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Penalty methods for the numerical solution of American multi-asset option problems

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Abstract

We derive and analyze a penalty method for solving American multi-asset option problems. A small, non-linear penalty term is added to the Black–Scholes equation. This approach gives a fixed solution domain, removing the free and moving boundary imposed by the early exercise feature of the contract. Explicit, implicit and semi-implicit finite difference schemes are derived, and in the case of independent assets, we prove that the approximate option prices satisfy some basic properties of the American option problem. Several numerical experiments are carried out in order to investigate the performance of the schemes. We give examples indicating that our results are sharp. Finally, the experiments indicate that in the case of correlated underlying assets, the same properties are valid as in the independent case.

MSC: 65M06; 65M12; 65Y20; 91B28

Keywords: American options; Penalty methods; Numerical solution

1. Introduction

American derivatives are popular trading instruments in present-day financial markets. We consider American put options where the payoff depends on more than one underlying asset. Such option prices can be modeled by higher dimensional generalizations of the original Black–Scholes equation [1]. The purpose of this paper is to extend the penalty method discussed in [19] to multi-asset American put option problems.

Various numerical techniques can be applied to price multi-variate derivatives. Higher dimensional generalizations of lattice binomial methods can be used, cf. [2], where European options based on three underlying options are solved numerically. Another way of pricing multi-asset derivatives is by the Monte Carlo simulation techniques, cf. [12]. In a wide range of scientific fields, finite element and finite difference methods (FEM and FDM) are popular. For studies of FEM and FDM concerning the numerical valuation of financial derivatives see, e.g., [11,27,6,26,4,5].

The idea behind the penalty method for multi-asset option models is similar to the method described in [19]. American put options can be exercised at any time before expiry. This introduces a free and moving boundary

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problem. By adding a certain penalty term to the Black–Scholes equation, we extend the solution to a fixed domain. Furthermore, this term forces the solution to stay above the payoff function at expiry. Throughout the last decade, a number of papers addressing penalty schemes for American options have been published, see [10,15,9,17,7] and references therein.

The number of spatial degrees of freedom in the Black–Scholes equation equals the number of underlying assets involved in the contract. This means that the spatial dimension can be of order $O(10)$ (or even higher). Furthermore, as will be explained below, in order to solve an n-dimensional option problem one must typically solve a series of Black–Scholes equations with spatial dimensions $n-1, n-2, \ldots, 1$, leading to a very CPU demanding procedure. Consequently, it is necessary to design efficient numerical schemes for such problems. Stable higher order methods for the Black and Scholes equation have been introduced by Voss et al. [23] and Khaliq et al. [15]. Mesh-free methods based on radial basis functions may also reduce the computational efforts significantly, see Fasshauer et al. [9].

The present study is motivated by the scheme introduced by Forsyth and Vetzal in [28] for American options with stochastic volatility. In their work they add a source term to the discrete equations. Our method represents a refinement of their work in the sense that the penalty term is added to the continuous equation. For independent underlying assets, this leads to restrictions regarding the magnitude of the penalty term as well as conditions for the discretization parameters. Also, by choosing a semi-implicit finite difference discretization, we avoid solving non-linear algebraic equations and thereby enhance the overall computational efficiency.

We present numerical experiments illustrating the properties of the schemes. In the case of correlated underlying assets, we have been unable to derive proper bounds on the numerical solutions. However, numerical experiments indicate that similar properties are present in such cases.

This paper is organized as follows: In Section 2 we describe the multi-asset Black–Scholes equation, along with the penalty formulation of the problem. The boundary conditions corresponding to zero values of the underlying assets are obtained by solving lower dimensional Black–Scholes equations. In Section 4, numerical schemes for the two-factor model problem are derived, starting with specifying the two-factor model problem. First, an explicit scheme is presented, and then both a semi-implicit and a fully implicit scheme are defined. Analysis of these schemes are carried out in Section 4, under the assumption that the underlying assets are independent. Restrictions regarding the time step size and the penalty term are then provided for all three schemes. In the last section of this paper, we present a series of numerical experiments, starting with comparing the fully implicit and the semi-implicit schemes with respect to computational efficiency. In Section 5, we show that the numerical experiments indicate that for our model data, the restrictions derived in Section 4 for independent assets are also valid when the underlying assets are correlated. Finally, we make some conclusive remarks in Section 6.

2. American multi-asset option problems

The multi-dimensional version of the Black–Scholes equation takes the form

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - D_i)S_i \frac{\partial P}{\partial S_i} - rP = 0, \tag{1}$$

see e.g. [8,16] or [24]. Here, P is the value of the contract, S_i is the value of the ith underlying asset, n is the number of underlying assets, $\rho_{i,j}$ is the correlation between asset i and asset j, r is the risk-free interest rate and D_i is the dividend yield paid by the ith asset.

The value of an American option at the time T of expiry of the contract is readily known as a function of the underlying assets. That is, $P(S_1, S_2, \ldots, S_n, T)$ is known and we want to use (1) to compute P throughout the time interval $[0, T]$. This means that $P(S_1, S_2, \ldots, S_n, T)$ provides a final condition and that we seek to solve this PDE backwards in time. The plus sign in front of the second-order term in (1) will thus not cause any stability problems. A precise mathematical formulation of the problem will be presented below.

For a majority of multi-asset option models the payoff function at expiry is

$$P(S_1, S_2, \ldots, P_n, T) = \phi(S_1, \ldots, S_n) = \max\left(E - \sum_{i=1}^{n}\alpha_i S_i, 0\right), \tag{2}$$

where E and $\alpha_1, \dots, \alpha_n$ are given constants, see [16]. We will in this paper consider put options, i.e.

$$E, \alpha_1, \dots, \alpha_n \geq 0.$$

Notice that the American early exercise feature of the contract imposes the constraint

$$P(S_1, \dots, S_n, t) \geq \phi(S_1, \dots, S_n)$$

on the solution for all admissible values of $S_1, \dots, S_n$ and t.

