An algebra R is simple if $R^2 \neq 0$ and R has no ideals except 0 and R.

Group Representation

By S_n we mean the group of all permutations on n objects. In this paper permutations act on positions, not on elements. The permutation $\pi = (1234)$ acting on the ordered four-tuple $[a, b, c, d]$ changes the order to $[d, a, b, c]$. We write this as $[a, b, c, d]_\pi = [d, a, b, c]$. Similarly, $[x_1, x_2, x_3, x_4]_\pi = [x_4, x_1, x_2, x_3]$. $[x_1, x_2, x_3, x_4]_\pi$ is *not* equal to $[x_{1\pi}, x_{2\pi}, x_{3\pi}, x_{4\pi}] = [x_2, x_3, x_4, x_1]$. If σ and τ are elements of S_n, then $[x_1, x_2, x_3, \ldots, x_n]_{\sigma\tau} = [[x_1, x_2, x_3, \ldots, x_n]_\sigma]_\tau$ by definition of composition of permutations.

If F is a field and G is a finite group, the set of all formal sums $\{\sum_{g \in G} \gamma_g g \mid \gamma_g \in F\}$ is a finite-dimensional algebra over F under the operations:

$$\gamma \sum_{g \in G} \gamma_g g = \sum_{g \in G} (\gamma\gamma_g) g,$$

$$\sum_{g \in G} \gamma_g g + \sum_{g \in G} \lambda_g g = \sum_{g \in G} (\gamma_g + \lambda_g) g.$$

Multiplication of the basis is defined to correspond to group multiplication. It is extended to sums by linearity. Thus, for $h \in G$,

$$h \sum_{g \in G} \lambda_g g = \sum_{g \in G} \lambda_g (hg) = \sum_{g' \in G} \lambda_{(h^{-1}g')} g'.$$

If the characteristic of F is zero or if it does not divide the order of G, then the group algebra is semisimple, and if F is large enough, the group algebra is isomorphic to a direct sum of matrix algebras over F. By G_n we mean the group algebra on S_n.

In this paper we make use of the group algebra over F of the symmetric groups S_3 and S_4. We now give the representation that is used throughout the paper. $G_3 \cong F_{1\times 1} \oplus F_{1\times 1} \oplus F_{2\times 2}$. We indicate the three components by M_1, M_2, and M_3, respectively. $\pi \in S_3$ is mapped to $\pi_1 \oplus \pi_2 \oplus \pi_3$ where $\pi_1 = 1$, $\pi_2 = \operatorname{sgn} \pi$, and π_3 is given by Table I.

TABLE I

π	I	(12)	(13)	(23)	(123)	(132)
π_3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

TABLE II

π	π_3	π_4	π	π_3	π_4
(12)	$\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$	I	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$
(13)	$\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & -1 & 0\\ 0 & -1 & 0\\ 0 & -1 & 1\end{pmatrix}$	(123)	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}$
(14)	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & -1\end{pmatrix}$	(124)	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0 & 1\\ -1 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}$
(23)	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$	(132)	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1 & 0\\ 1 & -1 & 0\\ 0 & -1 & 1\end{pmatrix}$
(24)	$\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$	(134)	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & -1 & 0\\ 0 & -1 & 1\\ 0 & -1 & 0\end{pmatrix}$
(34)	$\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$	(142)	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & -1\\ 1 & 0 & -1\end{pmatrix}$
(1234)	$\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}$	(143)	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}$
(1243)	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0 & 1\\ -1 & 0 & 0\\ -1 & 1 & 0\end{pmatrix}$	(234)	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$
(1324)	$\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1 & 1\\ 1 & -1 & 0\\ 0 & -1 & 0\end{pmatrix}$	(243)	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$
(1342)	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1 & 0\\ 0 & -1 & 1\\ 1 & -1 & 0\end{pmatrix}$	(12)(34)	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0 & 0\\ -1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix}$
(1423)	$\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & -1\\ 0 & 0 & -1\\ 1 & 0 & -1\end{pmatrix}$	(13)(24)	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -1 & 1\\ 0 & -1 & 0\\ 1 & -1 & 0\end{pmatrix}$
(1432)	$\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 0 & -1\\ 1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}$	(14)(23)	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & -1\\ 1 & 0 & -1\\ 0 & 0 & -1\end{pmatrix}$

$G_4 \cong F_{1\times 1} \oplus F_{1\times 1} \oplus F_{2\times 2} \oplus F_{3\times 3} \oplus F_{3\times 3}$. We indicate these components as M_1, M_2, M_3, M_4, and M_5, respectively. $\pi \in S_4$ is mapped to $\pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4 \oplus \pi_5$ where $\pi_1 = 1$, $\pi_2 = \operatorname{sgn} \pi$, π_3 and π_4 are given in Table II, and $\pi_5 = (\operatorname{sgn} \pi)\, \pi_4$.

The structure of group algebras may be found in [2]. Tables I and II were constructed from the images of the generators given in [2].

TECHNIQUE

Let us use the symbol $R^{\langle n\rangle}$ to mean the Cartesian product of n factors of R to distinguish it from the previously defined R^n. If R is a nonassociative algebra and $f\colon R^{\langle n\rangle} \to R$, for $\pi \in S_n$, by f_π, we mean to evaluate f after first permuting the arguments by π. Thus $f_\pi(x_1, x_2, ..., x_n) = f([x_1, x_2, ..., x_n]_\pi)$. If one has a function $f\colon R^{\langle n\rangle} \to R$ and n elements of R, $x_1, x_2, ..., x_n$, then any element $g = \sum_{\pi\in S_n} \gamma_\pi \pi$ of the group algebra G_n determines a specific element of R. The element $s(g)$ determined by g is

$$s(g) = \sum_{\pi\in S_n} \gamma_\pi f_\pi(x_1, x_2, ..., x_n).$$

It is easy to show that if $s(g) = s(g') = 0$ and $\gamma \in F$, then $s(g + g') = s(\gamma g) = 0$, and so the set $\{g \in G_n \mid s(g) = 0\}$ is a subspace of G_n. What we have is a notational device. Instead of continually writing and rewriting the function f and the arguments $x_1, x_2, ..., x_n$, we can manipulate their representation in G_n, where all computations can be done with matrices.

The above device is useful; however, the following is of greater significance. If $f\colon R^{\langle n\rangle} \to R$ is given, an element $g \in G_n$ is called an identity if $s(g) = 0$ for all possible choices of $x_1, x_2, ..., x_n$ in R. We further generalize our definition of identity to the following. If H is a subspace of R, then g is called an identity mod H if $s(g) \in H$ for all choices of $x_1, x_2, ..., x_n$ in R. If $g = \sum_{\pi\in S_n} \gamma_\pi \pi$ is an identity mod H and σ is any permutation in S_n, then $s(\sigma g) = \sum_{\pi\in S_n} \gamma_\pi f([x_1, x_2, ..., x_n]_{\sigma\pi}) = \sum_{\pi\in S_n} \gamma_\pi f_\pi([x_1, x_2, ..., x_n]_\sigma) \in (\sum_{\pi\in S_n} \gamma_\pi f_\pi)(R^{\langle n\rangle}) \subseteq H$. Thus, g is an identity mod H implies σg is an identity mod H for any permutation σ.

If we let $L = \{g \in G_n \mid g \text{ is an identity mod } H\}$, it is easy to show that L is a subspace of G_n. By the last sentence in the previous paragraph, L is a left ideal of G_n as well. If the identity element I of G_n is in L, then $f_I = f$ and f maps $R^{\langle n\rangle}$ into H. If $I - \pi$ is in L, then $f \equiv f_\pi \bmod H$. The advantage of this is that G_n is a direct sum of matrices; if $G_n = M_1 \oplus M_2 \oplus \cdots M_k$, then $L = L_1 \oplus L_2 \oplus \cdots L_k$, and we can deal with each summand independently. To show $I \in L$, it is only necessary to find an invertible element in each L_i. To show $f_I \equiv f_\pi$ requires only to check that each component $I_i - \pi_i$ is in L_i for $i = 1, 2, ..., k$.

