Generalized $G$-$KKM$ Theorems in Generalized Convex Spaces and Their Applications\textsuperscript{1}

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Some new generalized $G$-$KKM$ and generalized $S$-$KKM$ theorems are proved under the noncompact setting of generalized convex spaces. As applications, some new minimax inequalities, saddle point theorems, a coincidence theorem, and a fixed point theorem are given in generalized convex spaces. These theorems improve and generalize many important known results in recent literature.

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1. INTRODUCTION

Recently, Park and Kim \textsuperscript{12–14} introduced the concept of generalized convex (or $G$-convex) space which includes many classes of topological spaces with various convex structure appearing in nonlinear analysis as special cases. The concept has become an adequate and important tool for studying various problems in nonlinear analysis.

In 1991, Chang and Zhang \textsuperscript{2} introduced a class of generalized $KKM$ mappings in topological vector spaces and gave some generalized $KKM$ theorems and their applications to variational inequalities. In 1996, Ding \textsuperscript{5} introduced a new class of generalized $H$-$KKM$ mappings from a nonempty set $X$ to a $H$-space $(Y, \Gamma_d)$ without linear structure, established some new generalized $H$-$KKM$ theorems, and gave the applications to

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coincidence theorems and generalized games. Recently, Tan [16] introduced a class of generalized \(G\)-KKM mappings from a nonempty set \(X\) to a \(G\)-convex space which includes the classes introduced in [2, 5] as special cases. Under the assumption that the \(G\)-convex hull \(G\)-co(A) is compact for each nonempty finite subset \(A\) of \(G\)-convex space \((Y, \Gamma)\), he gave some new generalized \(G\)-KKM theorems and their applications to minimax inequalities and saddle point problems. Line and Chang [11] introduced a class of generalized \(S\)-KKM mappings from a nonempty set \(X\) to a \(G\)-convex space \(Y\), and some generalized \(S\)-KKM theorems and their applications were given in topological vector spaces.

Motivated and inspired by the above research, in this paper we first introduce the new classes of generalized \(G\)-KKM and generalized \(S\)-KKM mappings from a nonempty set \(X\) to a nonempty subset of a \(G\)-convex space which include the classes of set-valued mappings defined in [2, 5, 11, 17] as special cases. Some new generalized \(G\)-KKM and generalized \(S\)-KKM theorems are established under the noncompact setting of \(G\)-convex space. As applications of our results, some new minimax inequalities, saddle point theorems, a coincidence theorem, and a fixed point theorem are proved in \(G\)-convex spaces. These theorems improve and generalized many important known results in recent literature.

2. PRELIMINARIES

Let \(X\) and \(Y\) be two nonempty sets. We denote by \(2^Y\) and \(\mathcal{A}(X)\) the family of all subsets of \(Y\) and the family of all nonempty finite subsets of \(X\), respectively. For any \(A \in \mathcal{A}(X)\), we denote by \(|A|\) the cardinality of \(A\). Let \(\Delta_n\) be the standard \(n\)-dimensional simplex with vertices \(e_0, e_1, \ldots, e_n\). If \(J\) is a nonempty subset of \(\{0, 1, \ldots, n\}\), we denote by \(\Delta_J\) the convex hull of the vertices \(\{e_j: j \in J\}\).

The following notions were introduced by Ding [4, 5]. For a topological space \(X\), a subset \(A\) of \(X\) is said to be compactly open (resp., compactly closed) in \(X\) if for any nonempty compact subset \(K\) of \(X\), \(A \cap K\) is open (resp., closed) in \(K\). For any given subset \(A\) of \(X\), we define the compact closure and the compact interior of \(A\), denoted by \(\text{ccl}(A)\) and \(\text{cint}(A)\), as

\[
\text{ccl}(A) = \bigcap \{B \subset X: A \subset B \text{ and } B \text{ is compactly closed in } X\},
\]

and

\[
\text{cint}(A) = \bigcup \{B \subset X: B \subset A \text{ and } B \text{ is compactly open in } X\},
\]

respectively. It is easy to see that \(\text{cint}(A)\) (resp., \(\text{ccl}(A)\)) is compactly open (resp., compactly closed) in \(X\) and for each nonempty compact
subset $K$ of $X$ with $A \cap K \neq \emptyset$, we have $cl(A) \cap K = cl_k(A \cap K)$ and $cint(A) \cap K = int_k(A \cap K)$ where $cl_k(A \cap K)$ and $int_k(A \cap K)$ denote the closure and the interior of $A \cap K$ in $K$, respectively. It is clear that a subset $A$ of $X$ is compactly open (resp., compactly closed) in $X$ if and only if $cint(A) = A$ (resp., $cl(A) = A$). Let $X$ be a set and $Y$ be a topological space. A mapping $G: X \to 2^Y$ is said to transfer compactly open-valued (resp., transfer compactly closed-valued) on $X$ if for each $x \in X$ and for each nonempty compact subset $K$ of $Y$ with $G(x) \cap K \neq \emptyset$, $y \in G(x) \cap K$ (resp., $y \notin G(x) \cap K$) implies that there exists a point $x' \in X$ such that $y \in int_k(G(x') \cap K)$ (resp., $y \notin cl_k(G(x') \cap K)$). Clearly, each open-valued (resp., closed-valued) mapping $G: X \to 2^Y$ is transfer open-valued (resp., transfer closed-valued) (see Definitions 6 and 7 of Tian [17] where $X$ is assumed to be a topological space) and is also compactly open-valued (resp., compactly closed-valued). Each transfer open-valued (resp., transfer closed-valued) mapping $G: X \to 2^Y$ is transfer compactly open-valued (resp., transfer compactly closed-valued) and the inverse is not true in general.

The following notion of a generalized convex or $G$-convex space was introduced by Park and Kim [12–14]. $(E, D; \Gamma)$ is said to be a $G$-convex space if $E$ is a topological space, $D$ is a nonempty subset of $E$, and $\mathcal{A}(D) \to 2^E$ is such that

1. for each $A, B \in \mathcal{A}(D)$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$;
2. for each $A \in \mathcal{A}(D)$ with $|A| = n + 1$, there exists a continuous mapping $\phi_A: \Delta_n \to \Gamma(A)$ such that $B \in \mathcal{A}(A)$ with $|B| = J + 1$ implies $\phi_A(\Delta_J) \subset \Gamma(B)$, where $\Delta_J$ denotes the face of $\Delta_n$ corresponding $B \in \mathcal{A}(A)$.

