JOURNAL OF FUNCTIONAL ANALYSIS 65, 15–63 (1986)

Interpretation of AF C^*-Algebras in Łukasiewicz Sentential Calculus

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Communicated by A. Connes

Received November 9, 1984; accepted February 8, 1985

0. INTRODUCTION

Elliott [11, p. 30] remarked that the classification of AF C^*-algebras via dimension groups is combinatorial in nature. Taking this remark seriously, we shall give a criterion for the nonsimplicity of an AF C^*-algebra $\mathfrak{A}$ in terms of recursion-theoretic properties of the dimension group $K_0(\mathfrak{A})$: as will be shown in Theorem 6.1, appearance of (the noncommutative analogue of) Gödel's incompleteness [13] in $K_0(\mathfrak{A})$ is incompatible with $\mathfrak{A}$ being simple.

Gödel–Turing machinery can be naturally applied in this context, upon interpreting $K_0(\mathfrak{A})$ as a set of sentences in Łukasiewicz logic [33]. This is done in three steps, as follows:

1. In Theorem 1.3 we will show that every AF C^*-algebra $\mathfrak{A}$ can be embedded into a unique AF C^*-algebra $\mathfrak{A}_l$ such that $(K_0(\mathfrak{A}_l), [1_{\mathfrak{A}_l}]) \cong ((K_0(\mathfrak{A}))_l, [1_{\mathfrak{A}}]_l)$, where $(K_0(\mathfrak{A}))_l$ is the free lattice ordered group over $K_0(\mathfrak{A})$, and $[1_{\mathfrak{A}}]_l$ is the image of $[1_{\mathfrak{A}}]$ under the natural embedding of $K_0(\mathfrak{A})$ into $(K_0(\mathfrak{A}))_l$. See [9, 14, and 11] for K_0 of AF C^*-algebras, and [2, 34] for free lattice ordered abelian groups.

2. Given an arbitrary lattice ordered abelian group G with order unit u, letting $x^* = u - x$, $x \oplus y = u \wedge (x + y)$ and $x \cdot y = 0 \vee (x + y - u)$, we regard the unit interval $A = [0, u]$ of G as an MV algebra [4], i.e. (by 2.6) an algebra $(A, \oplus, \cdot, {}^*, 0, u)$, where $(A, \oplus, 0)$ is an abelian monoid, and where the following axioms hold: $x \oplus u = u$, $(x^*)^* = x$, $0^* = u$, $x \oplus x^* = u$, $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$, $x \cdot y = (x^* \oplus y^*)^*$. Specifically, we exhibit a functor Γ from lattice ordered abelian groups with order unit onto MV algebras, with the following property:

$$(G, u) \cong (G', u') \qquad \text{iff} \quad \Gamma(G, u) \cong \Gamma(G', u').$$

0022-1236/86 $3.00

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Upon restriction to totally ordered groups, Γ agrees with Chang's map [5]. In Theorem 3.9 we prove that Γ is an equivalence [23].

3. Defining $\tilde{\Gamma}(\mathfrak{A}) = \Gamma(K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$, by Elliott's fundamental result [11] together with the main theorem in [10] it follows that $\tilde{\Gamma}$ maps AF C^*-algebras with lattice ordered dimension group one–one onto countable MV algebras (see **1.1** and Theorem 3.12). Note that the domain of $\tilde{\Gamma}$ includes AF C^*-algebras with comparability of projections in the sense of Murray and von Neumann [12], e.g., the CAR algebra [3, 9]. In Theorem 3.14, using Theorem 1.2 we generalize 3.12 by mapping any arbitrary AF C^*-algebra $\mathfrak{A}$ into a pair (A, B), where $A = \tilde{\Gamma}(\mathfrak{A}_l)$ and $B \subseteq A$, in such a way that $\mathfrak{A} \cong \mathfrak{A}'$ holds iff there is an MV-algebra isomorphism ϕ of A onto A' with $\phi(B) = B'$.

The best excuse for the invariant $\tilde{\Gamma}(\mathfrak{A})$ is that, unlike lattice ordered abelian groups with order unit, MV algebras (i) are closed under subalgebras, quotients, and products [16], and (ii) have a distributive lattice structure naturally built in the algebraic structure (2.3). Moreover, (iii) the free MV algebra L with a denumerable set of free generators is isomorphic to an easily described MV algebra of continuous $[0, 1]$-valued functions over the Hilbert cube introduced by McNaughton [25]. The properties of these functions will be discussed in 4.13–17. Last, but not least, (iv) MV algebras are to many-valued logic as boolean algebras are to 2-valued logic: the above free MV algebra L is also the Lindenbaum algebra of the Łukasiewicz $\aleph_0$-valued sentential calculus [33, 31, 4, 17, 32] (see [24] for an essential bibliography).

Any countable MV algebra has the form $A \cong L/I$, letting I range over all ideals of L: stated otherwise, A is the Lindenbaum algebra L/I_Θ of some theory Θ in L, a theory being a set of sentences (5.1, 5.7). Hence there exists a unique map θ sending each AF C^*-algebra $\mathfrak{A}$ with lattice ordered K_0 into a nonempty set $\theta(\mathfrak{A})$ of theories in L, with the property that for any theory Θ, $\Theta \in \theta(\mathfrak{A})$ iff $\tilde{\Gamma}(\mathfrak{A}) \cong L/I_\Theta$. For any two such AF C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ we have

$$\mathfrak{A} \cong \mathfrak{B} \qquad \text{iff} \qquad \theta(\mathfrak{A}) = \theta(\mathfrak{B}) \qquad \text{iff} \qquad \theta(\mathfrak{A}) \cap \theta(\mathfrak{B}) \neq \varnothing.$$

For each consistent theory Θ in L there is a unique (up to isomorphism) AF C^*-algebra $\mathfrak{A}$ such that $\Theta \in \theta(\mathfrak{A})$. As an example, in Section 7 we shall explicitly write down a set Θ of sentences in Łukasiewicz logic corresponding to the CAR algebra.

Following the ideas of [13] we say that $\mathfrak{A}$ is *Gödel incomplete* iff there is a theory $\Theta \in \theta(\mathfrak{A})$ such that the set $\tilde{\Theta}$ of consequences of Θ is recursively enumerable but not recursive. In Theorem 6.1 we prove that if $\mathfrak{A}$ is Gödel incomplete then $\mathfrak{A}$ is not simple. Thus, purely combinatorial (actually, proof-theoretical) information on the invariant $\theta(\mathfrak{A})$ provides purely algebraic information on the AF C^*-algebra $\mathfrak{A}$.[1]

In C^*-mathematical physics, if it is true that nature does not have ideals [18, p. 852; 21, p. 468; 8, p. 85], then accordingly, every nonsimple C^*-algebra $\mathfrak{A}$—hence, by Theorem 6.1, every Gödel incomplete AF C^*-algebra—is an incomplete description of the physical reality, a possible completion of $\mathfrak{A}$ being any simple quotient $\mathfrak{A}/\mathfrak{J}$. However, while $\mathfrak{A}$ may have a recursively enumerable theory $\Theta = \tilde{\Theta} \in \theta(\mathfrak{A})$ (this being indeed the case of many explicit examples of AF C^*-algebras in the literature), the quotient $\mathfrak{A}/\mathfrak{J}$ need not inherit a recursively enumerable theory from Θ.

The Gödel incompleteness theorem for Peano arithmetic [13, 26] is a source of examples of the above phenomenon, already for abelian $\mathfrak{A}$, as shown in Example 6.4. In Example 6.5 we exhibit a Gödel complete primitive nonsimple AF C^*-algebra, thus solving a problem posed by the referee.

In Section 8 we study the freeness properties of the AF C^*-algebra $\mathfrak{M}$ defined by $\tilde{\Gamma}(\mathfrak{M}) \cong L$. Using the fact that the maximal ideal space of $\mathfrak{M}$ is homeomorphic to the Hilbert cube (8.1), we shall prove in Theorem 8.4 that every primitive ideal in $\mathfrak{M}$ is essential. We then conclude this paper by characterizing unital AF C^*-algebras with comparability of projections as those C^*-algebras which are quotients of $\mathfrak{M}$ by some primitive and essential ideal (8.8).

The prehistory of the present paper is in [29], where the author introduced a noncommutative framework for certain model-theoretical notions and their generalizations considered, e.g., in [27 and 28]. However, this paper is independent of [27–29].

The author wishes to express his gratitude to the referee and to A. Connes for their advice in the writing of the final version of this paper.

1. Canonical Embedding into AF C^*-Algebras with Lattice-ordered K_0

(1.1) Following [3] we say that a C^*-algebra $\mathfrak{A}$ is approximately finite-dimensional (AF) iff $\mathfrak{A}$ is the inductive limit of an increasing sequence of finite-dimensional C^*-algebras, all with the same unit. We refer to [9 or 14] for the definition of the functor K_0 from the category of AF C^*-algebras with C^*-algebra homomorphisms, to the category of coun-

[1] Theorem 6.1 is perhaps worth mentioning in connection with the problem of the applicability of many-valued logic outside mathematical logic: compare with J. Dieudonné, Present trends in pure mathematics, *Adv. in Math.* **27** (1978), 239.

table partially ordered abelian groups with order-preserving group homomorphisms.

For any AF C^*-algebra $\mathfrak{A}$, $K_0(\mathfrak{A})$ is a *dimension group*, i.e., a partially ordered group which is the direct limit of a directed set of simplicial groups—a partially ordered group being simplicial iff it is order-isomorphic to a free abelian group $\mathbb{Z}^n$ with coordinatewise ordering. Equivalently [10], a dimension group is a partially ordered abelian group G which is *directed* ($G = G^+ - G^+$), *unperforated* ($x \notin G^+ \Rightarrow \forall n \in \omega \setminus \{0\}$, $nx \notin G^+$), and has the (Riesz) *interpolation property* ($\forall a, b, c, d \in G$ with $a, b \leqslant c, d$, $\exists g \in G$ with $a, b \leqslant g \leqslant c, d$). An element u of a partially ordered group G is an *order unit* iff for every $a \in G$ there is $n \in \omega$ such that $a \leqslant nu$. Given the AF C^*-algebra $\mathfrak{A}$, the image $[1_{\mathfrak{A}}]$ of the unit of $\mathfrak{A}$ in $K_0(\mathfrak{A})$ is an order unit in $K_0(\mathfrak{A})$. Given a C^*-algebra morphism $\psi: \mathfrak{A} \to \mathfrak{B}$ between AF C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, if $\psi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$ then, letting $\phi = K_0(\psi)$, we have that $\phi([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$. By a morphism in the category of partially ordered abelian groups with order unit we shall mean an order-preserving group homomorphism which also preserves order units [14, p. 140]. Given any two such groups G and G' with order unit u and u', respectively, we let

$$(G, u) \cong (G', u')$$

mean that there is an isomorphism of partially ordered groups $\phi: G \to G'$ such that $\phi(u) = u'$.

1.2. THEOREM [11]. (i) *For every countable dimension group G with order unit u there is an* AF *C^*-algebra $\mathfrak{A}$ such that $(G, u) \cong (K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$.*

(ii) *Given two* AF *C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$ iff $(K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \cong (K_0(\mathfrak{B}), [1_{\mathfrak{B}}])$.*

(iii) *Given two* AF *C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and an order-preserving group homomorphism $\phi: K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ with $\phi([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$, there is a C^*-algebra homomorphism $\psi: \mathfrak{A} \to \mathfrak{B}$ such that $K_0(\psi) = \phi$ and $\psi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.*

Now let (G, u) be a dimension group with order unit u. Since G is unperforated then there exists *the* free lattice ordered group G_l over G [2, Appendix 2.6; 34, 2.7]. Uniqueness is obvious from the definition. Let $\eta: G \to G_l$ be the natural embedding, and $u_l = \eta(u)$.

1.3. THEOREM. *Let $\mathfrak{A}$ be an* AF *C^*-algebra. Then there is a unique (up to isomorphism)* AF *C^*-algebra $\mathfrak{A}_l$ such that $(K_0(\mathfrak{A}_l), [1_{\mathfrak{A}_l}]) \cong ((K_0(\mathfrak{A}))_l, [1_{\mathfrak{A}}]_l)$. Moreover, $\mathfrak{A}$ is isomorphic to a C^*-subalgebra of $\mathfrak{A}_l$.*

Proof. Let $(G, u) = (K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$. As an immediate consequence of the definition of G_l and $\eta: G \to G_l$, we have that G_l is abelian (since G is abelian), and G_l is generated by $\eta(G)$ as a lattice ordered group; therefore G_l is countable (since G is countable), and $u_l = \eta(u)$ is an order unit for G_l.