Let us use associator-dependent algebras as an example. An algebra is called an associator-dependent algebra if the associators satisfy an identity of the form:

$$\gamma_1(a, b, c) + \gamma_2(b, c, a) + \gamma_3(c, a, b) + \gamma_4(a, c, b) + \gamma_5(c, b, a) + \gamma_6(b, a, c) = 0 \quad (2)$$

for fixed scalars $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ in F. It is natural to examine such algebras with the function f being the associator (a, b, c). The element $\gamma_1 I + \gamma_2(132) + \gamma_3(123) + \gamma_4(23) + \gamma_5(13) + \gamma_6(12)$ is an element of the left ideal of identities. Identities of type (2) are equivalent if and only if their representations generate the same left ideal. To discover the various types of associator-dependent algebras, it is only necessary to examine the different left ideals of G_3. If L is the left ideal of identities, then $L = L_1 \oplus L_2 \oplus L_3$. Since $F_{1\times 1}$ is one-dimensional, either $L_1 = F_{1\times 1}$ or $L_1 = 0$. The same choices hold for L_2. L_3 can be 0, $F_{2\times 2}$, or a left ideal generated by a matrix of the form $\left(\begin{smallmatrix}\alpha & \beta \\ 0 & 0\end{smallmatrix}\right)$. If $\alpha \neq 0$, then letting

TABLE III

Generator	Common name
$1 \oplus 0 \oplus \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$	Third-power associativity $(a, a, a) \equiv 0$
$0 \oplus 1 \oplus \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$	Lie admissibility $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] \equiv 0$
$0 \oplus 0 \oplus \begin{pmatrix}1 & \lambda\\0 & 0\end{pmatrix}$	$(a, b, c) + (b, a, c) - (b, c, a) - (c, b, a)$ $+ \lambda\{(a, c, b) - (c, a, b) + (b, c, a) - (b, a, c)\} \equiv 0$
$1 \oplus 1 \oplus \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$	Cyclic law $(a, b, c) + (b, c, a) + (c, a, b) \equiv 0$
$1 \oplus 0 \oplus \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$	Flexibility $(a, b, a) \equiv 0$
$1 \oplus 0 \oplus \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$	Right alternativity $(b, a, a) \equiv 0$
$1 \oplus 0 \oplus \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	Left alternativity $(a, a, b) \equiv 0$
$1 \oplus 1 \oplus \begin{pmatrix}1 & \frac{1}{2}\\0 & 0\end{pmatrix}$	Third-power associativity $(a, a, a) \equiv 0$ and antiflexibility $(a, b, c) - (c, b, a) \equiv 0$
$1 \oplus 1 \oplus \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$	$(-1, 1)$ algebras
$1 \oplus 1 \oplus \begin{pmatrix}1 & \lambda\\0 & 0\end{pmatrix}$	(γ, δ) algebras (some exceptions)

$\lambda = \beta/\alpha$, the generator can be expressed as $\left(\begin{smallmatrix}1 & \lambda \\ 0 & 0\end{smallmatrix}\right)$. We specify associator-dependent algebras by giving a generator of the left ideal of identities. Table III contains a listing of the common associator-dependent algebras.

If an identity of R involves several different functions, we represent it using the direct sum of the group algebras. Suppose $[x_1, x_2, ..., x_n]$ are fixed elements of R, and $f_i : R^{\langle n \rangle} \to R$ for $i = 1, 2, ..., k$ are given. An element $(g_1, g_2, ..., g_k)$ of the direct sum of k copies of G_n is sent to $s(g_1, g_2, ..., g_k) = \sum_{i=1}^{k} s_i(g_i)$, where $s_i(g_i)$ is determined as before using the ith component g_i, $[x_1, x_2, ..., x_n]$, and the ith function. The set of all elements in $G_n^{\langle k \rangle}$ which are sent to the zero element of R is a subspace of $G_n^{\langle k \rangle}$. Again this is a valuable notational device. However, the study of identities where the $[x_1, x_2, ..., x_n]$ are not fixed is more useful. $G_n^{\langle k \rangle}$ is a module over G_n. If H is a subspace of R, let L be the set of elements in $G_n^{\langle k \rangle}$ whose representation always yields an element of H no matter which elements $[x_1, x_2, ..., x_n]$ are used. L is a subspace of $G_n^{\langle k \rangle}$ and L is also a submodule. It is not true that L can be decomposed $L = L_1 \oplus L_2 \oplus \cdots L_k$. If some $l \in L$ has the property that the ith component of l is the identity element of G_n, then f_i can be expressed as a linear combination of the remaining functions. If some $l \in L$ has the property that $l_i = 0$, then we have an identity which does not involve the ith function. In this way, we can reduce the number of functions as far as possible and express those removed as combinations of the remaining. When we use several functions, the factors of $G_n^{\langle k \rangle}$ are written horizontally, and each G_n is expressed as a direct sum of matrix algebras with the summands written vertically.

This approach is useful because ordinary matrix theory such as row operations and invertibility can be used. Also, we are able to handle identities involving large numbers of terms, without having to simplify them by letting two or more arguments be equal just to keep the expressions of reasonable length.

(γ, δ) Algebras

The definition of a (γ, δ) algebra varies with the author. In the original definition [1, Theorem 2], it is simply stated that an algebra satisfying Eq. (1) will be called a (γ, δ) algebra. It is not clear whether or not Albert tacitly intended that third-power associativity and/or almost right alternativity be included. We use Eq. (1) as our definition. This identity translates to $I + \gamma(12) + \delta(123)$ and has representation

$$(1 + \gamma + \delta) \oplus (1 - \gamma + \delta) \oplus \begin{pmatrix} 1 + \gamma - \delta & -\delta \\ -\gamma + \delta & 1 - \gamma \end{pmatrix}.$$

The restriction $\gamma^2 - \delta^2 + \delta = 1$ says that the rows of the 2×2 summand are dependent.

LEMMA 1. *(γ, δ) algebras are of type* $1 \oplus 1 \oplus \left(\begin{smallmatrix}1 & \lambda\\ 0 & 0\end{smallmatrix}\right)$ *with* $\lambda \neq 0, \frac{1}{2}, 2$. *The only exceptions are* $(-1, 0)$ *and* $(1, 0)$.

Proof. If a (γ, δ) algebra is not third-power associative, then $1 + \gamma + \delta = 0$; solving with $\gamma^2 - \delta^2 + \delta = 1$ gives that $(\gamma, \delta) = (-1, 0)$. The algebra $(-1, 0)$ has this for its representation:

$$0 \oplus 2 \oplus \begin{pmatrix}0 & 0\\ 1 & 2\end{pmatrix}.$$

This corresponds to $I - (12)$ or, equivalently, $(a, b, c) = (b, a, c)$ for all elements a, b, c.

If a (γ, δ) algebra is not Lie admissible, then $1 - \gamma + \delta = 0$ and solving with $\gamma^2 - \delta^2 + \delta = 1$ gives $(\gamma, \delta) = (1, 0)$. The algebra $(1, 0)$ has this representation:

$$2 \oplus 0 \oplus \begin{pmatrix}2 & 0\\ -1 & 0\end{pmatrix}.$$

This corresponds to $I + (12)$ or equivalently $(a, b, c) + (b, a, c) = 0$ for all elements a, b, c. This is the left alternative law.

Since the left ideal of $F_{2\times 2}$ contains

$$\begin{pmatrix}1 + \gamma - \delta & -\delta\\ -\gamma + \delta & 1 - \gamma\end{pmatrix},$$

it contains

$$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 + \gamma - \delta & -\delta\\ -\gamma + \delta & 1 - \gamma\end{pmatrix} = \begin{pmatrix}1 & 1 - \gamma - \delta\\ 0 & 0\end{pmatrix},$$

which is of the form $\left(\begin{smallmatrix}1 & \lambda\\ 0 & 0\end{smallmatrix}\right)$ for $\lambda = 1 - \gamma - \delta$.

It remains to be shown that if $(\gamma, \delta) \neq (\pm 1, 0)$, then $1 - \gamma - \delta \neq 0, \frac{1}{2}, 2$. One solves the equation $\gamma^2 - \delta^2 + \delta = 1$ with $1 - \gamma - \delta = 0, \frac{1}{2}, 2$. For $\lambda = \frac{1}{2}$, they are inconsistent. $\lambda = 0, 2$ yields $(\pm 1, 0)$ as the only possible solutions for (γ, δ); these are the excluded cases.