When $D = E$, we write $(E; \Gamma)$ in place of $(E, E; \Gamma)$. Let $(E, D; \Gamma)$ be a $G$-convex space and $X \subset E$. $X$ is said to be $G$-convex if for each $N \in \mathcal{A}(X \cap D)$, $\Gamma(N) \subset X$. The $G$-convex hull of $X$, denoted by $G-co(X)$, is defined by $G-co(X) = \bigcap \{B \subset E: X \subset B$ and $B$ is a $G$-convex subset of $E\}$. It is clear that for each $N \in \mathcal{A}(X \cap D)$, $\Gamma(N) \subset G-co(N)$ and the inverse inclusion relation is not true in general.

**Definition 2.1.** Let $X$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$ and $T: X \to 2^E$ be a set-valued mapping. $T$ is said to be a $G$-KKM mapping if for each $N \in \mathcal{A}(X \cap D)$, $\Gamma(A) \subset \bigcup_{x \in N} T(x)$.

Since $\Gamma(N) \subset G-co(N)$ for each $N \in \mathcal{A}(X \cap D)$, it is clear that each $G$-KKM mapping in the sense of Park [12] (see Definition 1.3 of Tan [16] with $X = D$) is a $G$-KKM mapping in the sense of Definition 1.1; the inverse is not true in general.

**Definition 2.2.** Let $X$ be a nonempty set and $Y$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$. $T: X \to 2^Y$ is said to be a generalized...
\textit{G-KKM} mapping if for each $A = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)$, there exists $B = \{y_1, \ldots, y_n\} \in \mathcal{A}(Y \cap D)$ such that for any subset $\{y_1, \ldots, y_i\}$ of $B$, $\Gamma((y_1, \ldots, y_i)) \subseteq \bigcup_{i=1}^{n} T(x_i)$.

The above notion of a generalized \textit{G-KKM} mapping generalizes the corresponding notions of Chang and Zhang [2] and Tan [16]. If $X$ is a nonempty subset of $Y = E$, then it is clear that each \textit{G-KKM} mapping must be a generalized \textit{G-KKM} mapping and the inverse is not true in general.

\textbf{Definition 2.3.} Let $X$ be a nonempty set, $Y$ be a nonempty subset of a $G$-convex subset $(E, D; \Gamma)$, and $T: X \to 2^Y$. $T$ is said to be a generalized \textit{S-KKM} mapping if for each $N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)$, there exist $y_i \in S(x_i) \cap D$, $i = 1, \ldots, n$ such that for any subset $\{y_1, \ldots, y_i\}$ of $\{y_1, \ldots, y_n\}$, $\Gamma((y_1, \ldots, y_i)) \subseteq \bigcup_{i=1}^{n} T(x_i)$.

Clearly, each generalized \textit{S-KKM} mapping is a generalized \textit{G-KKM} mapping and the above notion in Definition 7 of Lin and Chang [11].

\textbf{Definition 2.4.} Let $X$ be a nonempty set, $Y$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, and $f: X \times Y \to \mathbb{R} \cup \{\pm \infty\}$. For $\lambda \in \mathbb{R}$, $f(x, y)$ is said to be $\lambda$-generalized $G$-diagonally quasiconvex (resp., $\lambda$-generalized $G$-diagonally quasiconcave) in $x$ if for each $N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)$, there exists $\{y_1, \ldots, y_n\} \subset Y \cap D$ such that for any subset $\{y_1, \ldots, y_i\}$ of $\{y_1, \ldots, y_n\}$ and $y_0 \in \Gamma((y_1, \ldots, y_i))$, $\max_{1 \leq j \leq k} f(x_i, y_j) \geq \lambda$ (resp., $\min_{1 \leq j \leq k} f(x_i, y_j) \leq \lambda$).

The above notion is a generalization of the corresponding notion in Definition 1.7 of Tan [16].

\textbf{Definition 2.5.} Let $X$ be a nonempty set, $Y$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, $S: X \to 2^Y$, and $f: X \times Y \to \mathbb{R} \cup \{\pm \infty\}$ be mappings. For some $\lambda \in \mathbb{R}$, $f(x, y)$ is said to be $\lambda$-generalized $S$-diagonally quasiconvex (resp., $\lambda$-generalized $S$-diagonally quasiconcave) in $x$ if for each $N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)$, there exist $y_i \in S(x_i) \cap D$, $i = 1, \ldots, n$ such that for any subset $\{y_1, \ldots, y_i\}$ of $\{y_1, \ldots, y_n\}$ and $y_0 \in \Gamma((y_1, \ldots, y_i))$, $\max_{1 \leq j \leq k} f(x_i, y_j) \geq \lambda$ (resp., $\min_{1 \leq j \leq k} f(x_i, y_j) \leq \lambda$).

The notions in Definition 2.5 generalize the corresponding notions in Definition 7 of Lin and Chang [11].

\textbf{Definition 2.6.} Let $X$ be nonempty set, $Y$ be a topological space, and $f: X \times Y \to \mathbb{R} \cup \{\pm \infty\}$. For some $\lambda \in \mathbb{R}$, $f(x, y)$ is said to $\lambda$-transfer compactly lower (resp., upper) semicontinuous in $y$ if for each compact subset $K$ of $Y$ and for each $y \in K$, there exists a $x \in X$ such that $f(x, y) > \lambda$ (resp., $f(x, y) < \lambda$) implies that there exist an open neighborhood $N(y)$ of $y$ and a point $x' \in X$ such that $f(x', z) > \lambda$ (resp., $f(x', z) < \lambda$).
< \lambda) \) for all \( z \in N(y) \). It is easy to see that if a mapping \( T: X \to 2^Y \) is defined by \( T(x) = \{ y \in Y: f(x, y) \leq \lambda \} \) (resp., \( T(x) = \{ y \in Y: f(x, y) \geq \lambda \} \)) for some \( \lambda \in \mathbb{R} \), then \( T \) is transfer compactly closed-valued if and only if \( f(x, y) \) is \( \lambda \)-transfer compactly lower (resp., upper) semicontinuous in \( y \).