Since G_l is lattice ordered then G_l has the Riesz interpolation property, and is directed an unperforated [34, p. 188; 2, 1.3]. By the above mentioned characterization of dimension groups [10] and by Theorem 1.2(i), (ii), there is a unique AF C^*-algebra $\mathfrak{A}_l$ such that $(K_0(\mathfrak{A}_l), [1_{\mathfrak{A}_l}]) \cong (G_l, u_l)$. By Theorem 1.2(iii) there is a unit-preserving C^*-algebra homomorphism $\psi: \mathfrak{A} \to \mathfrak{A}_l$ such that $K_0(\psi) = \eta$. Since η is one–one, then ψ is one–one [14, Exercise 19J]. Thus ψ is a C^*-algebra isomorphism of $\mathfrak{A}$ onto a C^*-subalgebra of $\mathfrak{A}_l$, as required. ▮

(**1.4**) By an *l-group* we shall mean a lattice ordered abelian group. If (G, u) and (H, v) are l-groups with order unit u and v respectively, then a map $\lambda: G \to H$ is said to be a *unital l-homomorphism* iff λ is a group homomorphism and a lattice homomorphism such that $\lambda(u) = v$. Unital l-homomorphisms are precisely the morphisms in the category of l-groups with order unit. By an *l-ideal* in an l-group G we mean a convex subgroup J which is a sublattice of G. If G has an order unit u then the image u/J under the quotient map is an order unit of G/J, and the quotient map is a unital l-homomorphism of (G, u) onto $(G/J, u/J)$.

2. From Abelian Lattice Groups with Order unit to MV Algebras

In [4] Chang defined MV algebras as follows:

2.1. Definition. An MV *algebra* is an algebra $(A, \oplus, \cdot, {}^*, 0, 1)$, where A is a nonempty set, 0 and 1 are constant elements of A, $\oplus$ and $\cdot$ are binary operations, and * is a unary operation, satisfying the following axioms (where we let $x \vee y = (x \cdot y^*) \oplus y$ and $x \wedge y = (x \oplus y^*) \cdot y$):

$Ax1$	$x \oplus y = y \oplus x$	$Ax1'$	$x \cdot y = y \cdot x$
$Ax2$	$x \oplus (y \oplus z) = (x \oplus y) \oplus z$	$Ax2'$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$Ax3$	$x \oplus x^* = 1$	$Ax3'$	$x \cdot x^* = 0$
$Ax4$	$x \oplus 1 = 1$	$Ax4'$	$x \cdot 0 = 0$
$Ax5$	$x \oplus 0 = x$	$Ax5'$	$x \cdot 1 = x$
$Ax6$	$(x \oplus y)^* = x^* \cdot y^*$	$Ax6'$	$(x \cdot y)^* = x^* \oplus y^*$

$Ax7$ $x = (x^*)^*$

$Ax8$ $0^* = 1$

$Ax9$	$x \vee y = y \vee x$	$Ax9'$	$x \wedge y = y \wedge x$
$Ax10$	$x \vee (y \vee z) = (x \vee y) \vee z$	$Ax10'$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
$Ax11$	$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$	$Ax11'$	$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$.

Remark. We use $\oplus$ instead of Chang's original $+$, as the latter symbol denotes group addition in our paper; also, we write y^* instead of Chang's original notation $\bar{y}$, for typographical reasons. By a traditional abuse of notation we shall simply denote by A the whole MV algebra $(A, \oplus, ...)$, whenever this may cause no confusion. Following [4, p. 468] we shall consider multiplication $\cdot$ more binding than addition $\oplus$.

2.2. DEFINITION. For all $x, y \in A$ we write $x \leqslant y$ iff $x \vee y = y$.

2.3. THEOREM. (i) *The relation $\leqslant$ is a partial ordering over A; for all $x, y \in A$, $x \vee y$, and $x \wedge y$ are respectively the* sup *and the* inf *of the pair (x, y) with respect to $\leqslant$; also, for every $x \in A$, $0 \leqslant x \leqslant 1$.*

(ii) *Every* MV *algebra is a subdirect product of totally ordered* MV *algebras.*

(iii) *A is a distributive lattice with respect to the operations $\vee$ and $\wedge$.*

Proof. (i) [4, 1.11, 1.4]. (ii) is proved in [5, Lemma 3]. (iii) is immediate from (i) and (ii). ▮

In [5, p. 75] Chang defined a map from totally ordered abelian groups with order unit into totally ordered MV algebras. A natural generalization of Chang's map to lattice-ordered abelian groups with order unit is given by the following definition, as will be proved in Theorem 2.5.

2.4. DEFINITION. Let $G = (G, +, -, O_G, \vee_G, \wedge_G)$ be a lattice ordered abelian group with order unit u. We define $\Gamma(G, u) = (A, \oplus, \cdot, {}^*, 0, 1)$ by the following stipulations: $A = [O_G, u] = \{g \in G \mid O_G \leqslant g \leqslant u\}$, and, for all $x, y \in A$,

$$x \oplus y = u \wedge_G (x + y)$$

$$x^* = u - x$$

$$x \cdot y = (x^* \oplus y^*)^*$$

$$0 = O_G$$

$$1 = u.$$

Further, given a unital l-homomorphism $\theta\colon (G, u) \to (G', u')$, we define $\Gamma(\theta)\colon \Gamma(G, u) \to \Gamma(G', u')$ by $\Gamma(\theta) = \theta|_A =$ restriction of θ to A.

$\Gamma(\theta)$ is well defined, since θ is order-preserving. We shall now extend Chang's result [5, Lemma 4]: Following [4, p. 471] we say that a map $\mu\colon A \to A'$ is an *MV-homomorphism* iff $\mu(0) = 0'$, $\mu(1) = 1'$, and μ preserves the operations $\oplus$, $\cdot$, and $*$. In case μ is one–one onto A', then μ is an *isomorphism* of A onto A'. Of course, these notions are particular instances of the universal algebraic notions [16]. We refer to [23] for all category-theoretic concepts used in this paper.

2.5. THEOREM. *The map Γ is a functor from the category of lattice ordered abelian groups with order unit to the category of MV algebras. For any such group (G, u), the lattice operations on the unit interval $[O_G, u]$ of G agree with the lattice operations on the MV algebra $\Gamma(G, u)$, as given by* 2.1–2.3.

To prove the theorem we first give an equivalent reformulation (due to Mangani, *Boll. Un. Mat. Ital.* (4) **8** (1973), p. 68) of the definition of MV algebra.

2.6. LEMMA. *Let $(A, \oplus, \cdot, *, 0, 1)$ be an algebra where* 0 *and* 1 *are constant elements of A, $\oplus$, and $\cdot$ are binary operations on A, and $*$ is a unary operation on A, obeying the following axioms*:

P1 $(x\oplus y)\oplus z = x\oplus (y\oplus z)$,

P2 $x\oplus 0 = x$,

P3 $x\oplus y = y\oplus x$,

P4 $x\oplus 1 = 1$,

P5 $(x^*)^* = x$,

P6 $0^* = 1$,

P7 $x\oplus x^* = 1$,

P8 $(x^*\oplus y)^*\oplus y = (x\oplus y^*)^*\oplus x$,

P9 $x\cdot y = (x^*\oplus y^*)^*$.

Then A is an MV *algebra. Conversely, every* MV *algebra obeys axioms* P1–P9.

Proof of Lemma 2.6. We first prove that every MV algebra obeys P1–P9: Note that $P1 = Ax2$, $P2 = Ax5$, $P3 = Ax1$, $P4 = Ax4$, $P5 = Ax7$, $P6 = Ax8$, $P7 = Ax3$. Concerning P8, note that $(x^*\oplus y)^*\oplus y = (x^*\oplus (y^*)^*)^*\oplus y$, by $Ax7$; the latter expression is equal to $(x\cdot y^*)\oplus y$, by $Ax7$ and $Ax6'$, and hence equal to $x\vee y$ by definition of $\vee$, Definition 2.1. Similarly, using the commutativity of $\oplus$ ($Ax1$), one has $y\vee x = (x\oplus y^*)^*\oplus x$. Now $Ax9$ yields the desired conclusion. The validity of P9 is a consequence of $Ax6'$ and $Ax7$.

Conversely, we shall now prove that every algebra obeying P1–P9 is an MV algebra. Consider the following (Łukasiewicz) axioms:

A1 $x^*\oplus (y^*\oplus x) = 1$

A2 $(x^*\oplus y)^*\oplus ((y^*\oplus z)^*\oplus (x^*\oplus z)) = 1$

A3 $((x\cdot y^*)\oplus y)^*\oplus ((y\cdot x^*)\oplus x) = 1$

A4 $(x\oplus y^*)^*\oplus (y^*\oplus x) = 1$.

One immediately verifies that P1–P9 imply A1, A3, and A4. As for A2, using P1–P9 we have $(x^* \oplus y)^* \oplus (y^* \oplus z)^* \oplus x^* \oplus z = (x^* \oplus y)^* \oplus x^* \oplus (y^* \oplus z)^* \oplus z = (x^* \oplus y)^* \oplus x^* \oplus (y \oplus z^*)^* \oplus y = (x^* \oplus y)^* \oplus (x^* \oplus y) \oplus (y \oplus z^*)^* = 1 \oplus (y \oplus z^*)^* = 1$. Therefore, P1–P9 imply A1–A4. Arguing now as Chang does in [4, pp. 472–473] we conclude that P1–P9 imply all MV axioms.

2.7. LEMMA. [22] (i) *For any l-group G with order unit u, $\Gamma(G, u)$ is an MV algebra.*

(ii) *The lattice operations on G agree with the lattice operations on $\Gamma(G, u)$.*

Proof. (i) We prove that $\Gamma(G, u)$ satisfies P1–P9. Let x, y, z be arbitrary elements of $[O_G, u]$. Then we have

P2: $x \oplus 0 = u \wedge_G (x + O_G) = u \wedge_G x = x$, because $x \leqslant_G u$.

P3: $x \oplus y = u \wedge_G (x + y) = u \wedge_G (y + x) = y \oplus x$.

P1: $(x \oplus y) \oplus z = u \wedge_G (z + (x \oplus y)) = u \wedge_G (z + (u \wedge_G (x + y))) = u \wedge_G ((z + u) \wedge_G (z + x + y)) = (u \wedge_G (z + u)) \wedge_G (x + y + z) = u \wedge_G (x + y + z)$. Note that $z + u \;{}_G\!\geqslant O_G + u = u$, since $z \;{}_G\!\geqslant 0$. We have thus proved that $\oplus$ is associative.

P4: $x \oplus 1 = u \wedge_G (x + u) = u = 1$.

P5: $(x^*)^* = u - (u - x) = x$.

P6: $0^* = u - O_G = u = 1$.

P7: $x \oplus x^* = u \wedge_G (x + x^*) = u \wedge_G (x + u - x) = u = 1$.

P8: $(x^* \oplus y)^* \oplus y = u \wedge_G (y + (x^* \oplus y)^*) = u \wedge_G (y + u - (u \wedge_G (x^* + y))) = u \wedge_G (y + u - (u \wedge_G (u - x + y))) = u \wedge_G (y + u + (-u \vee_G (-u + x - y))) = u \wedge_G (y + ((u - u) \vee_G (u - u + x - y))) = u \wedge_G (y + (0 \vee_G (x - y))) = u \wedge_G ((y + 0) \vee_G (y + x - y)) = u \wedge_G (y \vee_G x) = y \vee_G x = x \vee_G y$, because $x, y \leqslant_G u$. This shows that x and y are interchangeable, whence P8 holds.

P9: $x \cdot y = (x^* \oplus y^*)^*$ by definition of Γ.

Thus $\Gamma(G, u)$ obeys P1–P9, hence by Lemma 2.6 it is an MV algebra.

(ii) Let $x, y \in A = \Gamma(G, u)$. By Definition 2.1, in the MV algebra A we have: $x \vee y = (x \cdot y^*) \oplus y = (x^* \oplus y)^* \oplus y$. The above proof that A obeys P8 now yields $x \vee y = x \vee_G y$. Further, by [4, Theorem 1.2(iii)] we obtain

$$x \wedge y = (x^* \vee y^*)^* = u - ((u - x) \vee_G (u - y)) = u + ((x - u) \wedge_G (y - u))$$
$$= (u + x - u) \wedge_G (u + y - u) = x \wedge_G y.$$

This completes the proof of Lemma 2.7.

2.8. *End of the Proof of Theorem* 2.5. In the light of Lemmas 2.6 and 2.7, there remains to be proved that whenever $\theta: (G, u) \to (G', u')$ is a unital l-homomorphism (1.4), $\Gamma(\theta)$ is an MV-homomorphism. To this purpose, let $\mu = \Gamma(\theta)$, and $(A, \oplus, \cdot, *, 0, 1) = \Gamma(G, u)$; similarly, let $(A', \otimes', \cdot', *', 0', 1') = \Gamma(G', u')$. We have already noted after Definition 2.4 that $\mu(A) \subseteq A'$, $\mu(0) = 0'$, and $\mu(1) = 1'$.

Claim 1. μ preserves $\oplus$. Indeed, for all $x, y \in A$ we have $\mu(x \oplus y) = \theta(u \wedge_G (x + y)) = \theta(u) \wedge_{G'} \theta(x + y) = u' \wedge_{G'} (\theta(x) +' \theta(y)) = u' \wedge_{G'} (\mu(x) +' \mu(y)) = \mu(x) \oplus' \mu(y)$.

Claim 2. μ preserves $*$. Indeed, for all $x \in A$, $\mu(x^*) = \theta(u - x) = u' -' \theta(x) = u' -' \mu(x) = (\mu(x))^{*'}$.

Claim 3. μ preserves multiplication. Immediate from Claims 1 and 2, since multiplication is definable in terms of $\oplus$ and $*$.