LEMMA 2. *Every algebra of type* $1 \oplus 1 \oplus \left(\begin{smallmatrix}1 & \lambda\\ 0 & 0\end{smallmatrix}\right)$ *where* $\lambda \neq 0, \frac{1}{2}, 2$ *is a* (γ, δ) *algebra where* $(\gamma, \delta) \neq (\pm 1, 0)$.

Proof. Let $(\gamma, \delta) = ((\lambda^2 - \lambda + 1)/(1 - 2\lambda), (\lambda(\lambda - 2))/(1 - 2\lambda))$. The representation of this algebra is

$$(2 - \lambda) \oplus \left(\frac{-3\lambda}{1 - 2\lambda}\right) \oplus \frac{1}{1 - 2\lambda}\begin{pmatrix}2 - \lambda & \lambda(2 - \lambda)\\ -(1 + \lambda) & -\lambda(1 + \lambda)\end{pmatrix}.$$

Clearly, if $\lambda \neq 0, \frac{1}{2}$, or 2, the left ideal generated by this is the same as that generated by $1 \oplus 1 \oplus \left(\begin{smallmatrix}1 & \lambda\\ 0 & 0\end{smallmatrix}\right)$.

We now examine the exceptional cases where $\lambda = 0, \frac{1}{2}, 2$.

The algebra $1 \oplus 1 \oplus \left(\begin{smallmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{smallmatrix}\right)$ corresponds to an antiflexible, third-power associative algebra. It is defined by $(a, b, c) + (b, c, a) + (c, a, b) \equiv 0$ and $(a, b, c) - (c, b, a) \equiv 0$. These algebras have not been studied very much; see [12, 13].

If $\lambda = 0$, the algebra corresponds closely to $(\gamma, \delta) = (1, 0)$. However, the $(1, 0)$ algebra under our definition is not Lie admissible. The $(1, 0)$ condition is the left alternative condition. The algebra $1 \oplus 1 \oplus \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is then the left alternative analog of a $(-1, 1)$ algebra. From [4] it is known that any such simple algebra is associative.

If $\lambda = 2$, the algebra corresponds closely to $(\gamma, \delta) = (-1, 0)$. However, the $(-1, 0)$ algebra under our definition is not third-power associative; it satisfies $(a, b, c)-(b, a, c) \equiv 0$. The algebra $1 \oplus 1 \oplus \left(\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ satisfies $(a, b, c)-(b, a, c) \equiv 0$ and $(a, b, c) + (b, c, a) + (c, a, b) \equiv 0$; these algebras are studied in [6].

LEMMA 3. *In a (γ, δ) algebra different from $(\pm 1, 0)$,*

$$(a, b, c) = -\gamma(b, a, c) - \delta(c, a, b)$$

and

$$(a, b, c) = \gamma(a, c, b) + (\delta - 1)(b, c, a).$$

Proof. The first statement is Eq. (1), the definition of (γ, δ) algebras. Since the (γ, δ) algebra is not of type $(\pm 1, 0)$, by Lemma 1 it corresponds to an algebra of type $1 \oplus 1 \oplus \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix}\right)$ and by Table III the cyclic law holds. Thus $(a, b, c) = -(b, c, a) - (c, a, b) = -(b, c, a) + \gamma(a, c, b) + \delta(b, c, a)$ by Eq. (1).

This lemma is used to move any particular element of an associator to the center position of the associator.

LEMMA 4. *Let R be a (γ, δ) algebra; $(\gamma, \delta) \neq (\pm 1, 0)$. Let P and Q be ideals of R. Then $PQ + QP$ is an ideal of R; in particular, P^n is an ideal for any positive integer n. The set of elements $\{a \in R \mid aP + Pa \subseteq Q\}$ is an ideal of R. It is called the annihilator of P* mod *Q.*

Proof. It is a consequence of Lemma 3.

LEMMA 5. *Let R be a (γ, δ) algebra, $(\gamma, \delta) \neq (\pm 1, 0)$. If S is a subset of R, then $S^{n+1} = SS^n + S^nS$.*

Proof. Let us assume the result is false for some positive integer n. If $n = 1$, the result is true. Let n be the smallest integer where it fails. By definition of S^{n+1}, there must be positive integers r and t such that $r + t = n + 1$ and $S^rS^t + S^tS^r \not\subseteq SS^n + S^nS$. We may assume of all possible choices of r and t, that r is minimal. Certainly $r \neq 1$. Thus $S^rS^t + S^tS^r \subseteq (SS^{r-1} + S^{r-1}S)\, S^t + S^t(SS^{r-1} + S^{r-1}S)$ by choice of $n \subseteq SS^{r+t-1} + S^{r-1}S^{t+1} + S^{t+1}S^{r-1} + S^{r+t-1}S + (S, S^{r-1}, S^t) + (S^{r-1}, S, S^t) + (S^t, S, S^{r-1}) + (S^t, S^{r-1}, S) \subseteq SS^n + S^nS +$

$(S, S^t, S^{r-1}) + (S^{r-1}, S^t, S)$ using minimality of r and Lemma 3 $\subseteq SS^n + S^nS$ using minimality of r. This contradiction shows that $S^{n+1} \subseteq S^nS + SS^n$ for all positive integers n.

HARD WORK

In this section we prove the major lemmas needed for the structure theory in the last section. We prove our theorems for algebras of type $1 \oplus 1 \oplus (\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix})$ where λ is eventually restricted to $\lambda \neq -1, 0, \frac{1}{2}, 2$. These correspond to (γ, δ) algebras excluding $(\pm 1, 0)$ and $(1, 1)$.

We introduce a list of identities which hold in algebras of type $1 \oplus 1 \oplus (\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix})$.

$$0 \equiv \bar{A}(a, b, c) = (a, b, c) + (b, a, c) - (b, c, a) - (c, b, a) \\ + \lambda\{(a, c, b) - (c, a, b) + (b, c, a) - (b, a, c)\} \quad (3)$$

$$0 \equiv \bar{B}(a, b, c) = (a, b, c) + (b, c, a) + (c, a, b) \quad (4)$$

$$0 \equiv [ab, c] + [bc, a] + [ca, b] \quad (5)$$

$$0 \equiv \bar{C}(a, b, c, d) = (ab, c, d) - (a, bc, d) + (a, b, cd) \\ -a(b, c, d) - (a, b, c)d \quad (6)$$

$$0 \equiv \bar{D}(a, b, c) = [ab, c] - a[b, c] - [a, c]\, b - (a, b, c) + (a, c, b) \\ - (c, a, b) \quad (7)$$

$$0 \equiv \bar{E}(a, b, c, d) = [a, (b, c, d)] - [b, (c, d, a)] + [c, (d, a, b)] \\ - [d, (a, b, c)] \quad (8)$$

$$0 \equiv \bar{F}(a, b, c) = [[a, b], c] + [[b, c], a] + [[c, a], b] \quad (9)$$

Proof. Equation (3) corresponds to the element $0 \oplus 0 \oplus (\begin{smallmatrix} 3 & 3\lambda \\ 0 & 0 \end{smallmatrix})$ which is contained in the left ideal generated by $1 \oplus 1 \oplus (\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix})$. Equation (4) corresponds to the element $3 \oplus 3 \oplus (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})$ which is contained in the left ideal generated by $1 \oplus 1 \oplus (\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix})$. Equation (5) is equivalent in any nonassociative algebra to Eq. (4). Equation (6) and Equation (7) hold in any nonassociative algebra. Equation (8) is a consequence of Equation (4) and Equation (6) because $\bar{E}(a, b, c, d) = -\bar{C}(a, b, c, d) + \bar{C}(b, c, d, a) - \bar{C}(c, d, a, b) + \bar{C}(d, a, b, c) + \bar{B}(ab, c, d) - \bar{B}(bc, d, a) + \bar{B}(cd, a, b) - \bar{B}(da, b, c)$. Equation (9) is a consequence of Equation (5). Equation (9) says the algebra is Lie admissible; i.e., under the Lie product, $[a, b] = ab - ba$, the algebra is a Lie algebra.