In the case of American options, the solution domain can be divided into two parts. In one region the price of the option satisfies the Black–Scholes equation and in the second subdomain it equals the payoff function ϕ. This leads to the linear complementarity form of the problem. Let $\mathcal{L}$ be the differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - D_i)S_i \frac{\partial}{\partial S_i} - r,$$

and

$$\begin{aligned}
&\Omega = \{(S_1, \dots, S_n);\ S_j > 0 \text{ for } j = 1, \dots, n\} = \mathbb{R}^n_+,\\
&\Omega_i = \{(S_1, \dots, S_{i-1}, 0, S_{i+1}, \dots, S_n);\ S_j \geq 0 \text{ for } j \neq i\},\\
&\mathbf{S} = (S_1, \dots, S_n).
\end{aligned}$$

If T represents the time of expiration of the contract, then the American put problem can be written in the form

$$(P - \phi), \mathcal{L}P = 0 \quad \text{in } \Omega \times [0, T], \tag{3}$$

$$\mathcal{L}P \leq 0 \quad \text{in } \Omega \times [0, T], \tag{4}$$

$$P(\mathbf{S}, t) \geq \phi(\mathbf{S}) \quad \text{in } \Omega \times [0, T], \tag{5}$$

$$P(\mathbf{S}, T) = \phi(\mathbf{S}) \quad \text{for all } \mathbf{S} \in \Omega, \tag{6}$$

$$P(\mathbf{S}, t) = g_i(\mathbf{S}, t) \quad \text{for all } \mathbf{S} \in \Omega_i \times [0, T] \text{ and } i = 1, \dots, n, \tag{7}$$

$$\lim_{S_i \to \infty} P(\mathbf{S}, t) = G_i(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n, t) \quad \text{for all } \mathbf{S} \in \Omega \times [0, T] \text{ and } i = 1, \dots, n, \tag{8}$$

and both P and its first derivatives must be continuous. Here $g_i(\cdot, \cdot)$ and $G_i(\cdot, \cdot)$ are given functions providing suitable boundary conditions. Typically, $g_i(\cdot, \cdot)$ is determined by solving the associated $n - 1$-dimensional American put problem and $G_i(\cdot, \cdot)$ is identical to zero. Further details can be found in Section 5. Until then, we will assume that the boundary conditions are consistent with the constraint imposed by the early exercise feature of the option, i.e. that $g_i(\cdot, \cdot)$ and $G_i(\cdot, \cdot)$ are consistent with the constraint (5).

2.1. A penalty method

Define the function

$$q(S_1, \dots, S_n) = E - \sum_{i=1}^{n} \alpha_i S_i.$$

As for American single-asset option problems, cf. [19], a penalty method for solving (3)–(8) can be defined as follows

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - D_i)S_i \frac{\partial P}{\partial S_i} - rP + \frac{\epsilon C}{P + \epsilon - q} = 0,$$

$$\mathbf{S} \in \Omega, t \in [0, T], \tag{9}$$

$$P(\mathbf{S}, T) = \phi(\mathbf{S}) \quad \text{for all } \mathbf{S} \in \Omega, \tag{10}$$

$$P(\mathbf{S}, t) = g_i(\mathbf{S}, t) \quad \text{for all } \mathbf{S} \in \Omega_i \times [0, T] \text{ and } i = 1, \dots, n, \tag{11}$$

$$\lim_{S_i \to \infty} P(\mathbf{S}, t) = G_i(S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n, t) \quad \text{for all } \mathbf{S} \in \Omega \times [0, T] \text{ and } i = 1, \dots, n, \tag{12}$$

where $0 < \epsilon \ll 1$ is a small parameter and C is a positive constant. Note that the penalty term

$$\frac{\epsilon C}{P + \epsilon - q}$$

is of order ϵ in regions where $P(\mathbf{S}, t) \gg q(\mathbf{S})$, and hence the Black–Scholes equation is approximately satisfied. On the other hand, as P approaches q this term is approximately equal to C assuring that the early exercise constraint (5) is not violated. In Section 4 we will prove that for a two-factor problem, with independent assets, a discrete analogue to (5) holds provided that $C \geq rE$.

3. Discretization

For the sake of simplicity we will define and analyze our numerical schemes for a two-factor model problem and use x and y, instead of the more conventional notation S_1 and S_2, to represent the asset prices. In principle, the numerical methods and analysis presented in this paper can be extended to general n-dimensional American option problems, provided that the payoff function at expiry is of the form (2). Nevertheless, the number of spatial dimensions in the modified Black–Scholes equation (9) equals n, and hence the number of degrees of freedom needed to discretize this equation grows rapidly with n. Furthermore, in order to determine proper boundary conditions, this procedure requires that a series of similar PDEs with spatial dimensions $n-1, n-2, \ldots, 1$ are solved. Appendix A contains a complete description of the three-factor case, i.e. $n = 3$.

For $n \geq 3$, the task of numerically determining the fair price of the option with our algorithms thus becomes CPU demanding. This raises the question whether suitable higher order schemes can be designed, see Voss et al. [23] and Khaliq et al. [15].

3.1. A two-factor model problem

We will consider the following penalty formulation of an American put problem with two underlying assets, i.e. $n = 2$,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 P}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 P}{\partial x \partial y}$$
$$+ (r - D_1)x\frac{\partial P}{\partial x} + (r - D_2)y\frac{\partial P}{\partial y} - rP + \frac{\epsilon C}{P + \epsilon - q} = 0, \quad x, y > 0, \ t \in [0, T), \tag{13}$$

$$P(x, y, T) = \phi(x, y), \quad x, y \geq 0, \tag{14}$$

$$P(x, 0, t) = g_1(x, t), \quad x \geq 0, \ t \in [0, T], \tag{15}$$

$$P(0, y, t) = g_2(y, t), \quad y \geq 0, \ t \in [0, T], \tag{16}$$

$$\lim_{x\to\infty} P(x, y, t) = G_1(y, t), \quad y \geq 0, \ t \in [0, T], \tag{17}$$

$$\lim_{y\to\infty} P(x, y, t) = G_2(x, t), \quad x \geq 0, \ t \in [0, T], \tag{18}$$

where

$$q(x, y) = E - (\alpha_1 x + \alpha_2 y), \qquad \phi(x, y) = \max(q(x, y), 0). \tag{19}$$

Recall that we want to solve this problem backwards in time starting with the final condition (14).