Claim 4. Γ preserves identities. Indeed, if $j: G \to G$ is the identity function on G, then $\Gamma(j) = j|_A: A \to A$ is the identity function on A.

Claim 5. Γ preserves composition. Assume we are given the following diagram: $(G, u) \to^{\phi} (G', u') \to^{\psi} (G'', u'')$. Then $\Gamma(\psi \circ \phi) = (\psi \circ \phi)|_A = \psi \circ (\phi|_A) = (\psi|_{A'}) \circ (\phi|_A) = \Gamma(\psi) \circ \Gamma(\phi)$.

The proof of Theorem 2.5 is now complete. ∎

2.9. *Remarks.* From now on we shall use $\vee$ and $\wedge$ (instead of $\vee_G$ and $\wedge_G$) to denote the lattice operations on G. The above theorem ensures that no confusion may arise with the lattice operations of the MV algebra $\Gamma(G, u) = (A, \oplus, ...)$. Thus, for example, let us show in the new notation that, for all $x_1, ..., x_n \in A$, $x_1 \oplus \cdots \oplus x_n = u \wedge (x_1 + \cdots + x_n)$. For $n = 2$, this is by definition. Proceeding by induction we have $x_1 \oplus \cdots \oplus x_{n+1} = u \wedge ((u \wedge (x_1 + \cdots + x_n)) + x_{n+1}) = u \wedge ((u + x_{n+1}) \wedge (x_1 + \cdots + x_{n+1})) = (u \wedge (u + x_{n+1})) \wedge (x_1 + \cdots + x_{n+1}) = u \wedge (x_1 + \cdots + x_{n+1})$, because $u \leqslant u + x_{n+1}$.

3. Properties of the Functor Γ

3.1. Proposition. *Let G be an l-group with order unit u. Let $(A, \oplus, \cdot, *, 0, 1) = \Gamma(G, u)$. Let x be an arbitrary element of G^+. Then we have*

(i) *There are $a_1, ..., a_n \in A$ such that $x = a_1 + \cdots + a_n$ and $a_i \oplus a_{i+1} = a_i$ for all $i = 1, ..., n - 1$;*

(ii) *If, in addition,* $y = b_1 + \cdots + b_n$ *with* $b_1, ..., b_n \in A$ *and* $b_i \oplus b_{i+1} = b_i$ *for all* $i = 1, ..., n-1$, *then the following identities hold*:

$$x \vee y = (a_1 \vee b_1) + \cdots + (a_n \vee b_n) \tag{1}$$

$$x \wedge y = (a_1 \wedge b_1) + \cdots + (a_n \wedge b_n). \tag{2}$$

Proof. (i) For each $m = 1, 2, ...$, define a_m by the following stipulation: $a_1 = x \wedge u$, $a_{m+1} = (x - a_1 - a_2 - \cdots - a_m) \wedge u$. A straightforward induction argument shows that each a_m belongs to $[0, u]$ and that for every $j \geqslant 1$, $a_1 + \cdots + a_j \leqslant x$. An easy computation shows that $a_i \oplus a_{i+1} = a_i$ for all $i \geqslant 1$. Since $x \leqslant nu$ for some $n \in \omega$, then $0 = a_{n+1} = a_{n+2} = \cdots$, as can be verified by representing G as a subdirect product of totally ordered groups [2, 4.1.8].

(ii) Again using [2, 4.1.8] write G as a subdirect product of totally ordered groups, $G \hookrightarrow \prod G_j$. Let us agree to denote by a_{ij} and b_{ij} the image in G_j of a_i and b_i, respectively. For notational simplicity we also let 0 and 1 respectively denote the image in G_j of the zero and of the order unit of G. As a consequence of our assumptions we have $0 \leqslant a_{ij}, b_{ij} \leqslant 1$, $x_j = a_{1j} + \cdots + a_{nj}$, $y_j = b_{1j} + \cdots + b_{nj}$, $a_{ij} \oplus a_{i+1,j} = a_{ij}$, and $b_{ij} \oplus b_{i+1,j} = b_{ij}$, where $\oplus$ refers to the addition operation in the MV algebra $\Gamma(G_j, 1)$. We observe that in the totally ordered group G_j the sequence $(a_{1j}, ..., a_{nj})$ has the form $(1, ..., 1, c, 0, ..., 0)$ for some $c \in [0, 1] \subseteq G_j$: as a matter of fact, since $a_{1j} \geqslant a_{2j} \geqslant \cdots \geqslant a_{nj}$, then by [4, 3.13] whenever $1 \neq a_{ij} = 0 \oplus a_{ij} = a_{i+1,j} \oplus a_{ij}$, it follows that $a_{i+1,j} = 0$. Similarly, $(b_{1j}, ..., b_{nj}) = (1, ..., 1, d, 0, ..., 0)$ for some $d \in [0, 1]$.

Claim. If $x_j \leqslant y_j$ then $a_{1j} \leqslant b_{1j}, ..., a_{nj} \leqslant b_{nj}$. For otherwise (absurdum hypothesis) if $a_{kj} > b_{kj}$ then $a_{k-1,j} = a_{k-2,j} = \cdots = a_{1j} = 1$, and $b_{k+1,j} = b_{k+2,j} = \cdots = b_{nj} = 0$, by the above discussion. Hence, $a_{ij} \geqslant b_{ij}$ for all $i = 1, ..., n$, and $a_{kj} > b_{kj}$, whence $x_j > y_j$, a contradiction. Our claim is settled. From the claim it follows that $(a_{1j} \vee b_{1j}) + \cdots + (a_{nj} \vee b_{nj}) = b_{1j} + \cdots + b_{nj} = y_j = x_j \vee y_j$, which establishes (1) in G_j, provided $x_j \leqslant y_j$. In case $x_j \geqslant y_j$ one similarly proves that (1) holds in G_j, by interchanging the roles of x_j and y_j. In conclusion, since the identity (1) holds in each G_j, then it holds in G. The proof of (2) is similar. ∎

3.2. DEFINITION. Given an l-group G with order unit u, let $(A, \oplus, \cdot, {}^*, 0, 1) = \Gamma(G, u)$. For every sequence $(w_1, ..., w_n)$ of elements of A, and every $a \in A$ we write $(w_1, ..., w_n) \sim a$ iff the following identities are simultaneously satisfied:

$$a^* \oplus w_2 \oplus \cdots \oplus w_n = w_1^*$$
$$w_1 \oplus a^* \oplus \cdots \oplus w_n = w_2^*$$
$$\vdots$$
$$w_1 \oplus w_2 \oplus \cdots \oplus a^* = w_n^*$$
$$w_1 \oplus w_2 \oplus \cdots \oplus w_n = a.$$

3.3. PROPOSITION. *Adopt the notation of the above definition. Then for all* $w_1, ..., w_n, a \in A$ *the following are equivalent*:

(i) $w_1 + \cdots + w_n = a$, *and*

(ii) $(w_1, ..., w_n) \sim a$.

Proof. Let $w_0 = a^* = u - a$. Assuming (i) we have $u = w_0 + w_1 + \cdots + w_n$. For each $i = 0, ..., n$ we have $w_i^* = u \wedge (u - w_i) = u \wedge (w_0 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_n) = w_0 \oplus \cdots \oplus w_{i-1} \oplus w_{i+1} \oplus \cdots \oplus w_n$, recalling Remarks 2.9. Therefore, (ii) holds. Conversely, if (ii) holds, then by 2.9 we have for all $i = 0, ..., n$,

$$u \wedge (w_0 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_n) = u - w_i,$$

whence by distributivity

$$(w_i - u + u) \wedge (w_i - u + w_0 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_n) = 0,$$

Letting $\sum w = w_0 + \cdots + w_n$ we have $w_i \wedge (-u + \sum w) = 0$, and, in particular, $0 \leqslant -u + \sum w$. Applying [2, 1.2.24] we obtain $\sum w \wedge (-u + \sum w) = 0$. Since $0 \leqslant -u + \sum w \leqslant \sum w$, we conclude that $-u + \sum w = 0$, i.e., $0 < -u + a^* + w_1 + \cdots + w_n = -a + w_1 + \cdots + w_n$. ▮

3.4. PROPOSITION. *If both* κ *and* λ *are unital l-homomorphisms of* (G, u) *into* (G', u'), *and* $\Gamma(\kappa) = \Gamma(\lambda)$, *then* $\kappa = \lambda$.

Proof. (Compare with [20, 1.5]). For every $x \in G^+$ there is $n \in \omega$ such that $x \leqslant nu$. By [2, 1.2.17] there are $a_1, ..., a_n \in [0, u]$ such that $x = a_1 + \cdots + a_n$. Since by hypothesis $\kappa | [0, u] = \lambda | [0, u]$, then $\kappa(x) = \sum \kappa(a_i) = \sum \lambda(a_i) = \lambda(x)$, whence $\kappa | G^+ = \lambda | G^+$. Since G is directed, each $z \in G$ has the form $z = x - y$ for some $x, y \in G^+$. Then, $\kappa(z) = \kappa(x) - \kappa(y) = \lambda(x) - \lambda(y) = \lambda(z)$. ▮

3.5. PROPOSITION. *Let* G *and* G' *be l-groups with order unit* u *and* u', *respectively. Assume* $\mu: \Gamma(G, u) \to \Gamma(G', u')$ *is an MV homomorphism. Then* $\mu = \Gamma(\lambda)$ *for some unital l-homomorphism* $\lambda: (G, u) \to (G', u')$.

Proof. For the moment, we define λ over G^+ as follows: Given an arbitrary $x \in G^+$, by Proposition 3.1(i) there are elements $a_1, ..., a_n \in [0, u]$ such that $x = a_1 + \cdots + a_n$. We stipulate

$$\lambda(x) = \mu(a_1) + \cdots + \mu(a_n). \tag{3}$$

Claim 1. λ is well defined over G^+, i.e., if also $x = b_1 + \cdots + b_m$ for some $b_1, ..., b_m \in [0, u]$ then $\mu(b_1) + \cdots + \mu(b_m) = \mu(a_1) + \cdots + \mu(a_n)$.

As a matter of fact, since $b_1 + \cdots + b_m = a_1 + \cdots + a_n$, then by the Riesz decomposition property [2, 1.2.16] there are elements $g_{ij} \in G^+$ (for $i = 1, ..., m$ and $j = 1, ..., n$) such that

$$a_j = \sum_i g_{ij} \qquad \text{and} \qquad b_i = \sum_j g_{ij}, \qquad \text{for all } i, j. \tag{4}$$

In particular, $g_{ij} \in [0, u]$. By Proposition 3.3 for each $i = 1, ..., m$ and $j = 1, ..., n$ the identities $(g_{1j}, ..., g_{mj}) \sim a_j$ and $(g_{i1}, ..., g_{in}) \sim b_i$ are satisfied in $\Gamma(G, u)$. Since μ is an MV homomorphism, and $a_j, b_i, g_{ij} \in [0, u]$, it follows that the corresponding identities $(g'_{1j}, ..., g'_{mj}) \sim a'_j$ and $(g'_{i1}, ..., g'_{in}) \sim b'_i$ are satisfied in $\Gamma(G', u')$, where we let $y' = \mu(y)$ for any $y \in [0, u]$. Again using Proposition 3.3 we obtain

$$a'_j = \sum_i g'_{ij} \qquad \text{and} \qquad b'_i = \sum_j g'_{ij}. \tag{4'}$$

Noting that a'_j, b'_i, $g'_{ij} \in [0', u']$ and using (4′) twice, we obtain $a'_1 + \cdots + a'_n = \sum_i g'_{i1} + \cdots + \sum_i g'_{in} = \sum_j g'_{1j} + \cdots + \sum_j g'_{mj} = b'_1 + \cdots + b'_m$, which settles Claim 1.

Claim 2. λ preserves addition over G^+, i.e., $\lambda(x + y) = \lambda(x) + \lambda(y)$ for all $x, y \in G^+$.

Indeed, any such x, y have a representation $x = a_1 + \cdots + a_n$, $y = b_1 + \cdots + b_m$ with $a_j, b_i \in [0, u]$, by Proposition 3.1(i). Now the conclusion follows from (3).

Claim 3. λ preserves $\vee$ over G^+, i.e., $\lambda(x \vee y) = \lambda(x) \vee \lambda(y)$ for all $x, y \in G^+$.

Indeed, using the full strength of Proposition 3.1(i) we can write $x = a_1 + \cdots + a_n$, $y = b_1 + \cdots + b_m$ with all a's and b's in $[0, u]$, and with the additional property that $a_j \oplus a_{j+1} = a_j$ and $b_i \oplus b_{i+1} = b_i$ for each $j = 1, ..., n-1$ and $i = 1, ..., m-1$. Appending zeros, if necessary, we can assume $n = m$ without loss of generality.