LEMMA 6. *If $\lambda \neq 0, 2$, any entry of an associator may be changed to the last position, according to the following identities.* $D = \lambda(2 - \lambda)$.

$$(a, b, c) = \frac{1 - 2\lambda}{D}(b, c, a) + \frac{1 - \lambda + \lambda^2}{D}(c, b, a),$$

$$(a, c, b) = \frac{1 - \lambda + \lambda^2}{D}(b, c, a) + \frac{1 - 2\lambda}{D}(c, b, a),$$

$$(b, a, c) = \frac{-1 + \lambda - \lambda^2}{D}(b, c, a) + \frac{-1 + \lambda^2}{D}(c, b, a),$$

$$(c, a, b) = \frac{-1 + \lambda^2}{D}(b, c, a) + \frac{-1 + \lambda - \lambda^2}{D}(c, b, a).$$

In particular,

$$(a, b, b) = \frac{1 - \lambda}{\lambda}(b, b, a),$$

$$(b, a, b) = -\frac{1}{\lambda}(b, b, a).$$

Proof. The table is obtained by solving the following system of linear equations: $\bar{B}(a, b, c) = 0$, $\bar{B}(a, c, b) = 0$, $\bar{A}(a, b, c) = 0$, $\bar{A}(b, a, c) = 0$.

We now begin using group representation very strongly. The functions $f(a, b, c, d)$ are expressions involving associators and commutators and are clear from the statements of the lemmas and theorems. When we are dealing with the left ideal of identities which hold for any choice of arguments, we indicate this by using R for the arguments of f. When we are dealing with the subspace of zeros which depends on a particular choice of arguments a, b, c, d, we indicate this by using $f(a, b, c, d)$. Rather than number our functions, we separate them with the symbol $\oplus$. Thus, where f, f', and f'' are functions, we write $f \oplus f' \oplus f''$, and it is understood that an element $g \oplus g' \oplus g''$ in $G_n^{\langle 3\rangle}$ applies g to f, g' to f' and g'' to f''. The representation of G_4 is $M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5$. In the representation of $G_4^{\langle 3\rangle}$, the representation of each summand has the corresponding five parts. $G_4 \oplus G_4 \oplus G_4$ is represented by:

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} \oplus \begin{bmatrix} M_1' \\ M_2' \\ M_3' \\ M_4' \\ M_5' \end{bmatrix} \oplus \begin{bmatrix} M_1'' \\ M_2'' \\ M_3'' \\ M_4'' \\ M_5'' \end{bmatrix}. \tag{10}$$

This display is crucial to efficient dealing with identities. The set L of all elements l of $G_4^{\langle 3\rangle}$, such that for any arguments $[a, b, c, d]$ the value of $s(l)$ is 0, forms a submodule of $G_4^{\langle 3\rangle}$ over G_4. Since it is always possible to find an element of G_4 which multiplies a particular row of a matrix by 1 and multiplies all other rows by 0, we can say that each row of the display is an identity of R. If one stays within the rows of the same M_i, one can perform any row operation and still maintain an identity of R. It is important to perform the same row operation on the entire row, which in this example extends across three separate matrices. The minimal identities correspond to rows of the display. The next larger unit of identities corresponds to rows which come from the same M_i factors. We often deal with rows corresponding to the same M_i and indicate it in this way:

In M_5, $M \oplus M' \oplus M''$ is an identity.

Here the M, M', and M'' are matrices. This is to be considered as a member of a complete display as given in (10) where all the nonlisted entries are zero. By the above discussion, an element with a display given in (10) is an identity if and only if $M_i \oplus M_i' \oplus M_i''$ is an identity for each i.

THEOREM 1. *The left ideal of identities for $R(R, R, R)$ contains*

$$1 \oplus 1 \oplus \begin{pmatrix} 1 & 1-\lambda \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1-\lambda & -1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The left-hand sides of $a\bar{A}(b, c, d) = 0$ and $a\bar{B}(b, c, d) = 0$ are, respectively: $I + (23) - (243) - (24) + \lambda\{(34) - (234) + (243) - (23)\}$ and $I + (243) + (234)$. The matrix representation of these two elements of G_4 is

$$0 \oplus 0 \oplus \begin{pmatrix} 3 & 3-3\lambda \\ 0 & 0 \end{pmatrix}$$
$$\oplus \begin{pmatrix} 1+\lambda & 1-2\lambda & -2+\lambda \\ 0 & 0 & 0 \\ -1-\lambda & -1+2\lambda & 2-\lambda \end{pmatrix} \oplus \begin{pmatrix} 1-\lambda & -1 & \lambda \\ -2+2\lambda & 2 & -2\lambda \\ 1-\lambda & -1 & \lambda \end{pmatrix}$$

and

$$3 \oplus 3 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

From the consideration of these representations the theorem is proved.

THEOREM 2. *If $\lambda \neq \frac{1}{2}$, the left ideal of identities of $[R, (R, R, R)]$ contains*

$$1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1-\lambda & -1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The left-hand sides of $[a, \bar{A}(b, c, d)] = 0$, $[a, \bar{B}(b, c, d)] = 0$, and $\bar{E}(a, b, c, d) = 0$ are, respectively, $I + (23) - (243) - (24) + \lambda\{(34) - (234) + (243) - (23)\}$, $I + (243) + (234)$, and $I - (1432) + (13)(24) - (1234)$. The representation of these three elements of G_4 is

$$0 \oplus 0 \oplus \begin{pmatrix} 3 & 3-3\lambda \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1+\lambda & 1-2\lambda & -2+\lambda \\ 0 & 0 & 0 \\ -1-\lambda & -1+2\lambda & 2-\lambda \end{pmatrix}$$
$$\oplus \begin{pmatrix} 1-\lambda & -1 & \lambda \\ -2+2\lambda & 2 & -2\lambda \\ 1-\lambda & -1 & \lambda \end{pmatrix}$$

$$3 \oplus 3 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$0 \oplus 4 \oplus \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & 2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Provided $\lambda \neq \frac{1}{2}$, the left ideal of identities of $[R, (R, R, R)]$ contains

$$1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1-\lambda & -1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\lambda = \frac{1}{2}$, it is only known that M_3 contains $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

COROLLARY 1. $[b, (b, b, a)] = [b, (b, a, b)] = [b, (a, b, b)] = 0.$

Proof. It is sufficient to observe that the representation of $[b, (b, b, a)]$, $[b, (b, a, b)]$ and $[b, (a, b, b)]$ is zero in M_3 and M_5.

COROLLARY 2. *If* $\lambda \neq \frac{1}{2}$, *then* $[a, (b, b, a)] = [a, (b, a, b)] = [a, (a, b, b)] = 0.$

Proof. $[a, (b, b, a)]$, $[a, (b, a, b)]$, $[a, (a, b, b)]$ are, respectively, $(I + (14))(I + (23))$, $(I + (13))(I + (24))$, $(I + (12))(I + (34))$. It is sufficient to observe that the representation of these elements is zero in M_5.

We also need the results contained in the following lemmas.

LEMMA 7. *If* $\lambda \neq \frac{1}{2}$, *then* $[a, (b, b, [a, b])] = 0.$

Proof. By Corollary 2, $[a, (b, b, [a, b])] = -[[a, b], (b, b, a)] =$ by Eq. (9) $[[b, (b, b, a)], a] + [[(b, b, a), a], b] = 0$ by Corollary 1 and Corollary 2.

LEMMA 8. *If* $\lambda \neq 0, \frac{1}{2}, 2$, *then* $[b, (b, a, [a, b])] = 0.$

Proof. From $\bar{E}(b, b, a, [a, b]) = 0$, Lemma 6, Lemma 7, and Corollary 2, one gets $[b, (a, [a, b], b)] = [b, (b, a, [a, b])] =$ by Corollary 2 and Lemma 6 $-\lambda(1 + \lambda)/D[b, (a, [a, b], b)]$. But then $3\lambda/D[b, (a, [a, b], b)] = 0 = [b, (b, a, [a, b])]$.

If we now consider the identity $\bar{C}(a, b, c, d) = 0$, by Lemma 6 we can write it down as

$$\begin{aligned}(a, b, cd) &+ \frac{1 - 2\lambda}{D}(c, d, ab) + \frac{1 - \lambda + \lambda^2}{D}\{(d, c, ab) + (a, d, bc)\} \\ &+ \frac{1 - \lambda^2}{D}(d, a, bc) - a(b, c, d) - d(a, b, c) \\ &+ [d, (a, b, c)] = 0.\end{aligned}$$

We write this identity using group algebra notation based on the functions $(R, R, RR) \oplus R(R, R, R) \oplus [R, (R, R, R)]$. The element of $G_4^{\langle 3 \rangle}$ is given below:

$$\begin{aligned}I &+ \frac{1 - 2\lambda}{D}(13)(24) + \frac{1 - \lambda + \lambda^2}{D}\{(1324) + (234)\} \\ &+ \frac{1 - \lambda^2}{D}(1234) \oplus -I - (1234) \oplus (1234).\end{aligned}$$

We now express this element of $G_4^{\langle 3 \rangle}$ using the matrix representation in the manner of (10) (Table IV).