Let, for given positive integers I, J and N,

$$\Delta x = \frac{x_\infty}{I+1}, \qquad \Delta y = \frac{y_\infty}{J+1}, \qquad \Delta t = \frac{T}{N+1}, \tag{20}$$

$$x_i = i\Delta x, \quad i = 0, \ldots, I+1, \tag{21}$$

$$y_j = j\Delta y, \quad j = 0, \ldots, J+1, \tag{22}$$

$$t_n = n\Delta t, \quad n = 0, \ldots, N+1, \tag{23}$$

$$q_{i,j} = q(x_i, y_j), \quad i = 0, \ldots, I+1 \text{ and } j = 0, \ldots, J+1, \tag{24}$$

$$P_{i,j}^n \approx P(x_i, y_j, t_n). \tag{25}$$

Here x_∞ and y_∞ are the upper boundaries of the truncated solution domain. Throughout this paper we will assume that $\Delta x = \Delta y = h$.

The discrete final condition and boundary conditions are defined in a straightforward manner

$$P_{i,j}^{N+1} = \max(q_{i,j}, 0), \quad i = 0, \dots, I+1 \text{ and } j = 0, \dots, J+1, \tag{26}$$

$$P_{i,0}^{n} = (g_1)_i^n, \quad i = 0, \dots, I+1 \text{ and } n = 0, \dots, N+1, \tag{27}$$

$$P_{0,j}^{n} = (g_2)_j^n, \quad j = 0, \dots, J+1 \text{ and } n = 0, \dots, N+1, \tag{28}$$

$$P_{i,J+1}^{n} = (G_1)_i^n, \quad i = 0, \dots, I+1 \text{ and } n = 0, \dots, N+1, \tag{29}$$

$$P_{I+1,j}^{n} = (G_2)_j^n, \quad j = 0, \dots, J+1 \text{ and } n = 0, \dots, N+1. \tag{30}$$

Here $(g_1)_i^n$, $(g_2)_j^n$, $(G_1)_i^n$, $(G_2)_j^n$ are discrete approximations of $g_1(x_i, t_n)$, $g_2(y_j, t_n)$, $G_1(x_i, t_n)$, $G_2(y_j, t_n)$, respectively. We let $(G_1)_i^n = (G_2)_j^n = 0$, whereas $(g_1)_i^n$ and $(g_2)_j^n$ are obtained by solving the corresponding 1D Black–Scholes equations.

In order to simplify the notation needed in this paper we introduce the finite difference operators

$$\mathcal{D}_{xx} Q_{i,j}^n = \frac{Q_{i+1,j}^n - 2Q_{i,j}^n + Q_{i-1,j}^n}{h^2}, \qquad \mathcal{D}_{yy} Q_{i,j}^n = \frac{Q_{i,j+1}^n - 2Q_{i,j}^n + Q_{i,j-1}^n}{h^2}, \tag{31}$$

$$\mathcal{D}_{xy} Q_{i,j}^n = \frac{Q_{i+1,j+1}^n - Q_{i,j+1}^n - Q_{i+1,j}^n + 2Q_{i,j}^n - Q_{i-1,j}^n - Q_{i,j-1}^n + Q_{i-1,j-1}^n}{2h^2}, \tag{32}$$

$$\mathcal{D}_{x} Q_{i,j}^n = \frac{Q_{i+1,j}^n - Q_{i,j}^n}{h}, \qquad \mathcal{D}_{y} Q_{i,j}^n = \frac{Q_{i,j+1}^n - Q_{i,j}^n}{h}, \tag{33}$$

$$\mathcal{D}_{t} Q_{i,j}^n = \frac{Q_{i,j}^n - Q_{i,j}^{n-1}}{\Delta t}, \tag{34}$$

where $\{Q_{i,j}^n\}_{i,j=0}^{I+1,J+1}$, for $n = 0, \dots, N+1$, is a discrete function defined on the mesh defined in Eqs. (20)–(23). Since we use upwind differences in (33), and a first-order approximation of the time derivative in (34), the truncation error of the resulting scheme is $\mathcal{O}(h, \Delta t)$. Throughout this paper we will assume[1] that

$$r \geq D_1, D_2,$$

and, hence we use an upwind differencing to discretize the transport terms in (13), cf. (33).

3.2. An explicit scheme

Assume that we know the solution at time step n, and we wish to compute P^{n-1}. Applying the space and time finite difference operators at time step n, the explicit scheme reads

$$\begin{aligned}
&\mathcal{D}_t P_{i,j}^n + \frac{1}{2}\sigma_1^2 x_i^2 \mathcal{D}_{xx} P_{i,j}^n + \frac{1}{2}\sigma_2^2 y_j^2 \mathcal{D}_{yy} P_{i,j}^n + \rho\sigma_1\sigma_2 x_i y_j \mathcal{D}_{xy} P_{i,j}^n \\
&\quad + (r - D_1) x_i \mathcal{D}_x P_{i,j}^n + (r - D_2) y_j \mathcal{D}_y P_{i,j}^n - r P_{i,j}^n + \frac{\epsilon C}{P_{i,j}^n + \epsilon - q_{i,j}} = 0,
\end{aligned}$$

for $i = 1, \dots, I$, $j = 1, \dots, J$ and $n = N+1, N, \dots, 1$. The final condition and boundary conditions are defined in (26)–(30).