The notions in Definition 2.6 generalize the corresponding notions introduced by Ding [4, p. 13].

3. GENERALIZED G-KKM THEOREMS

**Theorem 3.1.** Let \( X \) be a nonempty set and \( Y \) be a nonempty subset of a \( G \)-convex space \( (E, D, \Gamma) \). Let \( T: X \to 2^Y \setminus \{ \emptyset \} \) be such that each \( T(x) \) is compact closed.

(i) If \( T \) is a generalized \( G \)-KKM mapping, then for each \( N = \{ x_1, \ldots, x_n \} \in \mathcal{A}(X) \), \( \Gamma(\{ y_1, \ldots, y_n \}) \cap (\bigcap_{x \in N} T(x)) \neq \emptyset \) where \( \{ y_1, \ldots, y_n \} \) is the set in touch with \( \{ x_1, \ldots, x_n \} \) in Definition 2.2 of a generalized \( G \)-KKM mapping.

(ii) If the family \( \{ T(x) \in D: x \in X \} \) has the finite intersection property and \( \Gamma(y) = \{ y \} \) for each \( y \in D \), then \( T \) is a generalized \( G \)-KKM mapping.

**Proof.** (i) Suppose the conclusion of (i) is false. Then there exist \( N = \{ x_0, \ldots, x_n \} \in \mathcal{A}(X) \) and the set \( M = \{ y_0, \ldots, y_n \} \in \mathcal{A}(Y \cap D) \) in touch with \( N = \{ x_0, \ldots, x_n \} \) such that \( \Gamma(M) \cap (\bigcap_{x \in N} T(x)) = \emptyset \). It follows that

\[
\Gamma(M) \subset \bigcup_{x \in N} (Y \setminus T(x)). \tag{2.1}
\]

By the definition of a \( G \)-convex space, there is a continuous mapping \( \phi_M: \Delta_n \to \Gamma(M) \) such that for each \( B = \{ y_1, \ldots, y_k \} \in \mathcal{A}(M) \),

\[
\phi_M(\Delta_k) \subset \Gamma(B). \tag{2.2}
\]

Clearly, \( \phi_M(\Delta_n) \) is compact in \( Y \) and by (2.1), we have

\[
\phi_M(\Delta_n) = \bigcup_{i=0}^n \left( (Y \setminus T(x_i)) \cap \phi_M(\Delta_n) \right).
\]

Since each \( T(x) \) is compactly closed, we can assume that \( \{ y_i \}_{i=0}^n \) is the continuous partition of unity subordinated to the open cover \( \{ (Y \setminus T(x_i)) \} \)
Therefore there exists a hand, by the definition of with nonempty compactly closed KKM mapping, it follows from 2.2 that

\[ \{ y \in \phi_M(\Delta_n) : \psi(y) \neq 0 \} \subseteq (Y \setminus T(x)) \cap \phi_M(\Delta_n) \subseteq Y \setminus T(x), \]

(2.3)

and \( \sum_{i=0}^{n} \psi_i(y) = 1 \) for each \( y \in \phi_M(\Delta_n) \). Define a mapping \( \psi : \phi_M(\Delta_n) \to \Delta_n \) by

\[ \psi(y) = \sum_{i=0}^{n} \psi_i(y) e_i, \quad \forall y \in \phi_M(\Delta_n). \]

Then \( \psi \phi_M : \Delta_n \to \Delta_n \) has a fixed point \( z_0 \in \Delta_n \), that is, \( z_0 \in \psi \phi_M(z_0) \).

Let \( y_0 = \phi_M(z_0) \). Then

\[ z_0 = \psi(y_0) = \sum_{j \in J(y_0)} \psi_j(y_0) e_j \in \Delta_{J(y_0)}, \]

where \( J(y_0) = \{ j \in \{0, \ldots, n\} : \psi(y_0) \neq 0 \} \). Since \( T \) is a generalized G-KKM mapping, it follows from (2.2) that

\[ y_0 = \phi_M(z_0) \in \phi_M(\Delta_{J(y_0)}) \subseteq \Gamma(\{ y_j \}_{j \in J(y_0)}) \subseteq \bigcup_{j \in J(y_0)} T(x_j). \]

Therefore there exists a \( j \in J(y_0) \) such that \( y_0 \in T(x_j) \). On the other hand, by the definition of \( J(y_0) \), we have \( \psi_j(y_0) \neq 0 \) and it follows from (2.3) that \( y_0 \in Y \setminus T(x_j) \) which is a contradiction. Hence the conclusion of (i) holds.

(ii) Suppose \( (T(x) \cap D : x \in X) \) has the finite intersection property. Then for each \( \{ x_1, \ldots, x_n \} \in \mathcal{A}(X), \bigcap_{i=1}^{n} (T(x_i) \cap D) \neq \emptyset \) so that we can choose \( y^* \in \bigcap_{i=1}^{n} (T(x_i) \cap D) \) and let \( y_1 = \cdots = y_n = y^* \). Then for any subset \( \{ y_{i_1}, \ldots, y_{i_k} \} \) of \( \{ y_1, \ldots, y_n \} \), we have \( \Gamma(\{ y_{i_1}, \ldots, y_{i_k} \}) = \Gamma(y^*) = \{ y^* \} \) \( \subseteq \bigcup_{j=1}^{k} T(x_{i_j}) \); i.e., \( T \) is a G-KKM mapping.

Remark 3.1. Theorem 3.1 is a improving variant of Theorem 2.2 of Tan [16] and generalizes Theorem 3.1 of Ding [5], Theorem 3.1 of Chang and Zhang [2], and Theorem 1 of Chang and Ma [3] to G-convex spaces.

Theorem 3.2. Let \( X \) be a nonempty set and \( Y \) be a nonempty subset of a G-convex space \( (E, D, \Gamma) \). Let \( T : X \to 2^Y \) be a generalized G-KKM mapping with nonempty compactly closed values.

(i) If for some \( M \in \mathcal{A}(X), \bigcap_{x \in M} T(x) \) is compact, then \( \bigcap_{x \in X} T(x) \neq \emptyset \).

(ii) If \( \bigcap_{x \in X} (T(x) \cap D) \neq \emptyset \) and \( \Gamma(y) = \{ y \} \) for each \( y \in D \), then \( T \) is a generalized G-KKM mapping.
Proof. It is easy to see that the conclusions (i) and (ii) follow from Theorem 3.1.