Writing throughout z' for $\mu(z)$ whenever $z \in [0, u]$, we obtain

$$a'_j \oplus a'_{j+1} = a'_j \qquad \text{and} \qquad b'_j \oplus b'_{j+1} = b'_j, \tag{5}$$

since μ preserves all MV operations. Recalling Lemma 2.7(ii) we get

$$\begin{aligned}\lambda(x \vee y) &= \lambda((a_1 \vee b_1) + \cdots + (a_n \vee b_n)) && \text{by Proposition 3.1(ii),}\\ &= \lambda(a_1 \vee b_1) + \cdots + \lambda(a_n \vee b_n) && \text{by Claim 2,}\\ &= \mu(a_1 \vee b_1) + \cdots + \mu(a_n \vee b_n) && \text{by (3) and Claim 1,}\\ &= (a'_1 \vee b'_1) + \cdots + (a'_n \vee b'_n) && \text{because } \mu \text{ preserves } \vee \text{ over } [0, u].\end{aligned}$$

On the other hand, $\lambda(x) \vee \lambda(y) = \lambda(a_1 + \cdots + a_n) \vee \lambda(b_1 + \cdots + b_n) = (a'_1 + \cdots + a'_n) \vee (b'_1 + \cdots + b'_n) = (a'_1 \vee b'_1) + \cdots + (a'_n \vee b'_n)$ by Claim 2, (5) and Proposition 3.1(ii). This settles Claim 3.

Using Claims 1–3 we can apply [2, 1.4.5] to the effect that there exists exactly one l-homomorphism (which we also denote by λ) from G into G' extending λ. By (3) λ is unital since $\mu(u) = u'$. By definition of Γ, $\mu = \Gamma(\lambda)$. This completes the proof of the proposition. ∎

3.6. *Remark.* From [11; and 20, 1.5] it follows that if f is an order preserving map from $[0, u]$ into G' such that $f(x + y) = f(x) + f(y)$ whenever $x + y \in [0, u]$, then f uniquely extends to an order preserving group homomorphism $\hat{f}: G \to G'$. This holds in the general context of partially ordered groups with order unit.

In the present context of l-groups, the stronger assumption that μ preserves the MV structure over $[0, u]$ is used in Proposition 3.5 to prove that the (unique) extension λ of μ also preserves the lattice structure.

In [5, Lemma 6] Chang proved the following result:

3.7. PROPOSITION. *Let $A = (A, \oplus, \cdot, {}^*, 0, 1)$ be a totally ordered MV algebra. Then there is a totally ordered abelian group G with order unit u, such that $A \cong \Gamma(G, u)$. Furthermore, the cardinality* card G *of G obeys the following inequality*: card $G \leqslant \max(\omega, \text{card } A)$.

The following is a generalization of Chang's result (see also [22, Propositions 5 and 6]):

3.8. THEOREM. *Let $A = (A, \oplus, \cdot, {}^*, 0, 1)$ be an MV algebra. Then there exists an l-group G with order unit u, such that $A \cong \Gamma(G, u)$. Furthermore,* card $G \leqslant \max(\omega, \text{card } A)$.

Proof. By Theorem 2.3(ii) we can represent A as a subdirect product of totally ordered MV algebras, $A \subseteq \prod_{i \in I} A_i$. Using Proposition 3.7 we may regard each A_i as the MV algebra $A_i = \Gamma(G_i, u_i)$ on the unit interval $[O_i, u_i]$ of some totally ordered abelian group G_i with order unit u_i. We then have the canonical inclusions

$$A \subseteq \prod_{i \in I} A_i \subseteq \prod_{i \in I} G_i. \tag{6}$$

Each element $x \in \prod_{i \in I} G_i$ will be written $\{x_i\}_{i \in I}$, with $x_i \in G_i$ the ith coordinate of x. In each $A_i = \Gamma(G_i, u_i)$, the MV operations have the following form, for all $r, s \in A_i$

$$\begin{aligned} r \oplus s &= \min(u_i, r + s) \\ r^* &= u_i - r \\ r \cdot s &= (r^* \oplus s^*)^* = u_i - \min(u_i, u_i - r + u_i - s) = \max(O_i, r + s - u_i), \end{aligned} \tag{7}$$

where, of course, $+$ and $-$ are the group operations on G_i. In the light of (6) we define G as follows:

$$G = \text{lattice group generated by } A \text{ in } \prod_{i \in I} G_i, \tag{8}$$

and we let $u = u_G = \{u_i\}_{i \in I}$.

We shall prove that (G, u) obeys the requirements of our theorem. Evidently, G is a lattice ordered abelian group; to see that u is an order unit for G, let $x \in G$; then x is obtained from a finite number of elements of A by a finite number of applications of the lattice and group operations. By induction on the number of such operations one easily proves that there exists $n \in \omega$ such that $x \leqslant nu$. Thus u is an order unit for G.

We must now prove that $A \cong \Gamma(G, u)$: this will be done in 3.8.1–3.8.5 below. First, given a sequence $(a_1, ..., a_n)$ of elements of $\prod_{i \in I} G_i$ let us agree to say that the sequence is *good* iff $a_1, ..., a_n \in A$ and $a_m \oplus a_{m+1} = a_m$ $(m = 1, ..., n - 1)$.

3.8.1. LEMMA. (i) *Every good sequence is decreasing with respect to the MV order on A.*

(ii) *If, in addition, A is totally ordered, and $(a_1, ..., a_n)$ is good, then for all except possibly one $m = 1, ..., n$, we have that a_m is a member of the set $\{0, 1\}$.*

Proof. (i) $a_m = a_m \oplus a_{m+1} \geqslant a_{m+1}$ by [4, 1.10].

(ii) By [4, 3.13], since $0 \oplus a_m = a_{m+1} \oplus a_m$, if $0 \oplus a_m \neq 1$ (i.e., $a_m \neq 1$) then $0 = a_{m+1}$. Now by (i) we get the desired conclusion.

After the proof of Lemma 3.8.1 we define $A^+ \subseteq G$ by

$$A^+ = \{x \in G \mid x = a_1 + \cdots + a_n \text{ for some good sequence } (a_1, ..., a_n)\}, \tag{9}$$

where $+$ denotes addition on the group $\prod_{i \in I} G_i$.

3.8.2. LEMMA. *For all* $x, y \in A^+$, $x+y \in A^+$.

Proof. We first consider the case $x = a \in A$, $y = b \in A$.

Claim 1. The sequence $(a \oplus b, a \cdot b)$ is good, and $(a \oplus b) + (a \cdot b) = x + y$.

Indeed, write $a = \{a_i\}_{i \in I}$, $b = \{b_i\}_{i \in I}$ and examine $A_i \subseteq G_i$. Since G_i is totally ordered two cases are possible:

Case 1. $a_i + b_i \leqslant u_i$. Then $(a_i \oplus b_i) \oplus (a_i \cdot b_i) = a_i \oplus b_i \oplus O_i$ by (7); moreover, $(a_i \oplus b_i) + (a_i \cdot b_i) = a_i + b_i = x_i + y_i$, again by (7), which settles the case under consideration.

Case 2. $a_i + b_i > u_i$. Then $(a_i \oplus b_i) \oplus (a_i \cdot b_i) = u_i \oplus (a_i \cdot b_i) = u_i = a_i \oplus b_i$ by (7). Moreover, $(a_i \oplus b_i) + (a_i \cdot b_i) = u_i + \max(O_i, a_i + b_i - u_i) = a_i + b_i = x_i + y_i$. Since our claim holds in every coordinate G_i, then the claim is proved.

We now consider the general case $x = (a_1 + \cdots + a_n)$, $y = (b_1 + \cdots + b_m)$, with both $(a_1, ..., a_n)$ and $(b_1, ..., b_m)$ good sequences. We may limit attention to the case $m = 1$, i.e., $y = b \in A$. We define $a'_1, a'_2, ..., a'_{n+1} \in A$ as follows:

$$\begin{aligned}
a'_1 &= a_1 \oplus b \\
a'_2 &= a_2 \oplus a_1 \cdot b \\
a'_3 &= a_3 \oplus a_2 \cdot a_1 \cdot b \\
&\vdots \\
a'_n &= a_n \oplus a_{n-1} \cdot a_{n-2} \cdot \cdots \cdot a_1 \cdot b \\
a'_{n+1} &= 0 \oplus a_n \cdot a_{n-1} \cdot \cdots \cdot a_1 \cdot b.
\end{aligned}$$

Then, in the light of our claim, we have

$$\begin{aligned}
a'_1 &= a_1 + b - a_1 \cdot b \\
a'_2 &= a_2 + a_1 \cdot b - a_2 \cdot a_1 \cdot b \\
a'_3 &= a_3 + a_2 \cdot a_1 \cdot b - a_3 \cdot a_2 \cdot a_1 \cdot b \\
&\vdots \\
a'_n &= a_n + a_{n-1} \cdot a_{n-2} \cdot \cdots \cdot a_1 \cdot b + a_n \cdot a_{n-1} \cdot \cdots \cdot a_1 \cdot b \\
a'_{n+1} &= 0 + a_n \cdot \cdots \cdot a_1 \cdot b - 0.
\end{aligned}$$

This immediately shows that $a'_1 + \cdots + a'_{n+1} = a_1 + \cdots + a_n + b = x + y$.

Claim 2. $(a'_1, ..., a'_{n+1})$ is a good sequence.

Indeed, by definition, $a'_1,\dots, a'_{n+1} \in A$. Now, $a'_m \oplus a'_{m+1} = a_{m+1} \oplus a_m \oplus a_{m-1} \cdot \cdots \cdot a_1 \cdot b \oplus a_m \cdot \cdots \cdot a_1 \cdot b$. Applying now Claim 1 to the pair $(a_m \oplus a_{m-1} \cdot \cdots \cdot a_1 \cdot b,\ a_m \cdot \cdots \cdot a_1 \cdot b)$, we obtain $a'_m \oplus a'_{m+1} = a_{m+1} \oplus a_m \oplus a_{m-1} \cdot \cdots \cdot a_1 \cdot b = a_m \oplus a_{m-1} \cdot \cdots \cdot a_1 \cdot b = a'_m$, which settles our second claim, and completes the proof of Lemma 3.8.2.

3.8.3 LEMMA. *For all $x \in A^+$, $x \wedge u \in A$.*

Proof. Write $x = a_1 + \cdots + a_n$, with $(a_1,\dots, a_n)$ good.

Claim. $x \wedge u = a_1$.

Indeed, write $x = \{x_i\}_{i \in I}$, $a_1 = \{a_{1i}\}_{i \in I}$, $a_2 = \{a_{2i}\}_{i \in I},\dots$, and examine $A_i \subseteq G_i$. Three cases are possible:

Case 1. $a_{1i} = u_i$. Then $x_i \geqslant u_i$, hence $x_i \wedge u_i = u_i = a_{1i}$.

Case 2. $a_{1i} = O_i$. Then $a_{2i} = a_{3i} = \cdots a_{ni} = O_i$ by Lemma 3.8.1(ii), hence $x_i = O_i$, whence $x_i \wedge u_i = O_i = a_{1i}$.

Case 3. $a_{1i} \neq O_i,\ u_i$. Then $a_{21} = \cdots = a_{ni} = O_i$ by Lemma 3.8.1(ii), hence $x_i \wedge u_i = a_{1i} \wedge u_i = a_{1i}$.

Having proved our claim in each coordinate G_i, the lemma is proved. Note that the MV lattice operations are defined in terms of the MV algebraic operations, and the lattice group operations on G_i agree with the MV lattice operations on $A_i = \Gamma(G_i, u_i)$ by Theorem 2.5. Hence, the lattice operations on G agree with those on A.

Following common usage we let x_+ be short for $x \vee O$.

3.8.4 LEMMA. *If $x, y \in A^+$ then $(x - y)_+ \in A^+$.*

Proof. *Claim* 1. If $a, b \in A$ then $(a - b)_+ = a \cdot b^* \in A$.

As a matter of fact, in $A_i \subseteq G_i$ we have: $a_i \cdot b_i^* = (a_i^* \oplus b_i^{**})^* = u_i - \min(u_i, u_i - a_i + b_i) = \max(O_i, a_i - b_i) = (a_i - b_i) \vee O_i$. Since Claim 1 holds in each coordinate, then it holds in A.

Claim 2. Assume $(a_1,\dots, a_n)$ good, and $b \in A$. Then

$$(a_1 + a_2 + \cdots + a_n - b)_+ = (a_1 - b)_+ + a_2 + \cdots + a_n.$$

As a matter of fact, let us examine A_i. Three cases are possible:

Case 1. $a_{1i} = u_i$. Then $a_{1i} \geqslant b_i$, and hence, $(a_{1i} - b_i)_+ + a_{2i} + \cdots + a_{ni} = a_{1i} - b_i + a_{2i} + \cdots + a_{ni} = (a_{1i} + \cdots + a_{ni} - b_i)_+$.

Case 2. $a_{1i} = O_i$. Then by Lemma 3.8.1(i), (ii) we have $a_{2i} = \cdots = a_{ni} = O_i$ and $(a_{1i} - b_i)_+ + a_{2i} + \cdots + a_{ni} = (-b_i \vee O_i) + a_{2i} + \cdots + a_{ni} = O_i$. On the other hand, $(a_{1i} + \cdots + a_{ni} - b_i)_+ = -b_i \vee O_i = O_i$.

Case 3. $a_{1i} \neq O_i, u_i$. Then by Lemma 3.8.1(ii) we have $a_{2i} = \cdots = a_{ni} = O_i$ and $(a_{1i} - b_i)_+ + a_{2i} + \cdots + a_{ni} = (a_{1i} - b_i)_+$. On the other hand, $(a_{1i} + \cdots + a_{ni} - b_i)_+ = (a_{1i} - b_i)_+$. The proof of Claim 2 is complete.