Defining

$$\begin{aligned}
F(V_1, V_2, V_3, V_4, V_5, V_6, V_7, q, x, y) &= e(x,y)V_1 + [b(y) - e(x,y)]V_2 + [a(x) - e(x,y)]V_3 \\
&\quad + [1 - 2a(x) - 2b(y) + 2e(x,y) - c(x) - d(y) - r\Delta t]V_4 \\
&\quad + [a(x) - e(x,y) + c(x)]V_5 + [b(y) - e(x,y) + d(y)]V_6 + e(x,y)V_7 + \frac{\epsilon C \Delta t}{V_4 + \epsilon - q},
\end{aligned} \tag{35}$$

[1] If $r \leq D_1$, or $r \leq D_2$ we preserve upwind differencing by replacing (33) with the proper finite difference operator.

where

$$a(x) = \frac{1}{2}\frac{\Delta t}{h^2}\sigma_1^2 x^2, \qquad b(y) = \frac{1}{2}\frac{\Delta t}{h^2}\sigma_2^2 y^2, \qquad c(x) = (r - D_1)\frac{\Delta t}{h}x,$$
$$d(y) = (r - D_2)\frac{\Delta t}{h}y, \qquad e(x, y) = \frac{1}{2}\frac{\Delta t}{h^2}\rho\sigma_1\sigma_2 xy, \tag{36}$$

this scheme can be written in the form

$$P_{i,j}^{n-1} = F(P_{i-1,j-1}^n, P_{i,j-1}^n, P_{i-1,j}^n, P_{i,j}^n, P_{i+1,j}^n, P_{i,j+1}^n, P_{i+1,j+1}^n, q_{i,j}, x_i, y_j). \tag{37}$$

3.3. Semi-implicit and fully implicit schemes

The implicit and semi-implicit methods are obtained by applying the spatial finite difference operators at time step n and the time difference at time step $n + 1$,

$$\mathcal{D}_t P_{i,j}^{n+1} + \frac{1}{2}\sigma_1^2 x_i^2 \mathcal{D}_{xx} P_{i,j}^n + \frac{1}{2}\sigma_2^2 y_j^2 \mathcal{D}_{yy} P_{i,j}^n + \rho\sigma_1\sigma_2 x_i y_j \mathcal{D}_{xy} P_{i,j}^n$$
$$+ (r - D_1)x_i \mathcal{D}_x P_{i,j}^n + (r - D_2)y_j \mathcal{D}_y P_{i,j}^n - r P_{i,j}^n + \frac{\epsilon C}{P_{i,j}^{n+1/2} + \epsilon - q_{i,j}} = 0, \tag{38}$$

for $i = 1, \ldots, I$, $j = 1, \ldots, J$ and $n = N, N - 1, \ldots, 0$, where we define $P_{i,j}^{n+1/2} = P_{i,j}^{n+1}$ in the semi-implicit scheme and $P_{i,j}^{n+1/2} = P_{i,j}^n$ in the fully implicit method. As for the explicit scheme, the final condition and boundary conditions are defined in equations Eqs. (26)–(30).

Some simple algebraic manipulations show that this scheme can be written in the form

$$e_{i,j} P_{i-1,j-1}^n + [b_j - e_{i,j}]P_{i,j-1}^n + [a_i - e_{i,j}]P_{i-1,j}^n - [1 + 2a_i + 2b_j - 2e_{i,j} + c_i + d_j + r\Delta t]P_{i,j}^n$$
$$+ [a_i - e_{i,j} + c_i]P_{i+1,j}^n + [b_j - e_{i,j} + d_j]P_{i,j+1}^n + e_{i,j} P_{i+1,j+1}^n = -P_{i,j}^{n+1} - \frac{\epsilon C\Delta t}{P_{i,j}^{n+1/2} + \epsilon - q_{i,j}}, \tag{39}$$

where

$$a_i = a(x_i), \qquad b_j = b(y_j), \qquad c_i = c(x_i),$$
$$d_j = d(y_j), \qquad e_{i,j} = e(x_i, y_j).$$

Note that the semi-implicit scheme $P_{i,j}^{n+1/2} = P_{i,j}^{n+1}$ gives a system of linear algebraic equations, whereas the fully implicit scheme $P_{i,j}^{n+1/2} = P_{i,j}^n$ leads to a system of non-linear equations.

4. Analysis in the case of independent assets

In this section we will prove that our schemes satisfy the early exercise constraint. Our analysis will only cover the case of independent assets, i.e. we will assume that

$$\rho = 0$$

throughout this section. Unfortunately we have not been able to derive similar results in the correlated case. However, such problems will be addressed by numerical experiments in Section 5.

4.1. Analysis of the explicit scheme

Theorem 1. *Assume that $\rho = 0$ and that $C \geq rE$. Then the approximate option values generated by the explicit scheme (37) satisfy*

$$P_{i,j}^n \geq \max(q(x_i, y_j), 0), \quad i = 0, \ldots, I + 1, \; j = 0, \ldots, J + 1 \tag{40}$$

and $n = N+1, N, \ldots, 0$, *provided that*

$$\Delta t \le \frac{h^2}{\sigma_1 x_\infty^2 + \sigma_2 y_\infty^2 + (r - D_1)hx_\infty + (r - D_2)hy_\infty + rh^2 + \frac{C}{\epsilon}h^2}. \tag{41}$$

Proof. In the case of independent assets the function F, defined in Eq. (37), takes the form

$$\begin{aligned} F(V_1, V_2, V_3, V_4, V_5, V_6, V_7, q, x, y) &= b(y)V_2 + a(x)V_3 + [1 - 2a(x) - 2b(y) - c(x) - d(y) - r\Delta t]V_4 \\ &\quad + [a(x) + c(x)]V_5 + [b(y) + d(y)]V_6 + \frac{\epsilon C \Delta t}{V_4 + \epsilon - q}, \end{aligned} \tag{42}$$

i.e. $e(x, y) = 0$ for all $x, y \ge 0$, see (36). Clearly, for all $x, y \ge 0$

$$\frac{\partial F}{\partial V_2}, \frac{\partial F}{\partial V_3}, \frac{\partial F}{\partial V_5}, \frac{\partial F}{\partial V_6} \ge 0, \tag{43}$$

and for $V_4 \ge q$

$$\frac{\partial F}{\partial V_4} \ge 0, \tag{44}$$

provided that Δt satisfies (41).