Remark 3.2. Theorem 3.2 generalizes Theorem 2.3 of Tan [16] in the following ways: (1) \((Y, \Gamma)\) is replaced by a nonempty subset of a \(G\)-convex space \((E, D; \Gamma)\), (2) our class of generalized \(G\)-KKM mappings includes the class defined in [16] as a proper subclass, (3) the assumption that \(G\)-co\(A\) is compact for each \(A \in \mathcal{A}(Y)\) is dropped.

Theorem 3.3. Let \(X\) be nonempty set, \(Y\) be a nonempty subset of a \(G\)-convex space \((E, D; \Gamma)\), \(K\) be nonempty compact subset of \(Y\), and \(T, S: X \to 2^Y \setminus \{\emptyset\}\) be such that

(i) \(T\) is a generalized \(S\)-KKM mapping with nonempty compact closed values,

(ii) for each \(N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)\), there is a compact \(G\)-convex subset \(L_N\) of \(Y\) containing \(\{y_1, \ldots, y_n\}\) such that

\[
L_N \cap \left( \bigcap_{x \in S^{-1}(L_N)} T(x) \right) \subset K,
\]

where \(\{y_1, \ldots, y_n\}\) is the set in touch with \(\{x_1, \ldots, x_n\}\) in the definition of a generalized \(S\)-KKM mapping.

Then \(K \cap (\bigcap_{x \in X} T(x)) \neq \emptyset\).

Proof. We first show that the family \(\{T(x) \cap K: x \in X\}\) has the finite intersection property. For any \(N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)\), let \(L_N\) be the compact \(G\)-convex subset of \(Y\) in condition (ii). Define a mapping \(F: S^{-1}(L_N) \to 2^{L_N}\) by

\[
F(x) = T(x) \cap L_N, \quad \forall \ x \in S^{-1}(L_N).
\]

Since each \(T(x)\) is compactly closed and \(L_N\) is compact, therefore each \(F(x)\) is closed in \(L_N\) for each \(x \in S^{-1}(L_N)\). Since \(T\) is a generalized \(S\)-KKM mapping, for each \(N = \{x_1, \ldots, x_m\} \in \mathcal{A}(S^{-1}(L_N))\), let \(\{y_1, \ldots, y_m\} \subset (Y \cap D)\) be the set in touch with \(\{x_1, \ldots, x_m\}\) in Definition 2.3. Then, by condition (ii), \(\{y_1, \ldots, y_m\} \subset L_N\). Since \(L_N\) is a \(G\)-convex subset of \(Y\), we have that for any subset \(\{y_1, \ldots, y_i\}\) of \(\{y_1, \ldots, y_n\}\), \(\Gamma(\{y_1, \ldots, y_i\}) \subset L_N\). Noting that \(T\) is a generalized \(S\)-KKM mapping, we obtain that \(F\) is also a generalized \(S\)-KKM mapping. Since each generalized \(S\)-KKM mapping is a generalized \(G\)-KKM mapping, it follows from the compactness of \(L_N\) and Theorem 3.2 that \(\bigcap_{x \in S^{-1}(L_N)} F(x) \neq \emptyset\). Let \(y \in \bigcap_{x \in S^{-1}(L_N)} F(x) = L_N \cap \ldots\).
(\bigcap_{x \in S^{-1}(L_x)} T(x))$. Then $y \in K$ by the condition (ii). Hence we have

\[ y \in K \cap \left( \bigcap_{x \in S^{-1}(L_x)} T(x) \right) \subseteq \bigcap_{x \in N} (T(x) \cap K). \]

This shows that the family $\{T(x) \cap K \mid x \in X\}$ has the finite intersection property. Since $K$ is compact and each $T(x)$ is compactly closed, we must have $K \cap \left( \bigcap_{x \in X} T(x) \right) \neq \emptyset$. This completes the proof.

Remark 3.3. Theorem 3.2 generalizes Theorem 1 of Lin and Chang [11] and Theorem 3.2 of Chang and Zhang [2] from the topological vector space to $G$-convex space.

Corollary 3.1. Let $X$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$ and $T \colon X \to 2^E$ be a generalized $G$-KKM mapping with nonempty compactly closed values.

1. If for some $M \in \mathcal{A}(X)$, $\bigcap_{x \in M} T(x)$ is compact, then $\bigcap_{x \in X} T(x) \neq \emptyset$.

2. If for each $N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)$, there exists a compact $G$-convex subset $L_N$ of $E$ containing $\{y_1, \ldots, y_n\}$ and a nonempty compact subset $K$ of $E$ such that

\[ L_N \cap \left( \bigcap_{x \in L_N \cap X} T(x) \right) \subseteq K, \]

where $\{y_1, \ldots, y_n\}$ is the set corresponding to $\{x_1, \ldots, x_n\}$ in the definition of a generalized $G$-KKM mapping, then $K \cap \left( \bigcap_{x \in X} T(x) \right) \neq \emptyset$.

Proof. Since $X$ is a subset of $E$, by letting $Y = E$ and $S \colon X \to 2^E$ be the identity mapping on $X$, then the conclusions (1) and (2) of Corollary 3.1 hold from Theorem 3.2 and Theorem 3.3, respectively.


Theorem 3.4. Let $X$ be a nonempty set and $Y$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$. Let $T \colon X \to 2^Y$ be a set-valued mapping with nonempty transfer compactly closed values.

1. If $T$ is a generalized $G$-KKM mapping and for some $M \in \mathcal{A}(X)$, $\bigcap_{x \in M} \text{ccl} T(x)$ is compact, then $\bigcap_{x \in X} T(x) \neq \emptyset$.