Claim 3. For every $x \in \prod_{i \in I} G_i$ and $b_1, ..., b_n \in A$ we have

$$(x - b_1 - \cdots - b_n)_+ = (\cdots((x - b_1)_+ - b_2)_+ - \cdots - b_n)_+.$$

Proof. By induction on $n \geqslant 1$. The case $n = 1$ is trivial. Now $((\cdots(x - b_1)_+ - \cdots - b_n)_+ - b_{n+1})_+ = ((x - b_1 - \cdots - b_n)_+ - b_{n+1}) \vee 0 = ((x - b_1 - \cdots - b_{n+1}) \vee (0 - b_{n+1})) \vee 0 = (x - b_1 - \cdots - b_{n+1})_+$. This completes the proof of Claim 3.

We now conclude the proof of Lemma 3.8.4 as follows: Given $x, y \in A^+$ write $x = a_1 + \cdots + a_n$, $y = b_1 + \cdots + b_t$, with $(a_1, ..., a_n)$ and $(b_1, ..., b_t)$ both being good sequences, and $t = n$ without loss of generality. By Claim 3 we have $(x - y)_+ = (\cdots((x - b_1)_+ - b_2)_+ - \cdots - b_n)_+$. Note that $(x - b_1)_+ = (a_1 - b_1)_+ + a_2 + \cdots + a_n = a_1 \cdot b_1^* + a_2 + \cdots + a_n$, by Claims 2 and 1. Therefore, $(x - b_1)_+ \in A^+$ by Lemma 3.8.2. Let $x_1 = (x - b_1)_+$. By the same argument we obtain $(x_1 - b_2)_+ \in A^+$. Iterating this for n times we finally see that $(x - y)_+ \in A^+$.

3.8.5. *End of Proof of Theorem* 3.8. The set $H = A^+ - A^+ = \{g \in G \mid g = x - y \text{ for some } x, y \in A^+\}$ is a subgroup of G, by Lemma 3.8.2. Moreover, $g \in H$ implies $g_+ \in H$, by Lemma 3.8.4. By [2, 2.1.2], H is an l-subgroup of G, whence $H = G$ by definition (8) of G. Since $G = A^+ - A^+$, Lemma 3.8.4 also shows that $G^+ = A^+$. Thus if $g \in G$ and $0 \leqslant g \leqslant u$, then $g \in A^+$, and hence $g \wedge u = g \in A$ by Lemma 3.8.3. Therefore, recalling (7) at the beginning of the proof of Theorem 3.8, we have $A = \Gamma(G, u)$. The final statement in Theorem 3.8 concerning the cardinality of G is an immediate consequence of G being generated by A. ▮

3.9. THEOREM. *The functor Γ is an equivalence between the category of l-groups with order unit, and the category of MV algebras.*

Proof. In the light of [23, IV Theorem 1], it suffices to prove that Γ is full, faithful, and that every MV algebra is isomorphic to some MV algebra in the range of Γ. This is done in Proposition 3.5, Proposition 3.4, and Theorem 3.8, respectively. ▮

As an immediate consequence of Theorem 3.9 we have the following

3.10. COROLLARY. *The functor Γ from l-groups with order unit to MV algebras has the following property: For every MV algebra A there exists an l-group G with order unit u such that $A \cong \Gamma(G, u)$; (G, u) is uniquely determined by A, up to isomorphism.*

The results of Section 1 now motivate the following

3.11. DEFINITION. We define the map $\tilde{\Gamma}$ from AF C^*-algebras with lattice ordered K_0 into MV algebras, by writing $\tilde{\Gamma}(\mathfrak{A}) = \Gamma(K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$, for any such C^*-algebra $\mathfrak{A}$.

3.12. THEOREM. (i) *For every* AF C^**-algebra* $\mathfrak{A}$ *with lattice ordered* K_0, $\tilde{\Gamma}(\mathfrak{A})$ *is a countable MV algebra.*

(ii) *Given any two such* AF C^**-algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$, *we have*

$$\mathfrak{A} \cong \mathfrak{B} \qquad \textit{iff} \quad \tilde{\Gamma}(\mathfrak{A}) \cong \tilde{\Gamma}(\mathfrak{B}).$$

(iii) *For every countable MV algebra* A *there is an* AF C^**-algebra* $\mathfrak{A}$ *with lattice ordered* K_0, *such that* $A \cong \tilde{\Gamma}(\mathfrak{A})$.

Proof. (i) By Elliott's theorem (see 1.1 and 1.2) together with Theorem 2.5.

(ii) One side of the bi-implication is trivial. The other side follows from Theorems 1.2(ii) and Corollary 3.10.

(iii) By Theorem 3.8 and Theorem 1.2(i). ∎

The above theorem only deals with AF C^*-algebras with lattice-ordered K_0. Using Theorem 1.3 we can apply MV algebras to the whole class of AF C^*-algebras, as follows: Given an AF C^*-algebra $\mathfrak{A}$, let $(G, u) = (K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$. Following Section 1, let G_l be the free lattice-ordered group over G, let $\eta: G \to G_l$ be the natural embedding, and $u_l = \eta(u)$. The triplet $((G, u), \eta, (G_l, u_l))$ is uniquely determined by $\mathfrak{A}$. Let $A = \Gamma(G_l, u_l) = (A, \oplus, \cdot, {}^*, 0, 1)$, and let $B = \eta(G) \cap A$.

3.13. PROPOSITION. *Adopt the above notation: then we have*

(i) $B \subseteq A$, $0 \in B$, $1 = u_l \in B$.

(ii) B *has the Riesz interpolation property with respect to the MV order on* A.

(iii) *If* $x \in B$ *then* $x^* \in B$.

(iv) *If* $w_1, \ldots, w_n \in B$ *and* $a \in A$, *with* $a \sim (w_1, \ldots, w_n)$, *then* $a \in B$ (*see* 3.2 *for the definition of* $\sim$).

Proof. (i) Immediate, since η is an ordered group homomorphism which preserves order units.

(ii) By Elliott's theorem (see Sect. 1), (G, u) has the Riesz interpolation property; by Theorem 2.5, the MV order on A agrees with the group order on G_l.

(iii) Immediate, since $\eta(G)$ is closed under the group operations of G_l.

(iv) By 3.3, if $(w_1,..., w_n) \sim a$ then $w_1 + \cdots + w_n = a$, where $+$ is addition in G_l. Now $\eta(G)$ is closed under $+$, and $a \in A$, whence $a \in B$. ▮

3.14. THEOREM. *Adopt the above notation. Then the map sending each* AF C^**-algebra* $\mathfrak{A}$ *into the pair* (B, A) *has the following properties*:

(i) $A = \tilde{\Gamma}(\mathfrak{A}_l)$, *and* B *is a subset of* A *containing* 0, 1, *closed under* $*$, *and having the Riesz interpolation property with respect to the MV order on* A.

(ii) *For any two* AF C^**-algebras* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *we have*: $\mathfrak{A} \cong \mathfrak{A}'$ *iff there is an MV isomorphism* ϕ *of* A *onto* A' *such that* $\phi(B) = B'$.

Proof. (i) By Theorem 1.3 we get $A = \Gamma(G_l, u_l) \cong \Gamma((K_0(\mathfrak{A}))_l, [1_{\mathfrak{A}}]_l) \cong \Gamma(K_0(\mathfrak{A}_l), [1_{\mathfrak{A}_l}]) = \tilde{\Gamma}(\mathfrak{A}_l)$. By Proposition 3.13, B has the required properties.

(ii) If $\mathfrak{A} \cong \mathfrak{A}'$ then (G, u) and (G', u') are isomorphic as ordered groups with order unit; let ψ be an isomorphism. We can identify G with a subgroup of G_l, and identify G' with a subgroup of G'_l. Let $\psi_l: G_l \to G'_l$ be the induced l-isomorphism [34, 2.10]. Clearly, ψ_l preserves order units, and $\psi_l(G) = G'$. Therefore, the MV isomorphism $\Gamma(\psi_l)$ obeys the requirements of the theorem. Conversely,. let $((G, u), \eta, (G_l, u_l))$ be the triplet, where η is the order-embedding of $(G, u) = (K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$ into (G_l, u_l), with $u_l = \eta(u)$, as given by the Weinberg theorem. Let (H, u_l) denote the partially ordered group with order unit u_l, generated in G_l by $B = \eta(G) \cap [0, u_l]$, with the order induced from G_l. Since the Riesz decomposition property holds for $\eta(G)$, and u_l is an order unit for G_l, then each element of $(\eta(G))^+$ is a sum of elements of B. Moreover, each element $x \in \eta(G)$ has the form $x = y - z$ for some $y, z \in (\eta(G))^+$, because $\eta(G)$ is directed. We have just proved the first identity in the following line:

$$(H, u_l) = (\eta(G), u_l) \cong (G, u) = (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]). \tag{10}$$

Let now $\phi: A \to A'$ be the assumed MV isomorphism with $\phi(B) = B'$, where $A = \Gamma(G_l, u_l) = [0, u_l]$. Then by Theorem 3.9, ϕ can be uniquely extended to an l-isomorphism $\hat{\phi}: G_l \to G'_l$. Since $\phi(B) = B'$, the restriction of $\hat{\phi}$ to H is an isomorphism of the partially ordered groups H and H', where H' = group generated by B' in G'_l. Therefore, $(H, u_l) \cong (H', u'_l)$, whence by (10) we get $(K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \cong (K_0(\mathfrak{A}'), [1_{\mathfrak{A}'}])$. By Theorem 1.2 we have $\mathfrak{A} \cong \mathfrak{A}'$. ▮

(3.15) Recall [4] that an *ideal* in an MV algebra A is a subset $I \subseteq A$

such that (i) $0 \in I$, (ii) $x, y \in I \to x \oplus y \in I$, and (iii) $x \in I$, $y \in A \to x \cdot y \in I$. Equivalently, (iii) may be replaced by (iii′) $x \in I$, $x \geqslant y \in A \to y \in I$. Following common usage, we say that an ideal I is *proper* iff $I \neq A$. The *quotient* A/I is now defined in the usual way [4, 4.3]. Instead of A/I we may write A/R, where R is the *congruence relation* associated with I [4, p. 484]. Explicitly, $R \subseteq A^2$ is defined by xRy iff $x^* \cdot y \oplus x \cdot y^* \in I$.

3.16. COROLLARY. *Let A_ω be the free MV algebra with a denumerable set of free generators. Then for every* AF *C^*-algebra $\mathfrak{A}$ with lattice ordered K_0, there is a proper ideal $I \subseteq A_\omega$ such that $\tilde{\Gamma}(\mathfrak{A}) \cong A_\omega/I$. Conversely, for every proper ideal $I \subseteq A_\omega$ there is an* AF *C^*-algebra $\mathfrak{A}$ with lattice ordered K_0, such that $\tilde{\Gamma}(\mathfrak{A}) \cong A_\omega/I$. Moreover, $\mathfrak{A}$ is uniquely determined by I, up to isomorphism.*

Proof. Immediate from Theorem 3.12. ∎

(3.17) The involutive map sending each $B \subseteq A$ into $B^* = \{x^* \mid x \in B\}$ induces a canonical bijection between ideals and filters: in detail, a *filter* in an MV algebra A is a subset $F \subseteq A$ such that (i) $1 \in F$, (ii) $x, y \in F \to x \cdot y \in F$, and (iii) $x \in F$, $y \in A \to x \oplus y \in F$. Equivalently, (iii) may be replaced by (iii′): $x \in F$, $x \leqslant y \in A \to y \in F$. For any filter F we shall write A/F^* to denote the quotient of A by the ideal $I = F^* = \{x^* \mid x \in F\}$. For any subset B of A, the filter F_B *generated* by B is the intersection of all filters on A containing B.

3.18. LEMMA. *If $B = \varnothing$, then $F_B = \{1\}$. If $\varnothing \neq B \subseteq A$, then F_B is the set of those $x \in A$ such that $y_1 \cdot \cdots \cdot y_n \leqslant x$ for suitable $y_1, ..., y_n \in B$.*

Proof. The first assertion is obvious. If $\varnothing \neq B$, then F_B must contain the set $H = \{x \in A \mid \exists y_1, ..., y_n \in B \text{ with } y_1 \cdot \cdots \cdot y_n \leqslant x\}$. Conversely, $1 \in H$, since $B \neq \varnothing$. Moreover, if $x, y \in H$ then $x \cdot y \in H$ by [4, 1.8]. Finally, $x \in H$, $x \leqslant y \in A$ implies $y \in H$ by definition of H. Thus H is a filter, hence $H \supseteq F_B$, whence $H = F_B$. ∎

4. LINDENBAUM ALGEBRAS OF ŁUKASIEWICZ LOGIC

(4.1) This subsection is devoted to a presentation of the Łukasiewicz $\aleph_0$-valued sentential calculus [33, 31, 4]: the latter will provide an efficient tool for applying Corollary 3.10 to AF C^*-algebras.