TABLE IV[a]

	(R, R, RR)	$\oplus\ R(R, R, R)$	$\oplus\ [R, (R, R, R)]$
M_1	$\frac{2(2 - \lambda)}{D}$		
M_2	0		
M_3	$\frac{1}{D}\left[\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\lambda + \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}\lambda^2\right]$	$\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$	
M_4	$\frac{1}{D}\left[\begin{pmatrix} -1 & 0 & 2 \\ 0 & -2 & 2 \\ 1 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 & -3 \\ -1 & 5 & -1 \\ -3 & 3 & 2 \end{pmatrix}\lambda + \begin{pmatrix} 0 & -1 & 1 \\ 2 & -2 & 0 \\ 2 & -1 & -1 \end{pmatrix}\lambda^2\right]$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$	
M_5	$\frac{1}{D}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & -1 \\ -3 & 1 & 2 \end{pmatrix}\lambda + \begin{pmatrix} -2 & 3 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}\lambda^2\right]$	$\begin{pmatrix} -2 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

[a] In this table $\lambda \neq 0, \frac{1}{2}, 2$.

Theorem 1 gives us the identities for the function $R(R, R, R)$. If l_1 is the element listed in Theorem 1, then $0 \oplus l_1 \oplus 0$ will be an identity for $(R, R, RR) \oplus R(R, R, R) \oplus [R, (R, R, R)]$. Similarly, if l_2 is the identity of $[R, (R, R, R)]$ given in Theorem 2, then $0 \oplus 0 \oplus l_2$ is an identity for $(R,R,RR) \oplus R(R, R, R) \oplus [R, (R, R, R)]$. It should be noted that the representations of l_1 and l_2 in Theorems 1 and 2 are written horizontally and should be rewritten vertically to conform with usage in Table IV. Since $0 \oplus 0 \oplus l_2$ is an identity, we deduce that in M_4, $0 \oplus 0 \oplus I_{3\times 3}$ is an identity. This means that in M_4 any expression of the form $0 \oplus 0 \oplus M_{3\times 3}$ is an identity. This says that the entry of M_4 under $[R, (R, R, R)]$ can be ignored; no matter what it is, it maps to zero. This explains the blank listings appearing in Table IV. Here is an example of what we can learn from the table. From M_1 we deduce that $1 \oplus 0 \oplus 0$ is an identity; thus, $(x, x, x^2) = 0$.

We use the identity in Table IV and the identities in Theorems 1 and 2 to generate more identities. Let us represent the identity of M_5 by the three matrices $W \oplus X \oplus Y$. Then $X^{-1}W \oplus I \oplus X^{-1}Y$ is an identity. Multiplying this by

$$Z = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

gives $ZX^{-1}W \oplus Z \oplus ZX^{-1}Y$ is an identity. Since the matrix Z always maps to zero in M_5 using $R(R, R, R)$ by Theorem 1, we will have $ZX^{-1}W \oplus 0 \oplus ZX^{-1}Y$ is an identity. Now

$$ZX^{-1}Y = \tfrac{1}{2}\begin{pmatrix} -3 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We wish to replace $ZX^{-1}Y$ by a matrix which is invertible and still maintain an identity expression. By Theorem 2, in M_5 the matrix

$$K = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2-2\lambda & -2 & 2\lambda \end{pmatrix}$$

is an identity for $[R, (R, R, R)]$. Thus $ZX^{-1}W \oplus 0 \oplus K + ZX^{-1}Y$ is an identity. Multiplying by $(K + ZX^{-1}Y)^{-1}$ gives $T \oplus 0 \oplus I$ is an identity of R where $T = (K + ZX^{-1}Y)^{-1}\, ZX^{-1}W$.

$$T = \frac{1}{3D}\left[\begin{pmatrix} 4 & 0 & 0 \\ 4 & 0 & 0 \\ -8 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 3 \\ -13 & 3 & 3 \\ 14 & -6 & -6 \end{pmatrix}\lambda + \begin{pmatrix} -5 & 3 & 3 \\ 10 & -6 & -6 \\ -5 & 3 & 3 \end{pmatrix}\lambda^2\right].$$

LEMMA 9. *In M_5 $T \oplus 0 \oplus I$ is an identity of R based on $(R, R, RR) \oplus R(R, R, R) \oplus [R, (R, R, R)]$. $\lambda \neq 0, \tfrac{1}{2}, 2$.*

Going back to our original identity in M_5 of $W \oplus X \oplus Y$, if we multiply $T \oplus 0 \oplus I$ by $-Y$ and add the identities together, we get an identity of the form $W - YT \oplus X \oplus 0$. Multiplying this by X^{-1} gives $T' \oplus I \oplus 0$ is an identity where $T' = X^{-1}(W - YT)$.

$$T' = \frac{1}{3D}\left[\begin{pmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \\ -6 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 3 \\ -8 & 0 & 3 \\ 9 & 0 & -6 \end{pmatrix}\lambda + \begin{pmatrix} -2 & 0 & 3 \\ 5 & 0 & -6 \\ -3 & 0 & 3 \end{pmatrix}\lambda^2\right].$$

LEMMA 10. *In M_5, $T' \oplus I \oplus 0$ is an identity of R based on $(R, R, RR) \oplus R(R, R, R) \oplus [R, (R, R, R)]$. $\lambda \neq 0, \frac{1}{2}, 2$.*

LEMMA 11. *In M_5,*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0 \oplus 0$$

is an identity of R based on $(R, R, RR) \oplus R(R, R, R) \oplus [R, (R, R, R)]$. $\lambda \neq -1, 0, \frac{1}{2}, 2$.

Proof. If we multiply $T' \oplus I \oplus 0$ *in Lemma* 10 *by*

$$\begin{pmatrix} 1-\lambda & -1 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get

$$\frac{1}{3D}\begin{pmatrix} -4 + 3\lambda^2 - \lambda^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1-\lambda & -1 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0$$

is an identity. The middle summand is an identity on $R(R, R, R)$ by Theorem 1. The polynomial $-4 + 3\lambda^2 - \lambda^3 = -(\lambda + 1)(\lambda - 2)^2$ has -1 and 2 as roots. This means that in R_5 we have the identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0 \oplus 0.$$

THEOREM 3. *If $\lambda \neq 0, \frac{1}{2}, 2$, then $A = (R, R, R)$.*

Proof. From Table IV and Theorem 1, we can replace

$$\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} -1 & \lambda \\ 0 & -2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Since these replacement matrices are invertible, we can generate identities in M_3 and M_4 of the form $T_i \oplus I \oplus 0$. Lemma 10 gives an identity of that form in M_5. The blanks in M_1 and M_2 for $R(R, R, R)$ show that $0 \oplus 1 \oplus 0$ must be an identity there. We can say that in $G_4^{\langle 3\rangle}$ there is an identity of the form $g \oplus I \oplus 0$ where I is the identity of G_4. This says the function $R(R, R, R)$ can be expressed as the function (R, R, RR), which means that (R, R, R) absorbs multiplication on the left. By Eq. (6), (R, R, R) absorbs multiplication on the right; hence, (R, R, R) is itself an ideal. Thus $A = (R, R, R)$.

LEMMA 12. *If* $\lambda \neq \frac{1}{2}$, *then* $[d, (x, x, [c, d])] = [d, [c, (x, x, d)]]$.

Proof. By Corollary 2 and Eq. (9), $[d, (x, x, [c, d])] = -[[c, d], (x, x, d)] = [[d, (x, x, d)], c] + [[(x, x, d), c], d] = [d, [c, (x, x, d)]]$.