2. If there exists a nonempty compact subset $K$ of $Y$ and a mapping $S \colon X \to 2^Y$ such that $T$ is a generalized $S$-KKM mapping and for each $N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X)$, there is a compact $G$-convex subset $L_N$ of $Y$ containing
\{y_1, \ldots, y_n\} \text{ such that} \]
\[
L_N \cap \left( \bigcap_{x \in G^{-1}(L_N)} \text{ccl } T(x) \right) \subseteq K,
\]
where \{y_1, \ldots, y_n\} is the set in touch with \{x_1, \ldots, x_n\} in the definition of a generalized S-KKM mapping, then \( K \cap (\bigcap_{x \in X} T(x)) \neq \emptyset. \)

**Proof.** (i) Since \( T(x) \subseteq \text{ccl } T(x) \) for each \( x \in X \), therefore the mapping \( \text{ccl } T : X \rightarrow 2^Y \) defined by \( \text{ccl } T(x) = \text{ccl } T(x) \) is a generalized \( G-KKM \) mapping with nonempty compactly closed values. By (i) and the conclusion of Theorem 3.2, \( \bigcap_{x \in X} \text{ccl } T(x) \neq \emptyset. \) Since \( \bigcap_{x \in X} \text{ccl } T(x) \) is compact and \( T \) is transfer compactly closed-valued, it is easy to prove that \( \bigcap_{x \in X} T(x) = \bigcap_{x \in X} \text{ccl } T(x) \neq \emptyset. \)

(ii) By (ii) and Theorem 3.3, we have \( K \cap (\bigcap_{x \in X} \text{ccl } T(x)) \neq \emptyset. \) Now we show that \( K \cap (\bigcap_{x \in X} \text{ccl } T(x)) = K \cap (\bigcap_{x \in X} T(x)). \) Clearly, \( K \cap (\bigcap_{x \in X} T(x)) \subseteq K \cap (\bigcap_{x \in X} \text{ccl } T(x)). \) If \( K \cap (\bigcap_{x \in X} \text{ccl } T(x)) \subseteq K \cap (\bigcap_{x \in X} T(x)), \) then there exist a \( y \in \bigcap_{x \in X} (\text{ccl } T(x) \cap K) = \bigcap_{x \in X} \text{cl}_K(T(x) \cap K) \) and a \( x \in X \) such that \( y \notin T(x) \cap K. \) Since \( T \) is transfer compact closed-valued, there exists \( x' \in X \) such that \( y \notin \text{cl}_K(T(x') \cap K) \) which is a contradiction. Hence we must have \( K \cap (\bigcap_{x \in X} T(x)) = K \cap \bigcap_{x \in X} \text{ccl } T(x) \neq \emptyset \) and so the conclusion (ii) holds.

**Remark 3.5.** Theorem 3.4 is the improving versions of Theorem 3.2 and Theorem 3.3 which generalize Theorem 3.3 of Ding [5] in following ways: (1) \( X \) may not be a subset of \((E, D, \Gamma)\), (2) \((E, D, \Gamma)\) may not be a \( H \)-space, (3) the compactness assumption of each polytopes is dropped. Theorem 3.4 also generalizes Theorem 3.4 of Tan [16] and Theorem 1 of Lin and Chang [11], Theorem 2.1 of Chang [1], Theorems 2 and 3 of Tian [17], Theorem III of Lassonde [10], and Theorem 4 of Ky Fan [8] in many aspects.

4. SOME APPLICATIONS

In this section, by applying our generalized \( G-KKM \) theorems and generalized \( S-KKM \) theorems obtained in the above section, we shall show some new minimax inequalities, saddle point theorems, a coincidence theorem, and a fixed point theorem.

**Theorem 4.1.** Let \( X \) be a nonempty set, \( Y \) be a nonempty subset of a \( G \)-convex space \((E, D; \Gamma)\), and \( \lambda \in \mathbb{R}. \) Let \( f, g : X \times Y \rightarrow \mathbb{R} \cup \{\pm \infty\} \) be such that

\( i \) \hspace{10pt} for each \( (x, y) \in X \times Y, f(x, y) \leq g(x, y), \)

\( ii \) \hspace{10pt} \( f(x, y) \) is \( \lambda \)-transfer compactly lower semicontinuous in \( y. \)
(a) If \( g(x, y) \) is \( \lambda \)-generalized \( G \)-quasiconcave in \( x \) and there exists \( M \in \mathcal{R}(X) \) such that \( \bigcap_{x \in M} \text{ccl} \{ y \in Y : f(x, y) \leq \lambda \} \) is a compact subset of \( Y \), there exists a \( \hat{y} \in Y \) such that \( f(x, \hat{y}) \leq \lambda \) for all \( x \in X \).

(b) If there exist a nonempty compact subset \( K \) of \( Y \) and a mapping \( S: X \to 2^Y \) such that \( g(x, y) \) is \( \lambda \)-generalized \( S \)-diagonally quasiconcave in \( x \) and for each \( N = \{ x_1, \ldots, x_n \} \in \mathcal{R}(X) \), there exists a compact \( G \)-convex subset \( L_N \) of \( Y \) containing \( \{ y_1, \ldots, y_n \} \) such that

\[
L_N \cap \left( \bigcap_{x \in S^{-1}(L_N)} \text{ccl} \{ y \in Y : f(x, y) \leq \lambda \} \right) \subset K,
\]

where \( \{ y_1, \ldots, y_n \} \) is the set in touch with \( \{ x_1, \ldots, x_n \} \) in the definition of \( g(x, y) \) being \( \lambda \)-generalized \( S \)-diagonally quasiconcave in \( x \), then there exists a \( \hat{y} \in K \) such that \( f(x, \hat{y}) \leq \lambda \) for all \( x \in X \).

**Proof.** (a) Define mappings \( T, G: X \to 2^Y \) by

\[
T(x) = \{ y \in Y : f(x, y) \leq \lambda \} \quad \text{and} \quad G(x) = \{ y \in Y : g(x, y) \leq \lambda \}.
\]

It is easy to see from (i) that \( G(x) \subset T(x) \) for each \( x \in X \). Since \( g(x, y) \) is \( \lambda \)-generalized \( G \)-diagonally quasiconcave in \( x \), \( G \) is a generalized \( G \)-KKM mapping and so \( T \) is also a generalized \( G \)-KKM mapping. By (ii), \( T \) is transfer compactly closed-valued. By Theorem 3.4(i), \( \bigcap_{x \in X} T(x) \neq \emptyset \). Choose any \( \hat{y} \in \bigcap_{x \in X} T(x) \). Then \( f(x, \hat{y}) \leq \lambda \) for all \( x \in X \).