Stripping away inessentials, we let Σ be the four element set

$$\Sigma = \{C, N, X, |\}.$$

We say that C, N, X and | are the *symbols* of the *alphabet* Σ; we denote by Σ^* the set of all *words* over Σ, i.e., the set of all finite strings of symbols of Σ. Words of the form X, $X|$, $X||$,... are called *sentential variables* [33, p. 39] ("statement" variables in [4, p. 472]). C and N are called the *implication* and *negation* symbol, respectively. Following [33, p. 39] we define the set S of *sentences* to be the smallest subset of Σ^* having the following properties:

(i) each sentential variable belongs to S,

(ii) if $p, q \in S$, then $Cpq \in S$ and $Np \in S$,

where, e.g., Cpq denotes the word over Σ obtained by juxtaposing symbol C, word p, and word q, in the given order. Sentences are called "formulas" in [4]. For any $p \in S$ we denote by $\|p\|$ the *length* of p, i.e., the number of occurrences of symbols of Σ in p.

An *assignment* is a map $h: \omega \to [0, 1] \cap \mathbb{Q}$, where $\omega = \{0, 1, \ldots\}$. For any $p \in S$, the *truth value of p under h*, in symbols, $p(h)$, is defined by induction on the length of p as follows:

(i) if p is $X | \cdots |_{n \text{ times}}$, then $p(h) = h(n)$, $n \in \omega$;

(ii) if p is Cqr, then $p(h) = \min(1, 1 - q(h) + r(h))$;

(iii) if p is Nq, then $p(h) = 1 - q(h)$.

By induction on $\|p\|$ one easily shows that if all the sentential variables occurring in p belong to the set $\{X, X|, \ldots, X| \cdots |_{m \text{ times}}\}$ and h, k are two assignments such that $h(0) = k(0), \ldots, h(m) = k(m)$, then $p(h) = p(k)$: indeed, $p(h)$ only depends on the restriction of the map h to the set of those $i \in \omega$ such that $X| \cdots |_{i \text{ times}}$ occurs in p.

A sentence $p \in S$ is *valid* iff $p(h) = 1$ for all assignments h [4, p. 487].

4.2. PROPOSITION. *The following are valid sentences, for all $p, q \in S$:*

(i) Cpp;

(ii) $CpCqq$.

Proof. (i) For every assignment k we have: $(Cpp)(k) = \min(1, 1 - p(k) + p(k)) = 1$. (ii) $(CpCqq)(k) = \min(1, 1 - p(k) + (Cqq)(k)) = \min(1, 1 - p(k) + 1) = 1$, for every k. ∎

The following theorem goes back to Lindenbaum [33, p. 48]:

4.3. THEOREM. *Let us call* generalized assignment *any map $h: \omega \to [0, 1]$. For every $p \in S$ define the truth value $p(h)$ precisely as is done in* **(4.1)** *for assignments. Then for every $q \in S$, q is valid iff $q(k) = 1$ for all generalized assignments k.*

Proof. One direction is trivial, because every assignment is a generalized assignment. In the other direction, assume $q(k) \neq 1$ for some generalized assignment $k: \omega \to [0, 1]$. Let $n \in \omega$ be such that the set $R = \{X, X|,..., X| \cdots |_{n \text{ times}}\}$ contains the set of variables occurring in q. As already noted for assignments, even for generalized assignments we have that whenever $\tilde{k}: \omega \to [0, 1]$ satisfies $\tilde{k}(0) = k(0),..., \tilde{k}(n) = k(n)$, then $q(\tilde{k}) = q(k)$. Equip $[0, 1]^{n+1}$ with the natural product topology, and consider for each $p \in S$ the function $\hat{p}: [0, 1]^{n+1} \to [0, 1]$ defined by $\hat{p}(x_0,..., x_n) =$ (common) truth value $p(h)$ of p under any generalized assignment h such that $h(0) = x_0,..., h(n) = x_n$.

By induction on $\|p\|$ one easily proves that $\hat{p}$ is continuous. Turning to our $q(k) \neq 1$, we have $\hat{q}(k(0),..., k(n)) < 1$, hence by continuity of the map $\hat{q}$, there exists an open neighborhood $N \subseteq [0, 1]^{n+1}$ of $(k(0),..., k(n))$ such that $\hat{q}(x_0,..., x_n) < 1$ for all $(x_0,..., x_n) \in N$. Now, N contains some point $(y_0,..., y_n)$ with $y_0,..., y_n \in \mathbb{Q}$, hence, letting $\tilde{h}: \omega \to [0, 1] \cap \mathbb{Q}$ be the assignment defined by $\tilde{h}(0) = y_0,..., \tilde{h}(n) = y_n$, and $\tilde{h}(m) = 0$ for all $m > n$, we conclude that $q(\tilde{h}) \neq 1$, whence q is not valid. ▮

4.4. COROLLARY. *For all $p, q \in S$ we have that $p(k) = q(k)$ for each assignment k iff $p(h) = q(h)$ for each generalized assignment h.*

Proof. One direction is trivial. The other is proved by the continuity argument used above. ▮

It turns out that generalized assignments are more useful in topological contexts (see 4.13–4.17 below) while assignments are useful in proof-theoretic and in recursion-theoretic applications, as in the following well-known result, whose proof is included here for the sake of completeness. We refer to [26] and to [6] for all notions of mathematical logic used in the rest of this paper.

4.5. THEOREM. *The set of valid sentences is a recursive subset of Σ^*.*

Proof. The celebrated completeness theorem for the Łukasiewicz $\aleph_0$-valued sentential calculus [31, 4, 5] immediately implies that the set V of valid sentences is recursively enumerable (r.e.). It is also evident from the definition, that the set S of sentences is a recursive subset of Σ^*. Thus for the proof of the theorem it suffices to show that $S \backslash V$ is r.e. We now describe a Turing machine M yielding the desired recursive enumeration of $S \backslash V$: M enumerates all triplets $(p, R, \bar{y})$, where $p \in S$, $R = \{X, X|,..., X| \cdots |_{n \text{ times}}\}$ is such that all the variables occurring in p belong to R, and $\bar{y} = (y_0,..., y_n)$ is an element of the set $(\mathbb{Q} \cap [0, 1])^{n+1}$. For any such triplet, M computes $\hat{p}(y_0,..., y_n)$, where $\hat{p}$ is the function introduced in the proof of Theorem 4.3; note that $\hat{p}(y_0,..., y_n) \in \mathbb{Q}$ and that the restriction of $\hat{p}$ to $\mathbb{Q}^{n+1}$ is recursive. Finally, M outputs p iff $\hat{p}(y_0,\ldots, y_n) \neq 1$ for some triplet $(p, R, \bar{y})$. The set of sentences $\{p \in S \mid M \text{ outputs } p\}$ coincides with $S \setminus V$, and is r.e.: indeed, p is not valid iff $p(h) \neq 1$ for some assignment h, iff $\hat{p}(y_0,\ldots, y_n) \neq 1$ for some rational $(y_0,\ldots, y_n)$. ∎

4.6. PROPOSITION. *On the set S of sentences define the binary relation $\equiv$ by stipulating that $p \equiv q$ holds iff both Cpq and Cqp are valid sentences. It follows that*

(i) *for all $p, q \in S$, $p \equiv q$ iff $p(h) = q(h)$ for each (generalized) assignment h;*

(ii) *$\equiv$ is an equivalence relation on S; any two valid sentences are $\equiv$-equivalent.*

Proof. As in [4, 5.2], in the light of Corollary 4.4 we may limit attention to generalized assignments. Now we have

$$\begin{aligned} p \equiv q \quad & \text{iff } (Cpq)(h) = 1 = (Cqp)(h) \text{ for all } h\colon \omega \to [0, 1], \\ & \text{iff } \min(1, 1 - p(h) + q(h)) = 1 = \min(1, 1 - q(h) + p(h)) \text{ for all } h, \\ & \text{iff } q(h) - p(h) \geqslant 0 \text{ and } p(h) - q(h) \geqslant 0 \text{ for all } h, \\ & \text{iff } q(h) = p(h) \text{ for all } h. \end{aligned}$$

This proves (i); (ii) is now immediate. ∎

In the light of Proposition 4.6, we shall denote by $[p]$ the equivalence class of the sentence $p \in S$ with respect to $\equiv$; by $S/\equiv$ we shall denote the set of all such equivalence classes.

4.7. THEOREM. *Over the set $S/\equiv$ we define operations $\oplus$, $\cdot$, $*$, and constant elements 0 and 1, as follows:*

$$\begin{aligned} 1 &= [CXX] \\ 0 &= [NCXX] \qquad \text{and for all } p, q \in S, \\ [p] \oplus [q] &= [CNpq] \\ [p] \cdot [q] &= [NCpNq] \\ [p]^* &= [Np]. \end{aligned}$$

Then the algebra $L = (S/\equiv, \oplus, \cdot, *, 0, 1)$ is a countable MV algebra. Indeed, L is the free MV algebra with the set of free generators $\{[X], [X|], [X||], \ldots\}$. For every AF *$C^*$-algebra $\mathfrak{A}$ with lattice ordered K_0 there is a proper ideal $I \subseteq L$ such that $\tilde{\Gamma}(\mathfrak{A}) \cong L/I$. Conversely, for every proper ideal*

$I \subseteq L$ there is a (unique, up to isomorphism) AF C^*-algebra $\mathfrak{A}$ with lattice ordered K_0, such that $\tilde{\Gamma}(\mathfrak{A}) \cong L/I$.

Proof. Freeness of L is well known [4, 5]. The last two assertions are an immediate consequence of Corollary 3.16. ▮

4.8. *Remarks.* (i) The above MV algebra L is known under the name of *Lindenbaum algebra* of the $\aleph_0$-valued sentential calculus [17].

(ii) By Proposition 4.6(ii), the valid sentence CXX in the definition of 1 in L may be equivalently replaced by any other valid sentence.

4.9. PROPOSITION. (i) $[NNp]=[p]$; (ii) $[Cpq]=[p]^* \oplus [q]$; (iii) $[CNpNq]=[Cqp]$; (iv) *if* $[p]=[p']$ *then* $[Cqp]=[Cqp']$.

Proof. (i) Immediate from 4.7. (ii) By 4.7, $[p]^* \oplus [q] = [Np] \oplus [q] = [CNNpq]$. Thus it suffices to show that $[CNNpq]=[Cpq]$. For every assignment $h\colon \omega \to [0,1]$ we have: $(CNNpq)(h) = \min(1, 1-(NNp)(h)+q(h)) = \min(1, 1-p(h)+q(h)) = (Cpq)(h)$. By 4.6 and 4.7 we conclude that $CNNpq \equiv Cpq$, as required. (iii) $[CNpNq] = [p] \oplus [Nq]$ by definition. On the other hand, $[Cqp]=[q]^* \oplus [p] = [Nq] \oplus [p]$ by (ii). (iv) For every $h\colon \omega \to [0,1]$, $(Cpq)(h) = \min(1, 1-q(h)+p(h)) = \min(1, 1-q(h)+p'(h)) = (Cqp')(h)$; now recall 4.6. ▮

4.10. PROPOSITION. *For all $p, q \in S$ the following are equivalent:*

(i) $[p] \leqslant [q]$ *in the MV order on L (Definition 2.2);*

(ii) $p(h) \leqslant q(h)$ *for every (generalized) assignment h;*

(iii) *Cpq is valid.*

Proof. (iii)$\leftrightarrow$(ii). Cpq valid iff $(Cpq)(h)=1$ for every h, iff $1=\min(1, 1-p(h)+q(h))$ for every h, iff $0 \leqslant q(h)-p(h)$ for every h.

(iii)$\leftrightarrow$(i).

$[p] \leqslant [q]$ iff $[p]^* \oplus [q] = 1$ [4, 1.13] and Theorem 4.7,

iff $[Cpq]=[CXX]$, by 4.9(ii),

iff $(Cpq)(h)=1$ for all h, by 4.6 and 4.2,

iff Cpq is valid. ▮

4.11. PROPOSITION. *For any $q_1, \ldots, q_n, p \in S$ we have $Cq_1Cq_2 \cdots Cq_n p$ is valid iff $[q_1] \cdot \cdots \cdot [q_n] \leqslant [p]$.*

Proof. By induction on $n \geqslant 1$. The case $n=1$ is contained in

Proposition 4.10. Assuming now the proposition to hold up to n, we shall prove it for $n+1$. To this purpose we need the following abbreviation. For all $q, r \in S$ we let Lqr be short for $NCqNr$, whence

$$[Lqr] = [q] \cdot [r]. \tag{11}$$

Claim. $Cq_1 \cdots Cq_n p \equiv CLq_1 Lq_2 \cdots Lq_{n-1} q_n p$ $(n \geqslant 2)$.

Proof of Claim. By induction on n. *Basis*, $n=2$:

$$\begin{aligned}
[CLq_1 q_2 p] &= [Lq_1 q_2]^* \oplus [p] && \text{by 4.9(ii)},\\
&= ([q_1] \cdot [q_2])^* \oplus [p], && \text{by (11)},\\
&= [q_1]^* \oplus [q_2]^* \oplus [p], && \text{by } Ax6' \text{ in 2.1},\\
&= [q_1]^* \oplus ([q_2]^* \oplus [p]) &&\\
&= [q_1]^* \oplus [Cq_2 p] && \text{by 4.9(ii)},\\
&= [Cq_1 Cq_2 p], && \text{again by 4.9(ii)}.
\end{aligned}$$

Induction step. In the light of Proposition 4.9(iv) we have

$$\begin{aligned}
&Cq_1 Cq_2 \cdots Cq_n Cq_{n+1} p\\
&\quad \equiv Cq_1 CLq_2 Lq_3 \cdots Lq_n q_{n+1} p && \text{induction hypothesis}\\
&\quad \equiv CLq_1 Lq_2 \cdots Lq_n q_{n+1} p && (\text{by case } n=2),
\end{aligned}$$

which settles our claim.