LEMMA 13. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then the following identity holds in* R.

$$\begin{aligned} &(1 + \lambda)\{(b, a, [c, d]) + (b, c, [d, a]) + (b, d, [a, c])\} \\ &\quad +(-1 + 2\lambda)\{(c, a, [b, d]) + (c, b, [d, a]) + (c, d, [a, b])\} \\ &\quad + (-2 + \lambda)\{(d, a, [b, c]) + (d, b, [c, a]) + (d, c, [a, b])\} \\ &\quad + 2D/\lambda[a, (b, c, d)] = 0 \end{aligned} \tag{11}$$

Proof. From Lemmas 11 and 9 we know that in M_5

$$\frac{\lambda}{D}\begin{pmatrix} 1+\lambda & 1+\lambda & 1+\lambda \\ 1-2\lambda & 1-2\lambda & 1-2\lambda \\ -2+\lambda & -2+\lambda & -2+\lambda \end{pmatrix} \oplus I$$

is an identity based on $(R, R, RR) \oplus [R, (R, R, R)]$. Written as group elements, the identity is $\lambda/(2D)\{(1 + \lambda)(12) + (-1 + 2\lambda)(123) + (-2 + \lambda)(1234)\}\{I + (243) + (234)\}\{I - (34)\} \oplus I$.

Notice that I on the right-hand side of the matrix representation of the identity represents the element whose representation is $0 \oplus 0 \oplus 0_{2\times 2} \oplus 0_{3\times 3} \oplus I_{3\times 3}$. The I in the right-hand side of the equation written as group elements is the identity of G_4. This is possible by Theorem 2. The elements $0 \oplus 0 \oplus 0_{2\times 2} \oplus 0_{3\times 3} \oplus I_{3\times 3}$ and $1 \oplus 1 \oplus I_{2\times 2} \oplus I_{3\times 3} \oplus I_{3\times 3}$ represent the same element using the function $[R, (R, R, R)]$. From the equation written as group elements, one can express it in terms of the functions $(R, R, RR) \oplus [R, (R, R, R)]$. This is Eq. (11).

THEOREM 4. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then for any* $x \in R$, $(x, x, [x, R]) \subseteq U$.

Proof. From Corollary 1, Corollary 2, and Lemma 6, one calculates that

$$[d, (b, b, c)] = (3\lambda^2/D)[b, (d, c, b)]$$
$$[b, (b, d, c)] = -\lambda(1 + \lambda)/D[b, (d, c, b)].$$

Combining these equations gives

$$[b, (b, d, c)] = -(1 + \lambda)/(3\lambda)[d, (b, b, c)]. \tag{12}$$

In Eq. (11), letting $b = c$, we have

$$[a, (b, b, d)] = 3\lambda^2/(2D)\{(b, a, [d, b]) - (b, d, [a, b]) + (b, b, [a, d])\}.$$

From Eq. (9) and Corollary 1, $0 = \bar{F}(a, b, (b, b, d)) = [[a, b], (b, b, d)] + [b, [a, (b, b, d)]] =$ using Corollary 1, Corollary 2, and the previous sentence $-[d, (b, b, [a, b])] + 3\lambda^2/(2D)\{[b, (b, a, [d, b])] - [b, (b, d, [a, b])] =$ using Lemma 8 $-[d, (b, b, [a, b])] - 3\lambda^2/D[b, (b, d, [a, b])] =$ using Eq. (12) $-[d, (b, b, [a, b])] - \{3\lambda^2/D\}\{-(1 + \lambda)/(3\lambda)\}[d, (b, b, [a, b])]$. We have shown $\lambda(-1 + 2\lambda)/D[d, (b, b, [a, b])] = 0$. This shows that under our restrictions on λ, $(b, b, [a, b]) \in U$.

THEOREM 5. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then* $(x, x, [R, R]) \subseteq U$ *for all* $x \in R$.

Proof. The representation of $(x, x, [c, d])$ with respect to (R, R, RR) is

$$0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \oplus \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

By Lemma 11, all that is required is to show that in M_4,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

is in U. This follows from Theorem 4 since the representation of $(x, x, [x, d])$ is

$$0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -8 & 8 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

LEMMA 14. *If* $\lambda \neq 0$, *then* $(R, R, R) \subseteq \{(x, x, R) \mid x \in R\}$.

Proof. The algebras we are discussing have the representation $1 \oplus 1 \oplus \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix}\right)$. This corresponds to identities based on the function (R, R, R). If we consider identities modulo the subspace spanned by $H = \{(x, x, R) \mid x \in R\}$, we have that

$I + (12) = 2 \oplus 0 \oplus \left(\begin{smallmatrix} 2 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ is also an identity modulo H. This implies that the element $1 \oplus 1 \oplus \left(\begin{smallmatrix} 3 & \lambda \\ -1 & 0 \end{smallmatrix}\right)$ is an identity mod H; since $\lambda \neq 0$, this element is invertible. This means $(a, b, c)_I = (a, b, c) \in H$. Thus $(R, R, R) \subseteq H$.

LEMMA 15. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then* $[[R, R], A] = 0$.

Proof. By Theorem 3 and Lemma 14, it is enough to show that $[[R, R], (x, x, R)] = 0$. $[[R, R], (x, x, R)] \subseteq$ by Corollary 2 $[R, (x, x, [R, R])] = 0$ by Theorem 5.

LEMMA 16. $(x, U, x) = (U, x, x) = (x, x, U) = 0$ *for all* $x \in R$.

Proof. The first identity is a consequence of $\bar{D}(x, x, u) + \bar{B}(x, x, u) = 0$. The second follows from the first and from $\bar{A}(u, x, x) + 2\bar{B}(u, x, x) - \lambda\bar{B}(u, x, x) = 0$. The last is a consequence of the first two and $\bar{B}$.

LEMMA 17. *If* $\lambda \neq 0, 2$, *then* $0 = \bar{G}(a, b, u) = [a, b]\, u - [a, bu] - (3\lambda/D)(a, b, u)$ *for all* $a, b \in R$ *and* $u \in U$.

Proof. $0 = \bar{D}(u, b, a) = [ub, a] - u[b, a] - (u, b, a) + (u, a, b) - (a, u, b) = [ub, a] - u[b, a] - (3\lambda/D)(a, b, u)$ by Lemmas 6 and 16.

THEOREM 6. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then* $(R, [R, R], U) = 0$ *and* $(R, R, U) \subseteq U$.

Proof. If $a \in U$, and $b = c$, Eq. (11) reduces to $3\lambda(b, u, [b, d]) = 0$. This means $(R, U, [R, R])$ is an alternating function on its 1, 3, 4 arguments. By Lemmas 2, 3, and 16, $(U, R, [R, R])$ and $(R, [R, R], U)$ are alternating functions on their 2, 3, 4 and 1, 2, 3 arguments, respectively. By Lemma 16 and Eq. (12) with $c \in U$, we also have that $[R, (R, R, U)]$ is an alternating function on its first three arguments. By Eq. (5) we have that $[au, b] = [a, bu]$ for all $a, b \in R$, $u \in U$.

Returning to Eq. (11) and letting $d \in U$, the terms reduce to $(-6 + 3\lambda)(u, a, [b, c]) + (-1 + 2\lambda)(a, u, [b, c]) + (2D/\lambda)[a, (b, c, u)] = 0$. By Lemma 6 we get:

$$(2 + 5\lambda)(a, [b, c], u) + (2D/\lambda)[a, (b, c, u)] = 0. \tag{13}$$

From Lemma 17 and Eq. (9) we get the following. $0 = \bar{G}(a, [b, c], u) + \bar{G}(b, [c, a], u) + \bar{G}(c, [a, b], u) + [b, \bar{G}(c, a, u)] + [c, \bar{G}(a, b, u)] - \bar{F}(b, c, au) + \bar{F}(a, b, c)\, u = -(3\lambda/D)\{(a, [b, c], u) + (b, [c, a], u) + (c, [a, b], u) + [b, (c, a, u)] + [c, (a, b, u)]\}$. Since both $(a, [b, c], u)$ and $[a, (b, c, u)]$ are alternating functions on the first three arguments, we have $3(a, [b, c], u) + 2[a, (b, c, u)] = 0$. This relation compared to Eq. (13) gives $(R, [R, R], U) = 0$ and $[R, (R, R, U)] = 0$.

LEMMA 18. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then* $[R, (R, R, R)] \subseteq U$.