(b) Since \( g(x, y) \) is \( \lambda \)-generalized \( S \)-diagonally concave in \( x \), \( G \) is a generalized \( S \)-KKM mapping and so \( T \) is also a generalized \( S \)-KKM mapping. It is easy to see that all conditions of Theorem 3(ii) are satisfied. By Theorem 3.4(ii), \( K \cap (\bigcap_{x \in X} T(x)) \neq \emptyset \), i.e., there exists a \( \hat{y} \in K \) such that \( f(x, \hat{y}) \leq \lambda \) for all \( x \in X \).

**Remark 4.1.** Theorem 4.1(a) generalizes Theorem 3.1 and Theorem 3.2 of Tan [16] in the following ways: (1) \((Y, \Gamma)\) is replaced by a nonempty subset \( Y \) of a \( G \)-convex space \((E, D; \Gamma)\); (2) from one function to two functions; (3) the assumption that \( G \)-co(A) is compact for each \( A \in \mathcal{R}(X) \) is dropped; and Theorem 4.1(b) generalizes Theorem 9 of Lin and Chang [11] in several aspects.

**Corollary 4.1.** Let \( X \) be a nonempty set, \( Y \) be a nonempty subset of a \( G \)-convex space \((E, D; \Gamma)\), and \( f: X \times Y \to \mathbb{R} \cup \{ \pm \infty \} \) be such that for any \( \lambda \in \mathbb{R} \),

(i) \( f(x, y) \) is \( \lambda \)-transfer compactly lower semicontinuous in \( y \)

(ii) one of the conditions (a) and (b) in Theorem 4.1 holds.

Then there exists \( \hat{y} \in Y \) such that

\[
f(x, \hat{y}) \leq \sup_{x \in X} f(x, x), \quad \forall \ x \in X.
\]
Proof. By letting $f(x, y) = g(x, y)$ for all $(x, y) \in X \times Y$ and $\lambda = \sup_{x \in X} f(x, x)$, the conclusion of Corollary 4.1 follows from Theorem 4.1.

Remark 4.2. Corollary 4.1 generalizes Corollary 7 of Lin and Chang [11] in the following ways: (1) the convex subset $Y$ of a topological vector space is replaced by a nonempty subset $Y$ of a $G$-convex space $(E, D; \Gamma)$, (2) the lower semicontinuity of the function $y \mapsto \phi(x, y)$ is weakened, (3) the condition (9.2) of Theorem 9 in [11] is replaced by the weaker condition that $f(x, y)$ is $\lambda$-generalized $S$-diagonally quasiconcave in $x$ for any $\lambda \in \mathbb{R}$. Corollary 4.1 also generalizes the well-known minimax inequality of Ky Fan in [6] to $G$-convex spaces.

When $Y$ is a compact $G$-convex subset of $(E, D; \Gamma)$, Theorem 4.1 gives the following result.

Theorem 4.2. Let $X$ be a nonempty set, $Y$ be a nonempty compact $G$-convex subset of a $G$-convex space $(E, D; \Gamma)$, and $f : X \times Y \to \mathbb{R} \cap [\pm \infty]$ be such that

(i) $f(x, y)$ is 0-transfer lower semicontinuous in $y$,

(ii) $f(x, y)$ is 0-generalized $G$-diagonally quasiconcave in $x$.

Then there exists $\hat{y} \in Y$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Remark 4.3. Theorem 4.2 generalizes Theorem 3.3 of Tan [16] in several aspects and the well-known Ky Fan’s minimax inequality [7, Theorem 1] to $G$-convex space.

Theorem 4.3. Let $X$ and $Y$ be nonempty subsets of $G$-convex spaces $(F, D_1, \Gamma_1)$ and $(E, D_2, \Gamma_2)$, respectively, and $f : X \times Y \to \mathbb{R} \cup [\pm \infty]$ be such that

(i) $f(x, y)$ is 0-transfer compactly upper semicontinuous in $x$ and 0-transfer compactly lower semicontinuous in $y$,

(ii) $f(x, y)$ is 0-generalized $G$-diagonally quasiconcave in $x$ and 0-generalized $G$-diagonally quasiconvex in $y$,

(iii) there exist $M = \{x_1, \ldots, x_m\} \in \mathcal{R}(X)$ and $N = \{y_1, \ldots, y_n\} \in \mathcal{R}(Y)$ such that the set $\bigcap_{x \in N} \text{ccl} (\{y \in Y : f(x, y) \leq 0\})$ is compact in $Y$ and the set $\bigcap_{y \in M} \text{ccl} (\{x \in X : f(x, y) \geq 0\})$ is compact in $X$.

Then $f$ has a saddle point $(\hat{x}, \hat{y}) \in X \times Y$; i.e.,

$$f(x, \hat{y}) \leq f(\hat{x}, \hat{y}) \leq f(\hat{x}, y), \quad \forall (x, y) \in X \times Y.$$ 

In particular, we have

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y) = 0.$$
Proof. By Theorem 4.1(a) (where, \( f \equiv g \) and \( \lambda = 0 \)), there exists \( \hat{y} \in Y \) such that \( f(x, \hat{y}) \leq 0 \) for all \( x \in X \). Let \( g(y, x) = -f(x, y) \) for all \((y, x) \in Y \times X\). Then by Theorem 4.1(a) again, there exists \( \hat{x} \in X \) such that \( g(y, \hat{x}) \leq 0 \) for all \( y \in Y \). It follows that \( f(x, \hat{y}) \leq 0 \leq f(\hat{x}, y) \) for all \((x, y) \in X \times Y\). Hence \( f(\hat{x}, \hat{y}) = 0 \) and

\[
f(x, \hat{y}) \leq f(\hat{x}, \hat{y}) \leq f(\hat{x}, y), \quad \forall (x, y) \in X \times Y
\]

which implies

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{\hat{x} \in X} f(\hat{x}, \hat{y}) \leq \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]

Since \( \inf_{y \in Y} \sup_{x \in X} f(x, y) \geq \sup_{x \in X} \inf_{y \in Y} f(x, y) \) is always true, we have

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y) = f(\hat{x}, \hat{y}) = 0.
\]

Remark 4.4. Theorem 4.3 improves Theorem 4.1 of Tan [16] in the following ways: (1) \( X \) and \( Y \) may be subsets of \( G \)-convex spaces \((F, D; \Gamma)\) and \((E, D; \Gamma)\), respectively; (2) the compactness assumptions of \( G \)-co\((A)\) and \( G \)-co\((B)\) for each \( A \in \mathcal{A}(X) \) and \( \mathcal{A}(Y) \) are both dropped; (3) the lower continuity of \( y \rightarrow f(x, y) \) and the upper semicontinuity of \( y \rightarrow f(x, y) \) are both weakened. If \( X \) and \( Y \) are both compact \( G \)-convex, then the condition (iii) is satisfied automatically and so Theorem 4.3 also includes Theorem 4.2 of Tan [16] as a special case.