Assume that inequality (40) holds at time step t_n. From the definition (37) of our scheme, and inequalities (43) and (44), we find that

$$\begin{aligned} P_{i,j}^{n-1} &= F(0, P_{i,j-1}^n, P_{i-1,j}^n, P_{i,j}^n, P_{i+1,j}^n, P_{i,j+1}^n, 0, q_{i,j}, x_i, y_j) \\ &\ge F(0, q_{i,j-1}, q_{i-1,j}, q_{i,j}, q_{i+1,j}, q_{i,j+1}, 0, q_{i,j}, x_i, y_j). \end{aligned} \tag{45}$$

Recall the definition (19) of the payoff function q at time $t = T$ of the basket option. Thus,

$$\begin{aligned} q_{i,j-1} &= q_{i,j} + \alpha_2 h, \qquad & q_{i-1,j} &= q_{i,j} + \alpha_1 h, \\ q_{i+1,j} &= q_{i,j} - \alpha_1 h, \qquad & q_{i,j+1} &= q_{i,j} - \alpha_2 h, \end{aligned}$$

and consequently

$$\begin{aligned} P_{i,j}^{n-1} &\ge b_j \alpha_2 h + a_i \alpha_1 h + q_{i,j} - r\Delta t q_{i,j} - [a_i + c_i]\alpha_1 h - [b_j + d_j]\alpha_2 h + \frac{\epsilon C \Delta t}{q_{i,j} + \epsilon - q_{i,j}} \\ &= q_{i,j} - r\Delta t q_{i,j} - (r - D_1)\frac{\Delta t}{h} x_i \alpha_1 h - (r - D_2)\frac{\Delta t}{h} y_j \alpha_2 h + C\Delta t \\ &= q_{i,j} - r\Delta t q_{i,j} - r\Delta t(x_i \alpha_1 + y_j \alpha_2) + D_1 \Delta t x_i \alpha_1 + D_2 \Delta t y_j \alpha_2 + C\Delta t \\ &\ge q_{i,j} - r\Delta t q_{i,j} - r\Delta t(E - q_{i,j}) + C\Delta t, \end{aligned}$$

where we have used the definition (19) of q. Therefore, if $C \ge rE$ then

$$P_{i,j}^{n-1} \ge q_{i,j} + (C - rE)\Delta t \ge q_{i,j}.$$

Furthermore, from Eq. (42) and (45) and the assumption that $P_{i,j}^n$ satisfies (40), i.e. $P_{i,j}^n \ge 0$ and $P_{i,j}^n \ge q_{i,j}$, we find that

$$P_{i,j}^{n-1} \ge 0,$$

and, hence the desired result follows by induction. ■

4.2. Analysis of the semi-implicit and fully implicit schemes

Theorem 2. *For every* $C \ge rE$ *the approximate option prices* $\{P_{i,j}^n\}$ *defined by the fully implicit scheme* (38) *satisfy*

$$P_{i,j}^n \ge \max(q(x_i, y_j), 0), \quad i = 0, \ldots, I+1,\ j = 0, \ldots, J+1, \tag{46}$$

and $n = N+1, N, \ldots, 0$. Similarly, if $C \geq rE$, and in addition

$$\Delta t \leq \frac{\epsilon}{rE}, \tag{47}$$

the numerical option prices generated by the semi-implicit version of (38) satisfy the lower bound (46).

Proof. In a straightforward manner it follows that the difference

$$u_{i,j}^n = P_{i,j}^n - q_{i,j}$$

between the approximate option value $P_{i,j}^n$ and q, used in the payoff function at expiry (19) and (24), satisfies the equation

$$\begin{aligned}[1 + 2a_i + 2b_j + c_i + d_j + r\Delta t]u_{i,j}^n &= u_{i,j}^{n+1} + b_j u_{i,j-1}^n + a_i u_{i-1,j}^n + [a_i + c_i]u_{i+1,j}^n \\ &\quad + [b_j + d_j]\, u_{i,j+1}^n + \frac{\epsilon C \Delta t}{u_{i,j}^{n+1/2} + \epsilon - q_{i,j}} - r\Delta t E,\end{aligned}$$

cf. Eq. (39) (and recall that $\rho = 0$, i.e. $e(x, y) = 0$ for all $x, y \geq= 0$). Next, by defining

$$u^n = \min_{i,j} u_{i,j}^n$$

it follows that

$$\begin{aligned}[1 + 2a_i + 2b_j + c_i + d_j + r\Delta t]u^n &\geq u_{k,l}^{n+1} + b_j u^n + a_i u^n + [a_i + c_i]u^n \\ &\quad + [b_j + d_j]u^n + \frac{\epsilon C \Delta t}{u_{k,l}^{n+1/2} + \epsilon - q_{i,j}} - r\Delta t E,\end{aligned}$$

where k and l are indices such that $u_{k,l}^n = u^n$. Hence, we conclude that

$$[1 + r\Delta t]u^n \geq u_{k,l}^{n+1} + \frac{\epsilon C \Delta t}{u_{k,l}^{n+1/2} + \epsilon - q_{i,j}} - r\Delta t E. \tag{48}$$

Having established inequality (48) the result follows exactly as for the single-asset option problem analyzed in [19]. The rest of the proof is therefore omitted. ∎

5. Numerical experiments

We will now present a number of examples concerning options depending on two or three assets. In the derivation and analysis of the schemes above we only assumed that the boundary conditions fulfilled the constraint (5), imposed by the early exercise feature of the contract. Clearly, in order to perform numerical experiments we need to fully specify these boundary conditions. Since we are considering put options the contract gets worthless as the price of either of the assets tends to infinity, i.e. for a two-factor problem

$$\begin{aligned}&G_1(y, t) = 0, \quad y \geq 0,\ t \in [0, T],\\ &G_2(x, t) = 0, \quad x \geq 0,\ t \in [0, T],\end{aligned}$$

see Eqs. (17) and (18). Next, it follows from the lognormal distribution model of the assets, cf. e.g. [25], that if one of the assets is zero at time t^* then the asset will be worthless at any time $t \geq t^*$. Hence, we conclude that g_1 and g_2, in Eqs. (15) and (16), are the solutions of the associated single-asset American put problems,

$$\frac{\partial g_1}{\partial t} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 g_1}{\partial x^2} + (r - D_1)x\frac{\partial g_1}{\partial x} - rg_1 = 0 \quad \text{for } x > \bar{x}(t) \text{ and } 0 \leq t < T, \tag{49}$$

$$g_1(x, T) = \max(E - \alpha_1 x, 0) \quad \text{for } x \geq 0, \tag{50}$$

$$\frac{\partial g_1}{\partial x}(\bar{x}(t), t) = -\alpha_1, \tag{51}$$

$$g_1(\bar{x}(t), t) = E - \alpha_1 \bar{x}(t), \tag{52}$$

$$\lim_{x\to\infty} g_1(x,t) = 0, \tag{53}$$

$$\bar{x}(T) = E/\alpha_1, \tag{54}$$

$$g_1(x,t) = E - \alpha_1 x \quad \text{for } 0 \le x < \bar{x}(t), \tag{55}$$

and a similar problem for g_2. Here $\bar{x}(t)$ represents the free (and moving) boundary, see e.g. [8,16] or [25]. (Similar properties are valid in the case of basket options depending on $n = 3$ assets, see Appendix for a discussion).