Now we have $Cq_1 \cdots Cq_n p$ is valid iff $CLq_1 \cdots Lq_{n-1} q_n p$ is valid (Claim, 4.6) iff $[Lq_1 \cdots Lq_{n-1} q_n] \leqslant [p]$ (by 4.10) iff $[q_1] \cdot \cdots \cdot [q_n] \leqslant [p]$, by repeated application of (11) together with associativity of multiplication. ∎

4.12. Corollary. (i) *$Cq_1 Cq_2 \cdots Cq_n r$ is valid iff $Cq_1 Cq_2 \cdots Cq_n NNr$ is valid;*

(ii) *$Cq_1 Cq_2 \cdots Cq_n r$ valid implies $Cq_1 \cdots Cq_n Cq_{n+1} r$ valid;*

(iii) *If v is valid, then $Cq_1 Cq_2 \cdots Cq_n r$ is valid iff $Cq_1 Cq_2 \cdots Cq_n Cvr$ is valid.*

Proof. (i) Immediate from Propositions 4.11 and 4.9(i).

(ii) Immediate from Proposition 4.11, noting that $[q_1] \cdot [q_2] \cdot \cdots \cdot [q_n] \geqslant [q_1] \cdot [q_2] \cdot \cdots \cdot [q_n] \cdot [q_{n+1}]$ by monotony of multiplication [4, 1.8, or 1.10].

(iii) Using Proposition 4.11 and Remarks 4.8 we have that $Cq_1 \cdots Cq_n Cvr$ is valid iff $[q_1] \cdot \cdots \cdot [q_n] \cdot [v] \leqslant [r]$ iff $[q_1] \cdot \cdots \cdot [q_n] \leqslant [r]$ iff $Cq_1 \cdots Cq_n r$ is valid. ∎

In the rest of this section we deal with the McNaughton representation of the free MV algebra L. This representation will not be used until Section 8. We let $[0, 1]^n$ and $[0, 1]^\omega$ denote the product of n (resp., denumerably many) copies of the real unit interval with the product topology. Elements of the Hilbert cube $[0, 1]^\omega$ are nothing else but our generalized assignments of Theorem 4.3 and, accordingly, will be denoted by $h, k, \ldots$. As usual, $\mathbb{R}$ denotes the set of real numbers.

4.13. Definition [25, p. 2]. A function $f: [0, 1]^n \to \mathbb{R}$ is called a *McNaughton function over* $[0, 1]^n$ iff f obeys the following conditions:

(i) f is continuous, and

(ii) there are a finite number of distinct polynomials $\alpha_1, \ldots, \alpha_m$, each $\alpha_j = b_j + a_{1j}x_1 + \cdots + a_{nj}x_n$, where all a's and b's are integers, such that for every $(x_1, \ldots, x_n) \in [0, 1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x_1, \ldots, x_n) = \alpha_i(x_1, \ldots, x_n)$. ∎

In his original definition McNaughton also required that range$(f) \subseteq [0, 1]$. Compare with Theorem 4.15 below.

4.14. Proposition. *Call a function* $g: [0, 1]^\omega \to \mathbb{R}$ *a* McNaughton function over $[0, 1]^\omega$ *iff for some integer* $n \geqslant 1$ *there is a McNaughton function* f *over* $[0, 1]^n$ *such that for all* $h \in [0, 1]^\omega$ *we have* $g(h) = f(h(0), \ldots, h(n-1))$. *Then the McNaughton functions over* $[0, 1]^\omega$ *with pointwise operations form an l-group* M *of continuous functions, in which the constant* 1 *is an order unit.*

4.15. Theorem. *Up to isomorphism,* $(M, 1)$ *is the only l-group with order unit such that* $L \cong \Gamma(M, 1)$.

Proof. McNaughton [25, Theorem 2] proved that L is isomorphic to the MV algebra A given by those McNaughton functions over $[0, 1]^\omega$ whose range is contained in $[0, 1]$, with pointwise MV operations. Now, $A = \Gamma(M, 1)$. Uniqueness of $(M, 1)$ follows from Corollary 3.10. ∎

4.16. Corollary. (i) *There is a denumerable set* $Y \subseteq \Gamma(M, 1)$ *such that* $Y \cup \{1\}$ *generates* M, *and for every l-group* G *with order unit* u, *and every map* $\lambda: Y \to [0_G, u]$ *there is a unital l-homomorphism* $\bar{\lambda}: (M, 1) \to (G, u)$ *extending* λ.

(ii) *Property* (i) *characterizes* $(M, 1)$ *up to isomorphism.*

(iii) *H is a countable l-group with order unit w iff $(H, w) \cong (M/J, 1/J)$ for some l-ideal J of M.*

Proof. (i) For each $i \in \omega$ let the canonical projection $p_i: [0, 1]^\omega \to [0, 1]$ be defined by $p_i(h) = h(i)$ for all $h \in [0, 1]^\omega$. Identify $\Gamma(M, 1)$ and L using Theorem 4.15. The set $Y = \{p_0, p_1, ...\}$ is a free generating set in the free MV algebra L [4, 5]. Therefore λ can be uniquely extended to an MV homomorphism $\bar{\lambda}: L \to \Gamma(G, u)$. By Theorem 3.9 there is a unital l-homomorphism $\hat{\lambda}: (M, 1) \to (G, u)$ with $\Gamma(\hat{\lambda}) = \bar{\lambda}$. Since Y generates the MV algebra L, and the MV operations are definable in terms of the order unit 1 together with the l-group operations of M, it follows that $Y \cup \{1\}$ generates the l-group M.

(ii) The usual proof of uniqueness of free algebras [16, p. 163] can be easily adapted to $(M, 1)$ using the equivalence Γ (3.9).

(iii) This is immediate from (i). ∎

The l-group $(M, 1)$ of McNaughton functions over the Hilbert cube "separates points" in the following strong sense:

4.17. PROPOSITION. *Let $U \subseteq [0, 1]^\omega$ be open, and $k \in U$. Then there is an $f \in M$ such that $f(k) = 0$ and $f(h) = 1$ for all $h \in [0, 1]^\omega \setminus U$.*

Proof. Assume $U = \{h \in [0, 1]^\omega \mid m/n < h(0) < p/q\}$ for some $m, n, p, q \in \omega$. Then since $k \in U$ we have $m/n < k(0) < p/q$ and there exist m', n', p', and $q' \in \omega$ such that $m/n < m'/n' < k(0) < p'/q' < p/q$. Let $v: \mathbb{R} \to \mathbb{R}$ be the function defined by $v(x) = m' - n'x$. Then there is a natural number c such that $cv(x) \geqslant 1$ for all $x \leqslant m/n$ and $cv(x) \leqslant 0$ for all $x \geqslant m'/n'$. Similarly, letting $w(x) = -p' + q'x$, there exists $d \in \omega$ such that $dw(x) \geqslant 1$ for all $x \geqslant p/q$ and $dw(x) \leqslant 0$ for all $x \leqslant p'/q'$. Let $r: [0, 1] \to \mathbb{R}$ be the restriction to $[0, 1]$ of the function $((cv \vee 0) \wedge 1) \vee (dw \vee 0) \wedge 1)$. Then r is a McNaughton function over $[0, 1]$. In addition we have

$$r(k(0)) = 0, \qquad r(z) \geqslant 0 \qquad \text{for all } z \in [0, 1],$$

and

$$r(x) = 1 \qquad \text{for all } x \text{ with } x \leqslant m/n \text{ or } x \geqslant p/q.$$

The McNaughton function f over $[0, 1]^\omega$ defined by $f(h) = r(h(0))$ for all $h \in [0, 1]^\omega$, has the required properties.

In case $U = \{h \in [0, 1]^\omega \mid m_0/n_0 < h(0) < p_0/q_0, ..., m_t/n_t < h(t) < p_t/q_t\}$ for some $m_0, n_0, p_0, q_0, ..., m_t, n_t, p_t, q_t \in \omega$, since $k \in U$, then $m_i/n_i < k(i) < p_i/q_i$ for all $i = 0, ..., t$. Arguing as we have done in the first case, we exhibit McNaughton functions $r_0, \ldots, r_t$ over $[0, 1]$ obeying the following conditions, for all $i = 0, \ldots, t$:

$$r_i(k(i)) = 0, \qquad r_i(z) \geqslant 0 \qquad \text{for all } z \in [0, 1],$$

and

$$r_i(x) = 1 \qquad \text{for all } x \text{ with } x \leqslant m_i/n_i \text{ or } x \geqslant p_i q_i.$$

Define the function s: $[0, 1]^\omega \to \mathbb{R}$ by

$$s(h) = r_0(h(0)) + r_1(h(1)) + \cdots + r_t(h(t)) \qquad \text{for all} \quad h \in [0, 1]^\omega.$$

Then s is a McNaughton function over $[0, 1]^\omega$ having the following properties

$$s(k) = \sum r_i(k(i)) = 0, \qquad s(h) \geqslant 0 \qquad \text{for all } h \in [0, 1]^\omega,$$

and

$$s(h) \geqslant 1 \qquad \text{whenever } h \in [0, 1]^\omega \setminus U.$$

The function $f = 1 \wedge s$ has the required properties to settle the proposition in the case under discussion. In general, every open $U' \subseteq [0, 1]^\omega$ will contain a basic opn U of the form given above (second case), with $k \in U$. The function f constructed for U will be good for U', too. This completes the proof of our proposition. ∎

5. Lindenbaum Algebras of Theories in Łukasiewicz Logic

5.1. Following model-theoretic usage [6, 26] we call *theory* of L any subset of S. Given theory $\Theta \subseteq S$, in case $\Theta \neq \varnothing$, the set $\tilde{\Theta}$ of (syntactic) *consequences* of Θ is defined by:

$$\tilde{\Theta} = \{p \in S \mid \exists q_1, \ldots, q_n \in \Theta \text{ such that } Cq_1 Cq_2 \cdots Cq_n p \text{ is valid}\}.$$

In case $\Theta = \varnothing$, then we let $\tilde{\Theta}$ = set of all valid sentences. For any theory Θ we denote by $\Theta/\equiv$ the subset of L given by

$$\Theta/\equiv \; = \{[p] \in L \mid p \in \Theta\},$$

and we let F_Θ denote the filter generated by $\Theta/\equiv$ according to **3.17**, Lemma 3.18. Dually, the ideal I_Θ is defined by $I_\Theta = F_\Theta^* = \{[p] \in L \mid [p]^* \in F_\Theta\} = \{[p] \in L \mid [Np] \in F_\Theta\}$. For any theory Θ, the *Lindenbaum algebra* of Θ is the quotient L/I_Θ of L by the ideal I_Θ (compare with [6] for the 2-valued case). Note that I_Θ is a proper ideal iff $0 \neq 1$ in L/I_Θ.

5.2. PROPOSITION. *Given any two theories Θ and Φ we have*:

(i) *each valid sentence belongs to* $\tilde{\Theta}$;

(ii) $\Theta \subseteq \tilde{\Theta}$;

(iii) $\tilde{\tilde{\Theta}} = \tilde{\Theta}$; *in particular*, $\tilde{\tilde{\varnothing}} = \tilde{\varnothing} =$ *all valid sentences*;

(iv) $\Phi \subseteq \Theta$ *implies* $\tilde{\Phi} \subseteq \tilde{\Theta}$.

Proof. (i) If $\Theta = \varnothing$ then the conclusion immediately follows from the definition of $\tilde{\varnothing}$. If $\Theta \neq \varnothing$ then let $p \in \Theta$; for every valid sentence $t \in S$ we have: Cpt is valid iff $[p] \leqslant [t]$ iff $[p] \leqslant 1$, using Proposition 4.10 and Remarks 4.8. Hence Cpt is valid, whence $t \in \tilde{\Theta}$.

(ii) To avoid trivialities assume $\Theta \neq \varnothing$ and let $p \in \Theta$. Then $p \in \tilde{\Theta}$ because Cpp is valid (Proposition 4.2).