Theorem 4.4. Let \( X \) and \( Y \) be nonempty subsets of \( G \)-convex spaces \((E, D; \Gamma_1)\) and \((E, D; \Gamma_2)\), respectively, and \( f : X \times Y \rightarrow \mathbb{R} \cup \{\pm \infty\} \) be such that

(i) \( f(x, y) \) is \( 0 \)-transfer compactly lower semicontinuous in \( y \) and \( 0 \)-transfer compactly upper semicontinuous in \( x \),

(ii) there exist a nonempty compact subset \( K \) of \( Y \) and set-valued mappings \( S : X \rightarrow 2^Y \) such that \( f(x, y) \) is \( 0 \)-generalized \( S \)-diagonally quasiconcave in \( x \) and for each \( N = \{x_1, \ldots, x_n\} \in \mathcal{A}(X) \), there is a compact \( G \)-convex subset \( L_N \) of \( Y \) containing \( \{y_1, \ldots, y_n\} \) such that

\[
L_N \cap \bigcap_{x \in S^{-1}(L_N)} \text{ccl} \left( \{y \in Y : f(x, y) \leq 0\} \right) \subset K,
\]

where \( \{y_1, \ldots, y_n\} \) is the set in touch with \( \{x_1, \ldots, x_n\} \) in the definition of \( f(x, y) \) being \( 0 \)-generalized \( S \)-diagonally quasiconcave in \( x \),

(iii) there exists a nonempty compact subset \( C \) of \( X \) and a set-valued mapping \( T : Y \rightarrow 2^X \) such that \( f(x, y) \) is \( 0 \)-generalized \( T \)-diagonally quasiconvex in \( y \) and for each \( M = \{y_1, \ldots, y_m\} \in \mathcal{A}(Y) \), there is a compact \( G \)-convex
subset $L_M$ of $X$ containing \{x_1, \ldots, x_m\} such that
\[
L_M \cap \bigcap_{y \in T^{-1}(L_M)} \text{ccl} \left( \{x \in X : f(x, y) \geq 0\} \right) \subseteq C,
\]
where \{x_1, \ldots, x_m\} is the set in touch with \{y_1, \ldots, y_m\} in the definition of $f(x, y)$ being $0$-generalized $T$-diagonally quasiconvex in $y$.

Then $f$ has a saddle point $(\hat{x}, \hat{y}) \in X \times Y$, i.e.,
\[
f(x, \hat{y}) \leq f(\hat{x}, \hat{y}) \leq f(\hat{x}, y), \quad \forall (x, y) \in X \times Y.
\]
In particular, we have
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]

**Proof.** By (i), (ii), and Theorem 4.1(b) (where $f \equiv g$ and $\lambda = 0$), there exists $\hat{y} \in Y$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$. Let $g(y, x) = -f(x, y)$ for all $(y, x) \in Y \times X$. Then by (i), (iii) and Theorem 4.1(b), there exists $\hat{x} \in X$ such that $g(y, \hat{x}) \leq 0$ for all $y \in Y$. It follows that
\[
f(x, \hat{y}) \leq 0 \leq f(\hat{x}, y), \quad \forall (x, y) \in X \times Y.
\]
Hence we have
\[
f(x, \hat{y}) \leq f(\hat{x}, \hat{y}) = 0 \leq f(\hat{x}, y), \quad \forall (x, y) \in X \times Y,
\]
and
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]
Since $\inf_{y \in Y} \sup_{x \in X} f(x, y) \geq \sup_{x \in X} \inf_{y \in Y} f(x, y)$ is always true, we obtain
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]


**Theorem 4.5.** Let $(X, \Gamma_1)$ and $(Y, \Gamma_2)$ be two compact $G$-convex spaces and $f : X \times Y \to \mathbb{R} \cup \{\pm \infty\}$ be such that for any $\lambda \in \mathbb{R}$,

(i) $f(x, y)$ is $\lambda$-transfer lower semicontinuous in $y$ and $\lambda$-transfer upper semicontinuous in $x$,

(ii) $f(x, y)$ is $\lambda$-generalized $G$-diagonally quasiconcave in $x$.

Then
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]
In particular, if \( y \mapsto f(x, y) \) is lower semicontinuous on \( Y \) and \( x \mapsto f(x, y) \) is upper semicontinuous on \( X \), then \( f \) has a saddle point \((\hat{x}, \hat{y}) \in X \times Y\).

Proof. Clearly, the inequality

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \geq \inf_{y \in Y} \sup_{x \in X} f(x, y)
\]

is always true. In order to prove the equality holds, it is enough to show that

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]

If it is false, then there exists a \( \lambda \in \mathbb{R} \) such that

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) > \lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]

This shows that for each \((\hat{x}, \hat{y}) \in X \times Y\), there exists \((x, y) \in X \times Y\) such that

\[
f(x, y) > \lambda \quad \text{and} \quad f(\hat{x}, \hat{y}) < \lambda.
\]

(4.1)

Let \((X \times Y, \Gamma_1 \times \Gamma_2)\) be the product \(G\)-convex space of \(G\)-convex spaces \((X, \Gamma_1)\) and \((Y, \Gamma_2)\). Define a function \(g: (X \times Y) \times (X \times Y) \to \mathbb{R}\) by

\[
g((w, z), (x, y)) = \begin{cases} 1, & \text{if } f(w, y) > \lambda \text{ and } f(x, z) < \lambda, \\ 0, & \text{otherwise.} \end{cases}
\]

We claim that \(g((w, z), (x, y))\) is 0-transfer lower semicontinuous in \((x, y)\). Suppose that for each \((x, y) \in X \times Y\), there is a \((w, z) \in X \times Y\) such that \(g((w, z), (x, y)) > 0\). Then we have \(f(w, y) > \lambda\) and \(f(x, z) < \lambda\). Since \(f(x, y)\) is \(\lambda\)-transfer lower semicontinuous in \(y\), there exist a open neighborhood \(N_2(y)\) of \(y\) and a point \(x \in X\) such that

\[
f(x, v) > \lambda, \quad \forall \ v \in N_2(y).
\]

Since \(f(x, y)\) is \(\lambda\)-transfer upper semicontinuous in \(x\), there exist an open neighborhood \(N_1(x)\) of \(x\) and a point \(y' \in Y\) such that

\[
f(u, y') < \lambda, \quad \forall \ u \in N_1(x).
\]

Then \(N(x, y) = N_1(x) \times N_2(y)\) is an open neighborhood of \((x, y)\) in \(X \times Y\) and

\[
g((x', y'), (u, v)) > 0, \quad \forall (u, v) \in N(x, y).
\]

This shows that \(g((w, z), (x, y))\) is 0-transfer semicontinuous in \((x, y)\).