In the experiments below we applied the penalty method, derived for single-asset problems in [19], to compute an approximate solution of (49)–(55), i.e. to compute $(g_1)_i^n$ and $(g_2)_j^n$ in Eqs. (27) and (28).

The following model parameters were used in the experiments

$$r = 0.1,$$
$$E = 1.0,$$
$$T = 1.0.$$

Furthermore, for $n = 2$ we applied

$$\sigma_1 = 0.2, \quad \sigma_2 = 0.3,$$
$$\alpha_1 = 0.6, \quad \alpha_2 = 0.4,$$
$$D_1 = 0.05, \quad D_2 = 0.01,$$

and for $n = 3$

$$\sigma_1 = 0.2, \quad \sigma_2 = 0.3, \quad \sigma_3 = 0.2,$$
$$\alpha_1 = 0.3, \quad \alpha_2 = 0.3, \quad \alpha_3 = 0.4,$$
$$D_1 = 0.05, \quad D_2 = 0.01, \quad D_3 = 0.01.$$

The correlation parameters $\rho_{1,2}, \rho_{1,3}, \rho_{2,3}$ were zero in the independent cases. For options depending on two correlated assets a number of tests were performed with different correlation parameters, i.e. with $\rho = \rho_{1,2} = 0.05, 0.1, 0.15, \ldots, 0.95, 1.0$. Finally, $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.5$ in the correlated 3D example presented below.

In order to perform simulations, we must choose an upper limit for the solution domain, that is a domain where option values outside are regarded worthless. If not stated otherwise, we used $x_\infty = y_\infty = 4$ and $x_\infty = y_\infty = z_\infty = 4$ for two and three underlying assets, respectively.[2] The role of the size of the computational domain will be addressed in Section 5.2.

Numerical results for the fully implicit scheme are not provided, based on the lack of efficiency of the non-linear scheme examined in [19]. The implementation of the finite difference schemes was done within the PyCC framework [20].

In order to illustrate the properties stated in Theorems 1 and 2, we computed the difference between the numerical solutions and the early exercise constraint. That is, for a number of different values of ϵ, we computed

$$\phi = \min_{i,j,n}(P_{i,j}^n - \max(q_{i,j}, 0)) \tag{56}$$

for the 2D problems and

$$\phi = \min_{i,j,k,n}(P_{i,j,k}^n - \max(q_{i,j,k}, 0)) \tag{57}$$

for the three-factor cases (k is the index used in the third "asset dimension").

5.1. Independent assets

We first compared the explicit and semi-implicit schemes with respect to efficiency, i.e. we compared the CPU time for a given spatial resolution, choosing time step sizes according to (41) for the explicit scheme and (47) for the semi-implicit scheme.

[2] For options depending on three assets the computational domain was $[0, x_\infty] \times [0, y_\infty] \times [0, z_\infty]$.

Table 1
CPU time comparison of the explicit and semi-implicit schemes for a two-factor problem

h	N	ϵ	Explicit		Semi-implicit	
			CPU time (s)	Δt	CPU time (s)	Δt
0.1	1 681	0.01	0.4 s	1.2×10^{-3}	0.07	0.1
0.05	6 561	0.01	2.87 s	3.1×10^{-4}	0.43	0.1
0.01	160 801	0.01	$4.76 \times 10^3 s$	1.2×10^{-5}	83	0.1
0.1	1 681	0.001	0.47 s	1.1×10^{-3}	0.24	0.01
0.05	6 561	0.001	2.83 s	3.0×10^{-4}	0.86	0.01
0.01	160 801	0.001	$4.65 \times 10^3 s$	1.2×10^{-5}	123	0.01

Note that Δt satisfies the bounds given in (41) and (47). We used a uniform mesh size h with a total number of unknowns $N = I \cdot I$ in space, where I is the number of nodes in each spatial dimension. In all experiments $\phi = 0$, cf. (56).

Table 2
CPU times for the semi-implicit scheme applied to a three-factor model problem with uncorrelated assets

h	N	ϵ	Δt	CPU time (s)
0.1	59 319	0.01	0.1	7.3
0.05	493 039	0.01	0.1	127.4
0.1	59 319	0.001	0.01	14.6
0.05	493 039	0.001	0.01	263.5

We used a uniform mesh size h with a total number of unknowns $N = I^3$ in space, where I is the number of nodes in each spatial dimension. In all experiments $\phi = 0$, cf. (57).

The linear system of algebraic equations in the semi-implicit case was solved with the stable bi-conjugate gradient method (called Bi-CGSTAB), cf. e.g. [3,13,22]. We used a relative residual convergence criterion for the iterative solver, i.e. the iteration process was stopped as soon as $\|r_k\|/\|r_0\| \leq 10^{-4}$, where r_k represents the residual vector after k iterations.

The results obtained with two underlying assets are given in Table 1. We observe that the severe restrictions on the time step size in the explicit case make this scheme slow for fine grained meshes. On the other hand, we experienced a fast convergence of the iterative solver used in the semi-implicit case. Together with the mild restriction on the time step size, the latter method is thus the most attractive as the mesh is refined.

It is somewhat surprising that the CPU times needed by the semi-implicit scheme with $\Delta t = 0.01$ are only roughly twice as long as those required with $\Delta t = 0.1$. This is due to the fact that with $\Delta t = 0.01$ a very good initial guess for the iterative solver was available at each time step.

In Section 4 we showed that if certain conditions on the time step size and the penalty function are satisfied, then the early exercise constraint is fulfilled in a discrete sense. We wanted to test the sharpness of the properties expressed in Theorems 1 and 2 by violating these restrictions, looking for negative values of ϕ.