Proof. We show $[a, [b, (c, d, e)]] = 0$ for all $a, b, c, d, e \in R$. By Eq. (9), $[a, [b, (c, d, e)]] + [b, [(c, d, e), a]] + [(c, d, e), [a, b]] = 0$, and so, by Lemma 15, $[a, [b, (c, d, e)]] = [b, [a, (c, d, e)]]$. We can thus assume $a = b$. By Lemma 14 we may assume $c = d$. By Corollary 2, Lemma 12, and Theorem 5, $[a, [a, (c, c, e)]] = -[a, [e, (c, c, a)]] = -[a, (c, c, [e, a])] = 0$.

LEMMA 19. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then the following hold.*

1. $(a, b, [c, d]) - (c, d, [a, b]) \in U$ *for all* $a, b, c, d \in R$.
2. $(R, A, [R, R]) + (A, R, [R, R]) \subseteq U$.
3. $(R, [R, R], [R, R]) \subseteq U$.
4. $(R, R, [[R, R], R]) \subseteq U$.
5. $A[[R, R], R] + R(A[[R, R], R]) \subseteq U$.

Proof. We prove part 1. The representation of $(a, b, [c, d]) - (c, d, [a, b])$ with respect to (R, R, RR) is

$$0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}.$$

By Theorem 5, we get that the M_4 component is in U, and by Lemmas 18, 9, and 11, we get the M_5 component is in U. We prove part 2. $[R, A] \subseteq U$ by Lemma 18; by Theorem 6, $(R, R, [R, A]) \subseteq U$. The result $(R, A, [R, R]) \subseteq U$ follows from the first part of this lemma. The result that $(A, R, [R, R]) \subseteq U$ is proved similarly.

We prove part 3. From Theorem 5 and Lemma 6, we get that for any $x \in R$, $([R, R], x, x) + (x, [R, R], x) + (x, x, [R, R]) \subseteq U$. Let $a \in R$, and $c, c' \in [R, R]$. From the previous sentence, it is easy to show that (a, c, c') is an alternating function of its entries mod U. By $0 \equiv \bar{B}$, we conclude $(a, c, c') \in U$.

We prove part 4. Using part 1 of this lemma, we have $(a, b, [c, [d, e]]) - (c, [d, e], [a, b]) \in U$. The result follows from part 3. We prove part 5. From $\bar{D}(A, R, [R, R])$, Lemma 15, Lemma 6, and part 2 of this lemma, we get $A[[R, R], R] \subseteq U$. From this and part 4 of this lemma, we get $R(A[[R, R], R]) \subseteq U$.

THEOREM 7. *If* $\lambda \neq -1, 0, \frac{1}{2}, 2$, *then* $\langle[[R, R], R]\rangle A + A\langle[[R, R], R]\rangle$ *is an ideal contained in* U.

Proof. By Lemma 4 it is an ideal. We now show that it is contained in U. Let $T = \{u \in U \mid Ru \subseteq U\}$. By Theorem 6, T is an ideal of R. By part 5 of Lemma 19, $A[[R, R], R] \subseteq T$. Let $H = \{x \in R \mid xA + Ax \subseteq T\}$. H is an ideal by Lemma 4. We have just shown that $[[R, R], R] \subseteq H$ using Lemma 15. Since H is an ideal, $\langle[[R, R], R\rangle \subseteq H$. This proves the theorem.

LEMMA 20. *If* I *is an ideal and* $I \subseteq U$, *then* $\langle[[R, R], R]\rangle I = 0$. *We require* $\lambda \neq -1, 0, \frac{1}{2}, 2$.

Proof. By $\bar{D}(R, I, [R, R]) = 0$, Lemma 6, Lemma 16, and Theorem 6, $[R, [R, R]]\, I \subseteq [RI, [R, R]] + R[I, [R, R]] + (R, I, [R, R]) + (R, [R, R], I) + ([R, R], R, I) = 0$. By Lemma 4, $\langle [R, [R, R]]\rangle\, I = 0$.

LEMMA 21. *If I is an ideal and $I \subseteq U$, then $AI = 0$. We require $\lambda \neq -1$, 0, $\frac{1}{2}$, 2.*

Proof. Let $x \in I$. By Eq. (6) and Lemma 16,

$$(a^2, b, x) - (a, ab, x) - a(a, b, x) - (a, a, b)\, x = 0.$$

Then, by Lemma 17 and Theorem 6,

$$D/(3\lambda)\{[a^2, b]\, x - [a, ab]\, x - (a[a, b])x\} - (a, a, b)\, x = 0.$$

Now by Lemma 20 and by $[a^2, b] = [a, ab + ba]$ (Eq. (5)), we have

$$(D/3\lambda)\{\tfrac{1}{2}[a^2, b] - a[a, b]\}\, x - (a, a, b)\, x = 0. \tag{$*$}$$

From Eq. (7) we have

$$\begin{aligned}(-D/3\lambda)\{\tfrac{1}{2}\,[a^2, b] - \tfrac{1}{2}\, a[a, b] - \tfrac{1}{2}\,[a, b]a \\ - \tfrac{1}{2}\,(a, a, b) + \tfrac{1}{2}\,(a, b, a) - \tfrac{1}{2}\,(b, a, a)\}\, x = 0.\end{aligned}$$

By Lemma 20 and $\bar{B}$, this yields

$$(-D/3\lambda)\{\tfrac{1}{2}\,[a^2, b] - a[a, b] + (a, b, a)\}\, x = 0. \tag{$**$}$$

Adding $(*)$ and $(**)$, one gets

$$(-D/3\lambda)(a, b, a)\, x - (a, a, b)\, x = 0.$$

By Lemma 6, $2(1 - 2\lambda)/3\lambda(a, a, b)\, x = 0$. By Lemma 14 and Theorem 3, $Ax = 0$.

LEMMA 22. *If $\lambda \neq -1, 0, \frac{1}{2}, 2$ and $A^n \subseteq \langle [[R, R], R]\rangle$, then $A^{n+2} = 0$.*

Proof. Let $I = \langle [[R, R], R]\rangle\, A + A\langle [[R, R], R]\rangle$. By Theorem 7, I is an ideal and $I \subseteq U$. By Lemma 5, $A^{n+2} \subseteq A(AA^n + A^nA) + (AA^n + A^nA)\, A \subseteq AI + IA = 0$ by Lemma 21.

Since the restriction $\lambda \neq -1, 0, \frac{1}{2}, 2$ corresponds to (γ, δ) algebras excluding types $(\pm 1, 0)$ or $(1, 1)$, we have the following theorem.

THEOREM 8. *The kernel K of the natural homomorphism of a (γ, δ) algebra R onto the subdirect sum $R/A \oplus R/C$ is a nilpotent ideal (index $\leqslant 3$). Furthermore, $K^2 \subseteq U$. We restrict $(\gamma, \delta) \neq (\pm 1, 0)$ or $(1, 1)$.*

Proof. $K^2 \subseteq U$ by Theorem 7. $K^3 = 0$ by Lemma 21.

STRONGLY $(-1, 1)$ ALGEBRAS

In this section we show that the ideal generated by the associators of a finitely generated strongly $(-1, 1)$ algebra is a nilpotent ideal. This result is nearly contained in [4], and in this section most references will be to [4]. A $(-1, 1)$ algebra is an algebra which satisfies $1 \oplus 1 \oplus (\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix})$. This corresponds to a right alternative Lie admissible algebra. In associator form: $(a, b, c) + (a, c, b) = 0$ and $(a, b, c) + (b, c, a) + (c, a, b) = 0$ for all a, b, c in R. A strongly $(-1, 1)$ algebra satisfies $(a, b, c) + (a, c, b) = 0$ and $[a, [b, c]] = 0$ for all a, b, c in R. A strongly $(-1, 1)$ algebra is a $(-1, 1)$ algebra, since $[R, [R, R]] = 0$ implies $(a, b, c) + (b, c, a) + (c, a, b) = 0$.

We let $U = \{u \in R \mid [u, R] = 0\}$. We let $W = (R, R, U)$, and for any set S, $\langle S \rangle =$ the ideal generated by S.