Now we show that \(g((w, z), (x, y))\) is 0-generalized \(G\)-diagonally quasi-concave in \((w, z)\). By (ii), for each \((w_1, z_1), \ldots, (w_n, z_n) \in \mathcal{F}(X \times Y)\), there exist \(y_1, \ldots, y_n \in Y\) such that for each subset \(\{y_{i_1}, \ldots, y_{i_k}\}\) of
Let $F$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, $K$ be a nonempty compact subset of $E$, and $Z$ be a nonempty set. Let $F, G: X \to 2^Z$ be two set-valued mappings such that

(i) for each $x \in X$, the set $\{y \in X: F(y) \cap G(x) \neq \emptyset\}$ is $G$-convex,

(ii) the mapping $T: X \to 2^E$ defined by

$$T(x) = \{y \in X: F(x) \cap G(y) \neq \emptyset\}$$

is transfer compactly closed-valued.

---

THEOREM 4.6. Let $X$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, $K$ be a nonempty compact subset of $E$, and $Z$ be a nonempty set. Let $F, G: X \to 2^Z$ be two set-valued mappings such that

(i) for each $x \in X$, the set $\{y \in X: F(y) \cap G(x) \neq \emptyset\}$ is $G$-convex,

(ii) the mapping $T: X \to 2^E$ defined by

$$T(x) = \{y \in X: F(x) \cap G(y) \neq \emptyset\}$$

is transfer compactly closed-valued.

---

Remark 4.6. Theorem 4.5 improves Theorem 4.4 of Tan [16] by relaxing the continuity assumptions of $f(x, y)$ and, in turn, generalizes Sion’s generalization [15] of von Neumann’s minimax theorem [18].

Finally, as application of Corollary 3.1, we obtain the following coincidence theorem and fixed point theorem.

THEOREM 4.6. Let $X$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, $K$ be a nonempty compact subset of $E$, and $Z$ be a nonempty set. Let $F, G: X \to 2^Z$ be two set-valued mappings such that

(i) for each $x \in X$, the set $\{y \in X: F(y) \cap G(x) \neq \emptyset\}$ is $G$-convex,

(ii) the mapping $T: X \to 2^E$ defined by

$$T(x) = \{y \in X: F(x) \cap G(y) \neq \emptyset\}$$

is transfer compactly closed-valued.
(iii) for each $N \in \mathcal{A}(X \cap D)$, there exists a compact $G$-convex subset $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \notin \text{ccl } T(x)$,

(iv) for each $x \in K$, $F(x) \cap G(x) \neq \emptyset$.

Then there exist $\hat{x} \in K$ such that $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

Proof. The mapping $T : X \to 2^E$ defined in the condition (ii) is transfer compactly closed-valued. If $T$ is $G$-KKM mapping, the condition (iii) implies that the assumption in Corollary 3.1(2) is satisfied. By Corollary 3.1(2), $K \cap (\cap_{t \in X} T(x)) \neq \emptyset$. Taking $\hat{y} \in K \cap (\cap_{t \in X} T(x))$, we have $\hat{y} \in K$ and $F(x) \cap G(\hat{y}) = \emptyset$ for all $x \in X$ which contradicts condition (iv). Therefore $T$ is not a $G$-KKM mapping and so there exists $N \in \mathcal{A}(X \cap D)$ and $\hat{x} \in \Gamma(N)$ such that $\hat{x} \notin \bigcup_{x \in N} T(x)$. It follows that $F(x) \cap G(\hat{x}) \neq \emptyset$ for all $x \in N$ and hence $N \subseteq \{z \in X : F(z) \cap G(\hat{x}) \neq \emptyset\}$. By (i), we have

$$\hat{x} \in \Gamma(N) \subseteq \{z \in X : F(z) \cap G(\hat{x}) \neq \emptyset\},$$

which implies $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$. This completes the proof.

**Remark 4.7.** Theorem 4.6 generalizes Theorem 4.1 of Ding [5] in the following ways: (1) from $H$-space $(X, \{\Gamma_i\})$ to $G$-convex space $(E, D; \Gamma)$; (2) the compactness assumption of each polytope is dropped; (3) $Z$ may be any set without topological structure.

**Corollary 4.2.** Let $X$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, $K$ be a nonempty compact subset of $E$, and $G : X \to 2^E$ be such that

(i) for each $x \in K$, $G(x) \cap X \neq \emptyset$ and each $G(x)$ is $G$-convex,

(ii) for each $y \in X$, $G^{-1}(y)$ is transfer compactly open-valued,

(iii) for each $N \in \mathcal{A}(X \cap D)$, there exists a compact $G$-convex subset $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{cl } G^{-1}(x)$.

Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in G(\hat{x})$.

Proof. Let $Z = E$ and $F$ is the identity mapping on $X$. It is easy to check that conditions (i)–(iii) imply that all conditions of Theorem 4.6 are satisfied. By Theorem 4.6, there exists $\hat{x}$ such that $\hat{x} \in G(\hat{x})$.

**Remark 4.8.** If $X$ is a compact $G$-convex subset of $(E, D; \Gamma)$, then condition (iii) is satisfied trivially by putting $X = K = L_N$ for each $N \in \mathcal{A}(X \cap D)$. Corollary 4.2 generalizes Corollary 4.3 of Ding [5] in several aspects and, in turn, includes Corollaries 1 and 2 and Theorem 2 of Huang [9] as special cases.
REFERENCES