We considered this issue for basket options defined in terms of two independent assets. First the time step Δt was increased by 15% for the explicit scheme. When $\epsilon = 0.01$, $\phi = -4.8 \times 10^{49}$ which clearly violates (40).

We also broke the milder restriction for the semi-implicit scheme by choosing $\Delta t = 10^{-2}$ and $\epsilon = 10^{-4}$. Again, we experienced negative values of ϕ, i.e. $\phi = -9.7 \times 10^{-2}$.

Finally we subtracted 10% from the constant C in the penalty term, i.e. we chose $C = 0.9 \cdot rE$, in the semi-implicit case. We used $\epsilon = 10^{-2}$, $\Delta t = 10^{-2}$ and $h = 0.1$ in this experiment. The penalty term was thus weaker, exerting less force on the solution as it approached the payoff function at expiry. We obtained $\phi = -6.7 \times 10^{-4}$, hence we were not able to keep the solution in the proper state space.

The results obtained with the semi-implicit scheme for options depending on three independent assets are shown in Table 2. Compared with the 2D cases reported above, this problem is significantly more CPU demanding. Consequently, the computational load increases rapidly with the number n of “asset dimensions”. In fact, from a practical point of view, for $n \geq 4$ the techniques presented in this paper are CPU intensive.

Table 3
The relative deviation between the numerical approximations generated on the largest solution domain and the smaller domains

i	1	2	3
d_i (%)	0.29	0.12	0.09

In these experiments we used the mesh parameters $h = 0.05$ and $\Delta t = 0.1$.

Table 4
CPU times for three-factor semi-implicit simulations with correlation coefficients $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.5$

h	N	ϵ	Δt	CPU time (s)
0.1	59 319	0.01	0.1	6.0
0.05	493 039	0.01	0.1	101.0
0.1	59 319	0.001	0.01	13.1
0.05	493 039	0.001	0.01	218.4

We used a uniform mesh size h with a total number of unknowns $N = I^3$ in space, where I is the number of nodes in each spatial dimension. In all experiments $\phi = 0$, cf. (57).

5.2. Varying the size of the computational domain

The importance of the size of the computational domain was also investigated. More precisely, the two-factor model problem defined above, with independent assets, was solved on a number of domains $\Omega_i = \Omega_{x^i_\infty, y^i_\infty}$ for $i = 1, 2, 3, 4$, where

$$\begin{aligned} x^1_\infty &= y^1_\infty = 4, \\ x^2_\infty &= y^2_\infty = 6, \\ x^3_\infty &= y^3_\infty = 8, \\ x^4_\infty &= y^4_\infty = 10. \end{aligned}$$

Let P_i denote the numerical approximation of the solution P of (13)–(18) generated by the semi-implicit scheme on Ω_i for $i = 1, 2, 3, 4$. Table 3 contains the relative difference d_i between P_i and P_4 at time $t = 0$, measured in the L^2-norm, for $i = 1, 2, 3$:

$$d_i = \frac{\|P_4(\cdot, 0) - P_i(\cdot, 0)\|_{L^2(\Omega_i)}}{\|P_4(\cdot, 0)\|_{L^2(\Omega_i)}}.$$

These results indicate that the American option problem can be solved on a rather small domain. The size of the computational domain needed will of course depend on the involved parameters. For example, it increases as α_1 and α_2 decrease and E increases, cf. the definition (19) of the payoff function at expiry ϕ. This issue has been analyzed in a number of papers, see [14,18,17,21].

5.3. Correlated assets

The results presented in Section 4 were obtained for independent underlying assets. A number of numerical experiments were undertaken for correlated assets as well. They indicate that the early exercise constraint still holds, even though we are beyond the scope of our theoretical investigations. More precisely, for $n = 2$ we chose different values for the correlation parameter $\rho = \rho_{1,2}$ between assets S_1 and S_2; $\rho = 0.05, 0.1, 0.15, \ldots, 0.95, 1.0$. The experiments given in Table 1 were carried out with the new correlation parameter settings, and in all cases $\phi = 0$. Thus, the early exercise constraint was fulfilled for both schemes. A plot of the numerical solution computed with the semi-implicit scheme at time $t = 0$ with $\rho = 0.5$ is shown in Fig. 1.

Table 4 contains the results obtained for a basket option depending on $n = 3$ correlated underlying assets with $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.5$. The comments regarding the independent 3D cases presented above are valid for this example as well; the CPU power needed to solve the problem increases rapidly with n, and for $n \geq 4$ the present PDE methods are CPU demanding.

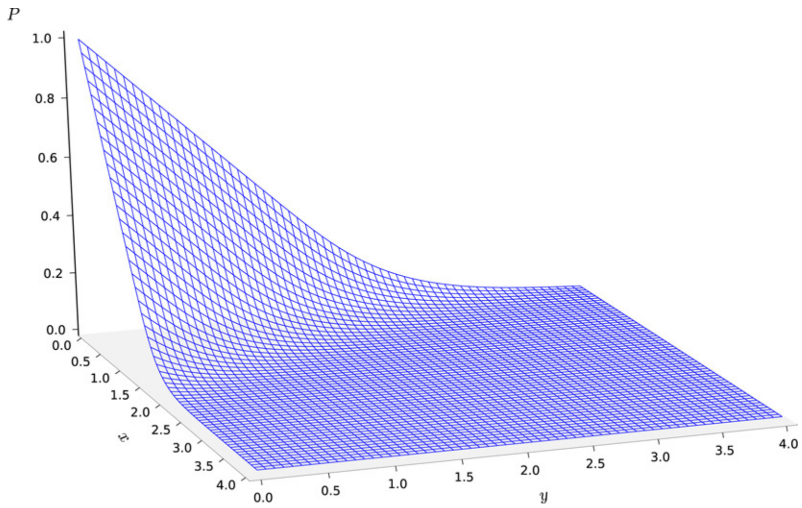

Fig. 1. A plot of the solution of the two-factor model problem computed with the semi-implicit scheme at time $t = 0$ with correlation $\rho = 0.5$. We used $\epsilon = 10^{-2}$, $x_\infty = y_\infty = 4.0$ and $h = \Delta t = 10^{-1}$.