LEMMA 23. *In a strongly $(-1, 1)$ algebra, the following hold.*

1. $W \subseteq U$ [4, Lemma 3]
2. $\langle (R, R, R) \rangle = (R, R, R)$ [4, Lemma 4]
3. $\langle W^i \rangle \subseteq W^i + W^i R$ [4, Lemma 11]
4. $\langle W^i \rangle \langle W^j \rangle \subseteq \langle W^{i+j} \rangle$ [4, Lemma 12]
5. *If R is generated by n elements, then* $W^{n+1} = 0$. [4, Lemma 13]
6. $A^2 \subseteq \langle W^1 \rangle$ [4, Lemma 14]
7. $(A, A, W^i) \subseteq \langle W^{i+1} \rangle$ [4, Lemma 15]
8. $(\langle W^i \rangle A) A \subseteq \langle W^{i+1} \rangle$ [4, Lemma 16]

LEMMA 24. *In a strongly $(-1, 1)$ algebra, $A^{2n} \subseteq \langle W^n \rangle$.*

Proof. The proof is by induction on n. If $n = 1$, the result is true by Lemma 23, part 6. Assume it is true for n. If J is any ideal, by Lemma 4, $T(J) = \{x \in R \mid xA + Ax \subseteq J\}$ is an ideal. We claim $W^n A + AW^n \subseteq T(\langle W^{n+1} \rangle)$. Since $W^n \subseteq U$, $AW^n = W^n A$. $A(AW^n) + (W^n A)\, A \subseteq A^2 W^n + (A, A, W^n) + (W^n A)\, A \subseteq \langle W^1 \rangle \langle W^n \rangle + (A, A, W^n) + (\langle W^n \rangle A)\, A$ by Lemma 23, part 6 $\subseteq \langle W^{n+1} \rangle$ by Lemma 23, parts 4, 7, and 8. We have shown that $W^n A + AW^n \subseteq T(\langle W^{n+1} \rangle)$. This means $W^n \subseteq T(T(\langle W^{n+1} \rangle))$; therefore $\langle W^n \rangle \subseteq T(T(\langle W^{n+1} \rangle))$. Now $A^{2n+2} = A(A^{2n} A + AA^{2n}) + (A^{2n} A + AA^{2n})A$ by Lemma 5 $\subseteq A(\langle W^n \rangle\, A + A \langle W^n \rangle) + (\langle W^n \rangle\, A + A \langle W^n \rangle)\, A \subseteq AT(\langle W^{n+1} \rangle) + T(\langle W^{n+1} \rangle)\, A \subseteq \langle W^{n+1} \rangle$.

LEMMA 25. *If R is a finitely generated strongly $(-1, 1)$ algebra, then A is a nilpotent ideal of R.*

Proof. If R is generated by n elements, then $A^{2n+2} \subseteq \langle W^{n+1} \rangle = 0$ by Lemmas 24 and 23, part 5.

Quasi-equivalence

If $(R, +, \cdot)$ is a (γ, δ) algebra, by defining a new product $\circ$ on R, $x \circ y = \mu xy + \nu yx$, one gets a new algebra. If $\mu \neq \pm \nu$ we call $(R, +, \circ)$ quasi-equivalent to $(R, +, \cdot)$. This property is symmetric since $(R, +, \cdot)$ is quasi-equivalent to $(R, +, \circ)$ for $\mu' = \mu/(\mu^2 - \nu^2)$ and $\nu' = -\nu/(\mu^2 - \nu^2)$. A subset I of R is an ideal of $(R, +, \cdot)$ if and only if I is an ideal of $(R, +, \circ)$. Also, I^n is the same set whether calculated in $(R, +, \cdot)$ or $(R, +, \circ)$. Thus, an ideal is locally nilpotent in $(R, +, \cdot)$ if and only if it is a locally nilpotent ideal in $(R, +, \circ)$.

We calculate that

$$(a, b, c)^\circ = \mu^2(a, b, c) - \nu^2(c, b, a) + \mu\nu\{(b, a, c) - (b, c, a) - (c, a, b) + (a, c, b)\} + \mu\nu[b, [a, c]].$$

Theorem 9 (Nikitin). *If $(R, +, \cdot)$ is any (γ, δ) algebra with $(\gamma, \delta) \neq (-1, 0)$, and $[R, [R, R]] = 0$, then R is quasi-equivalent to a strongly $(-1, 1)$ algebra $(R, +, \circ)$. Furthermore, $(R, R, R) = (R, R, R)^\circ$.*

Proof. Using $G_3^{\langle 2\rangle}$ based on $(R, R, R)^\circ \oplus (R, R, R)$, we have the identity $I \oplus -\mu^2 I + \nu^2(13) - \mu\nu\{(12) - (132) - (123) + (23)\}$. This has the representation

$$\begin{array}{ccc}
(R, R, R)^0 & \oplus & (R, R, R) \\
1 & & -\mu^2 + \nu^2 \\
1 & & -\mu^2 - \nu^2 + 4\mu\nu \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \begin{pmatrix} -(\mu + \nu)^2 & -\nu(\mu + \nu) \\ 0 & -(\mu - \nu)(\mu + \nu) \end{pmatrix}.
\end{array}$$

Multiplying M_3 by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -(\mu + \nu)^2 & -\mu(\mu + \nu) \\ 0 & 0 \end{pmatrix}.$$

To show that $(R, +, \circ)$ is a $(-1, 1)$ algebra requires that $1 \oplus 1 \oplus \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ be an identity. This will be an identity if

$$-\mu^2 + \nu^2 \oplus -\mu^2 - \nu^2 + 4\mu\nu \oplus \begin{pmatrix} -(\mu + \nu)^2 & -\mu(\mu + \nu) \\ 0 & 0 \end{pmatrix}$$

is an identity in $(R, +, \cdot)$. Since $[R, [R, R]] = 0$ and $(\gamma, \delta) \neq (-1, 0)$, we have that $1 \oplus 1 \oplus 0_{2\times 2}$ is an identity on $(R, +, \cdot)$. All that is required is to find a μ and a ν such that

$$-\mu(\mu + \nu)/(-(\mu + \nu)^2) = \lambda.$$

One possible choice is $\mu = \lambda$ and $\nu = 1 - \lambda$. These values of μ and ν are admissible. Since R is a (γ, δ) algebra, $\lambda \neq \frac{1}{2}$, and $\lambda \neq \frac{1}{2}$ implies $\mu \neq \pm\nu$.

Since $[a, b]^\circ = (\mu - \nu)[a, b]$, we deduce that in $(R, +, \circ)$, $[a, [b, c]]^\circ = (\mu - \nu)^2[a, [b, c]] = 0$; thus, $(R, +, \circ)$ is a strongly $(-1, 1)$ algebra.

Every associator in $(R, +, \circ)$ is a sum of associators of $(R, +, \cdot)$. Thus $(R, R, R)^\circ \subseteq (R, R, R)$. Since quasi-equivalence is symmetric, we have $(R, R, R) \subseteq (R, R, R)^\circ$.

THEOREM 10. *If R is a finitely generated (γ, δ) algebra with $(\gamma, \delta) \neq (\pm 1, 0)$ or $(1, 1)$, then (R, R, R) is a nilpotent ideal.*

Proof. Let $R\# = R/\langle [R, [R, R]]\rangle$. $R\#$ is finitely generated, and, by Theorem 9, $R\#$ is quasi-equivalent to a finitely generated strongly $(-1, 1)$ algebra. By Lemma 25, $\langle (R\#, R\#, R\#)^\circ\rangle^n = 0$ for some n. This means that in R, $(R, R, R)^n \subseteq \langle [R, [R, R]]\rangle$. By Lemma 22, $\langle (R, R, R)\rangle^{n+2} = 0$.

THEOREM 11. *If R is a (γ, δ) algebra, $(\gamma, \delta) \neq (\pm 1, 0)$ or $(1, 1)$, then (R, R, R) is a locally nilpotent ideal.*

Proof. By Theorem 3, (R, R, R) is an ideal. It suffices to show that if $S = \{(a_i, b_i, c_i) \mid i = 1, 2, \ldots, k\}$, then $S^n = 0$ for some n. If S' is the subalgebra generated by $\{a_i, b_i, c_i \mid i = 1, 2, \ldots, k\}$, certainly $S \subseteq (S', S', S')$, and by Theorem 10, there exists n such that $(S', S', S')^n = 0$. Thus $S^n = 0$ and the subalgebra generated by S is nilpotent.

THEOREM 12. *If R is a simple (γ, δ) algebra, $(\gamma, \delta) \neq (\pm 1, 0)$ or $(1, 1)$, then R is associative.*

Proof. See [16, Corollary 1 to Lemma 9] or [15, Theorem B]. A simple algebra cannot be locally nilpotent.

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