In this paper we have focused on the CPU time required by our methods. An equally important matter is the accuracy of the proposed penalty schemes. Do they converge towards the correct solution as $\epsilon, h, \Delta t \to 0$? If so, at what speed? These are delicate and difficult issues beyond the scope of the present text. Please see Forsyth and Vetzal [10] for further information.

Remark

In the case of correlated underlying assets we can construct a final condition that satisfies the early exercise constraint, but leads to a solution violating this constraint at the first time step. To see this, we consider a two-factor problem with parameters

$$\sigma_1 = \sigma_2 = \sigma,$$
$$E = 1,$$
$$\alpha_1 = \alpha_2 = 1/2,$$
$$\rho = 1/2.$$

Instead of using a final condition of the form (14), we study the problem obtained by defining

$$P(x, y, T) = \bar{P}(x, y) = \begin{cases} p^* & \text{for } (x, y) = (2, 6), \\ \phi(x, y) & (x, y) \neq (2, 6), \end{cases} \tag{58}$$

where ϕ is the function given in (19) and p^* is a positive constant. Note that $\bar{P}$ satisfies the early exercise constraint (5).

Assume that $x_\infty > 2$, $y_\infty > 6$, and let i^* and j^* be indices such that $(x_{i^*-1}, y_{j^*}) = (2, 6)$. Recall the form (37) of the explicit scheme. If $h \leq 1/2$ then

$$\begin{aligned} P_{i^*,j^*}^{n-1} &= F(0, 0, \bar{P}(x_{i^*-1}, y_{j^*}n), 0, 0, 0, 0, q_{i^*,j^*}, x_{i^*}, y_{j^*}) \\ &= F(0, 0, p^*, 0, 0, 0, 0, q_{i^*,j^*}, x_{i^*}, y_{j^*}) \\ &\leq -\frac{5}{8}\frac{\Delta t}{h^2}\sigma^2 p^* + \frac{\epsilon C \Delta t}{\epsilon + 3}. \end{aligned}$$

Hence,

$$P_{i^*,j^*}^{n-1} < 0$$

provided that

$$p^* > \frac{8h^2\epsilon C}{5\sigma^2(\epsilon+3)}. \tag{59}$$

Furthermore, if $h \ll 1$ then (59) is satisfied for rather small values of p^* and in such cases $\bar{P} \approx \phi$.

As far as we know, final conditions of the form (58) do not appear in mathematical finance. However, the example discussed above shows that our methods must be used with care for correlated assets: For each type of contracts the schemes must be tested thoroughly. In all our tests with the final condition (14), which is financially meaningful, our algorithms worked well and the early exercise constraint (5) was never violated.

6. Conclusion

We have presented a penalty method for solving multi-asset American put option problems. Explicit, semi-implicit and a fully implicit finite difference schemes utilizing a penalty term have been derived. For independent underlying assets, conditions on the discretization parameters and the penalty term have been established that assure that the numerical solution satisfies the constraint arising from the early exercise feature of the contract.

We have carried out several numerical experiments for the explicit and semi-implicit schemes. We prefer the semi-implicit scheme to the explicit one for fine grained meshes due to the computational efficiency of the semi-implicit scheme. The experiments indicate that the constraints derived in Section 4 are sharp. In the case of correlated underlying assets we have not achieved similar theoretical results. However, the experiments indicate that for our model parameters, the solutions of the explicit and semi-implicit schemes satisfy the early exercise constraint. Finally, we presented an example of a final condition that leads to the violation of the early exercise constraint for the explicit scheme in the correlated case.

The theoretical investigations presented in this paper address options depending on two underlying assets. This analysis can be extended in a rather straightforward manner to problems involving $n \geq 3$ assets. However, for such high-dimensional Black–Scholes PDEs our algorithms are CPU demanding and require the use of sophisticated numerical tools. Further information about this issue can be found in [23,15,10,9].

Appendix. The three-factor problem

Our penalty approach applied to the American put problem with three underlying assets leads to the following equations:

$$\begin{aligned}
&\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 P}{\partial y^2} + \frac{1}{2}\sigma_3^2 z^2 \frac{\partial^2 P}{\partial z^2} + \rho_{1,2}\sigma_1\sigma_2 xy \frac{\partial^2 P}{\partial x \partial y} + \rho_{1,3}\sigma_1\sigma_3 xz \frac{\partial^2 P}{\partial x \partial z} + \rho_{2,3}\sigma_2\sigma_3 yz \frac{\partial^2 P}{\partial y \partial z} \\
&\qquad + (r - D_1)x\frac{\partial P}{\partial x} + (r - D_2)y\frac{\partial P}{\partial y} + (r - D_3)z\frac{\partial P}{\partial z} - rP + \frac{\epsilon C}{P + \epsilon - q} = 0, \quad x, y, z > 0,\ t \in [0, T), \\
&P(x, y, z, T) = \phi(x, y, z), \quad x, y, z \geq 0, \\
&P(x, y, 0, t) = p_1(x, y, t), \quad x, y \geq 0,\ t \in [0, T], \\
&P(x, 0, z, t) = p_2(x, z, t), \quad x, z \geq 0,\ t \in [0, T], \\
&P(0, y, z, t) = p_3(y, z, t), \quad y, z \geq 0,\ t \in [0, T], \\
&\lim_{x\to\infty} P(x, y, z, t) = 0, \quad y, z \geq 0,\ t \in [0, T], \\
&\lim_{y\to\infty} P(x, y, z, t) = 0, \quad x, z \geq 0,\ t \in [0, T], \\
&\lim_{z\to\infty} P(x, y, z, t) = 0, \quad x, y \geq 0,\ t \in [0, T],
\end{aligned}$$

where

$$q(x, y, z) = E - (\alpha_1 x + \alpha_2 y + \alpha_3 z), \qquad \phi(x, y, z) = \max(q(x, y, z), 0).$$

The functions p_1, p_2 and p_3 can be determined by solving appropriate two-factor Black–Scholes equations. More precisely, p_1, p_2 and p_3 satisfy problems of the form (13)–(19). Consequently, the computation of the fair price of an American put option depending on three assets requires that three Black–Scholes equations with $n = 2$ spatial dimensions are solved. Finally, as explained in Section 5, Eqs. (13)–(19) involve the value of single-asset options.

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