(iii) Assume first $\Theta = \varnothing$: then $p \in \tilde{\tilde{\varnothing}}$ iff $Cq_1 \cdots Cq_n p$ is valid for suitable $q_1,..., q_n \in \tilde{\varnothing} =$ set of valid sentences, iff $[q_1] \cdot \cdots \cdot [q_n] \leqslant [p]$, iff $1 \leqslant [p]$, iff p is valid, by 4.11, 4.8, and 4.10. Thus, $\tilde{\tilde{\varnothing}} = \tilde{\varnothing}$, as required. If $\Theta \neq \varnothing$ then, in the light of (ii), it is sufficient to show that $\tilde{\tilde{\Theta}} \subseteq \tilde{\Theta}$. To this purpose, first note that by (i), $\tilde{\Theta} \neq \varnothing$ whence, by (ii), $\tilde{\tilde{\Theta}} \neq \varnothing$. If $p \in \tilde{\tilde{\Theta}}$ then $Cq_1 \cdots Cq_n p$ is valid, for suitable $q_1,..., q_n \in \tilde{\Theta}$; therefore, by 4.11, $[q_1] \cdot \cdots \cdot [q_n] \leqslant [p]$. For each q_i $(i = 1,..., n)$ there are $q^i_1,..., q^i_{m(i)} \in \Theta$ such that $Cq^i_1 \cdots Cq^i_{m(i)} q_i$ is valid, i.e., by 4.11, $[q^i_1] \cdot \cdots \cdot [q^i_{m(i)}] \leqslant [q_i]$. In conclusion, using monotony of multiplication [4, 1.10] we obtain:

$$\prod_{i=1}^{n} \prod_{j=1}^{m(i)} [q^i_j] \leqslant \prod_{i=1}^{n} [q_i] \leqslant [p],$$

which shows that $p \in \tilde{\Theta}$, by another application of 4.11. (iv) Obvious. ∎

5.3. PROPOSITION. *For every $p \in S$ and $\Theta \subseteq S$ the following are equivalent*:

(i) $[p] \in F_\Theta$;

(ii) $[p] \in F_{\tilde{\Theta}}$;

(iii) $p \in \tilde{\Theta}$.

Proof. In case $\Theta = \varnothing$ then $\tilde{\Theta} =$ all valid sentences, $\Theta/\equiv\, = \varnothing$, $F_\Theta = \{1\} \subseteq L$ (Lemma 3.18). Also, $F_{\tilde{\Theta}}$ is the filter generated by the set of all $[p]$ such that p is valid, i.e., the filter generated by the element $1 \in L$, by 4.8. Therefore, $F_\Theta = F_{\tilde{\Theta}} = \{1\}$, and $[p] \in \{1\}$ iff $[p] = 1$ iff p is valid iff $p \in \tilde{\Theta}$.

Now consider the case $\Theta \neq \emptyset$. (i) $\to$ (ii) holds by 5.2. (iii) $\to$ (ii) holds, i.e., $p \in \tilde{\Theta}$ implies $[p] \in F_{\tilde{\Theta}}$, as Cpp is a valid sentence (4.2). We now prove (i) $\leftrightarrow$ (iii):

$$\begin{aligned} [p] \in F_\Theta \quad & \text{iff } [p] \text{ belongs to the filter generated by } \Theta/\equiv, \\ & \text{iff } [p] \geqslant y_1 \cdot \cdots \cdot y_n, \text{ for suitable } y_i \in \Theta/\equiv, \\ & \text{iff } [p] \geqslant [q_1] \cdot \cdots \cdot [q_n], \text{ for suitable } q_i \in \Theta, \\ & \text{iff } Cq_1 \cdots Cq_n p \text{ is valid (4.11), iff } p \in \tilde{\Theta}. \end{aligned}$$

(ii) $\to$ (iii). $[p] \in F_{\tilde{\Theta}}$ iff $[q_1] \cdot \cdots \cdot [q_n] \leqslant [p]$ for suitable $q_i \in \tilde{\Theta}$ (arguing as in the above proof of (i) $\leftrightarrow$ (iii)). It follows that for all $i = 1, \ldots, n$, there exist $q^i_j \in \Theta$, as in the proof of 5.2(iii), such that $[q^i_1] \cdot \cdots \cdot [q^i_{m(i)}] \leqslant [q_i]$. Using monotony of multiplication [4, 1.10] we finally obtain

$$\prod_{i=1}^{n} \prod_{j=1}^{m(i)} [q^i_j] \leqslant \prod_{i=1}^{n} [q_i] \leqslant [p],$$

which shows that $p \in \tilde{\Theta}$, again by 4.11. ▮

5.4. COROLLARY. *For every* $\Theta \subseteq S$, $I_\Theta = I_{\tilde{\Theta}}$.

5.5. PROPOSITION. *Let D be a filter of L. Then there is a theory $\Theta \subseteq S$ such that $D = F_\Theta$. Furthermore, Θ may be so chosen that $\Theta = \tilde{\Theta}$.*

Proof. Define $\Theta = \{p \in S \mid [p] \in D\}$. Note that $\Theta \neq \emptyset$ since $1 \in D$, whence, say, $CXX \in \Theta$. We *claim* that $F_\Theta = D$. For every $p \in S$ we have by Proposition 5.3,

$$[p] \in F_\Theta \qquad \text{iff} \quad p \in \tilde{\Theta}. \tag{12}$$

On the other hand, by definition of Θ we have

$$[p] \in D \qquad \text{iff} \quad p \in \Theta. \tag{13}$$

Thus, to prove our claim it is sufficient to prove that $\Theta = \tilde{\Theta}$; since $\Theta \subseteq \tilde{\Theta}$, by Proposition 5.2, then it is sufficient to prove $\tilde{\Theta} \subseteq \Theta$. To this purpose, for every $p \in S$ we have

$$\begin{aligned} p \in \tilde{\Theta} \quad & \text{implies } [q_1] \cdot \cdots \cdot [q_n] \leqslant [p], \text{ for suitable } q_i \in \Theta, \text{ by 4.11}, \\ & \text{implies } y_1 \cdot \cdots \cdot y_n \leqslant [p], \text{ for suitable } y_i \in D, \text{ by (13)}, \\ & \text{implies } y \leqslant [p], \text{ for some } y \in D, \end{aligned}$$

since D is closed under products (3.12). A fortiori, $[p] \in D$, hence $p \in \Theta$ by (13). ▮

5.6. COROLLARY. *Let I be an ideal of L. Then there is a theory $\Theta \subseteq S$ such that $I_\Theta = I$ and $\Theta = \tilde{\Theta}$.*

The following result states that Lindenbaum algebras of theories in L yield the most general countable MV algebra.

5.7. COROLLARY. *For every countable MV algebra A there is a theory $\Theta \subseteq S$ such that $A \cong L/I_\Theta$.*

Proof. By Corollary 5.6 and Theorem 4.7. ▮

5.8. DEFINITION. Given a countable MV algebra A, we let $T(A)$ be defined by $T(A) = \{\Theta \subseteq S \mid A \cong L/I_\Theta\}$. We also let θ be the map which associates with each AF C^*-algebra $\mathfrak{A}$ with lattice ordered K_0, the set $\theta(\mathfrak{A}) = T(\tilde{\Gamma}(\mathfrak{A}))$.

5.9. THEOREM. *The map θ has the following properties*:

(i) *For every* AF *C^*-algebra $\mathfrak{A}$ with lattice ordered K_0, $\theta(\mathfrak{A})$ is a nonempty set of theories in the Łukasiewicz $\aleph_0$-valued sentential calculus. For every theory $\Theta \subseteq S$, we have*

$$\Theta \in \theta(\mathfrak{A}) \qquad \text{iff} \quad \tilde{\Gamma}(\mathfrak{A}) \cong L/I_\Theta.$$

(ii) *For any two* AF *C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with lattice ordered K_0, $\mathfrak{A} \cong \mathfrak{B}$ iff $\theta(\mathfrak{A}) = \theta(\mathfrak{B})$ iff $\theta(\mathfrak{A}) \cap \theta(\mathfrak{B}) \neq \emptyset$.*

(iii) *For every* consistent *theory $\Theta \subseteq S$ (i.e., $I_\Theta \neq L$) there is a (unique, up to isomorphism)* AF *C^*-algebra $\mathfrak{A}$ with lattice-ordered K_0, such that $\Theta \in \theta(\mathfrak{A})$.*

Proof. Immediate from Corollary 5.7 and Theorem 4.7. ▮

5.10. We now study concrete representations of L/I_Θ. Note that each element of L/I_Θ is an equivalence class of elements of L, each element of L being itself an equivalence class of sentences. We shall represent elements of L/I_Θ as equivalence classes of sentences. To this purpose, for every theory $\Theta \subseteq S$ we define the binary relation $\equiv_\Theta$ between sentences $p, q \in S$, as follows:

$$p \equiv_\Theta q \qquad \text{iff} \quad Cpq \in \tilde{\Theta} \qquad \text{and} \qquad Cqp \in \tilde{\Theta}.$$

If $\Theta = \varnothing$ then $\equiv_\Theta$ coincides with $\equiv$. Thus, unless otherwise stated, we shall assume $\Theta \neq \varnothing$.

5.11. PROPOSITION. *For every theory $\Theta \subseteq S$, the following hold:*

(i) *$\equiv_\Theta$ is an equivalence relation on S;*

(ii) *$\equiv_\Theta$ is coarser than $\equiv$ (i.e., $p \equiv q$ implies $p \equiv_\Theta q$);*

(iii) *$\equiv_\Theta$ preserves the negation symbol N ($p \equiv_\Theta q \to Np \equiv_\Theta Nq$);*

(iv) *$\equiv_\Theta$ preserves the implication symbol C ($p \equiv_\Theta p'$ and $q \equiv_\Theta q' \to Cpq \equiv_\Theta Cp'q'$).*

Proof. (i) $p \equiv_\Theta p$ holds, because Cpp is valid (4.2), hence, given $q \in \Theta$, $CqCpp$ is valid (4.2). Trivially, $\equiv_\Theta$ is symmetric. To prove transitivity, assume $p \equiv_\Theta q$ and $q \equiv_\Theta r$. Using 4.11 and 4.9(ii) we obtain

$$[w_1] \cdot \cdots \cdot [w_n] \leqslant [Cpq] = [p]^* \oplus [q] \qquad \text{for suitable } w_i \in \Theta,$$

and

$$[v_1] \cdot \cdots \cdot [v_m] \leqslant [Cqr] = [q]^* \oplus [r] \qquad \text{for suitable } v_j \in \Theta.$$

By monotony of the product [4, 1.10], [4, 3.1], 4.9(ii), and $Ax3'$ in 2.1, we have the following inequalities: $[w_1] \cdot \cdots \cdot [w_n] \cdot [v_1] \cdot \cdots \cdot [v_m] \leqslant [Cpq] \cdot [Cqr] = ([q]^* \oplus [r]) \cdot ([p]^* \oplus [q]) \leqslant [p]^* \oplus (([q]^* \oplus [r]) \cdot [q]) \leqslant [p]^* \oplus [r] \oplus [q] \cdot [q]^* = [p]^* \oplus [r] = [Cpr]$, which shows that $Cpr \in \tilde{\Theta}$; using the hypotheses that Crq, $Cqp \in \tilde{\Theta}$ one similarly proves that $Crp \in \tilde{\Theta}$; therefore $p \equiv_\Theta r$.

(ii) $p \equiv q$ iff Cpq and Cqp are both valid; letting $r \in \Theta$ we have a fortiori, $CrCpq$ and $CrCqp$ both valid (by 4.11, or 4.12(iii)), whence $p \equiv_\Theta q$.

(iii) By 5.3, $\{Cpq, Cqp\} \subseteq \tilde{\Theta}$ if and only if $\{[Cpq], [Cqp]\} \subseteq F_{\tilde{\Theta}}$. But by 4.9(iii), $\{[Cpq], [Cqp]\} = \{[CNpNq]\}, [CNqNp]\}$.

(iv) By hypothesis and 4.11, together with 4.9(ii) we can write, for suitable $v_1, \ldots, v_n, w_1, \ldots, w_m, r_1, \ldots, r_z, s_1, \ldots, s_t \in \Theta$:

$$[v_1] \cdot \cdots \cdot [v_n] \leqslant [Cpp'] = [p]^* \oplus [p'] \tag{14}$$

$$[w_1] \cdot \cdots \cdot [w_m] \leqslant [Cp'p] = [p']^* \oplus [p] \tag{15}$$

$$[r_1] \cdot \cdots \cdot [r_z] \leqslant [Cqq'] = [q]^* \oplus [q'] \tag{16}$$

$$[s_1] \cdot \cdots \cdot [s_t] \leqslant [Cq'q] = [q']^* \oplus [q]. \tag{17}$$

Now

$$\begin{aligned}[CCpqCp'q'] &= [Cpq]^* \oplus [Cp'q'] \\ &= ([p]^* \oplus [q])^* \oplus [p']^* \oplus [q'], && \text{by 4.9(ii)}, \\ &= [p] \cdot [q]^* \oplus [p']^* \oplus [q'], && \text{by } Ax6, 7 \text{ in 2.1}, \\ &\geqslant [q]^* \cdot ([p] \oplus [p']^*) \oplus [q'], && [4, 3.1], \\ &\geqslant ([q]^* \oplus [q']) \cdot ([p] \oplus [p']^*), && \text{by } [4, 3.1], \\ &\leqslant [w_1] \cdot \cdots \cdot [w_m] \cdot [r_1] \cdot \cdots \cdot [r_z] && \text{by (2), (3) and } [4, 1.10],\end{aligned}$$

which shows that $CCpqCp'q' \in \tilde{\Theta}$ in the light of 4.11. One similarly proves that $CCp'q'Cpq \in \tilde{\Theta}$ using (14) and (17). ∎

5.12. Given a theory $\Theta \subseteq S$, for every $p \in S$ let us denote by $\langle p \rangle$ the equivalence class of p with respect to $\equiv_\Theta$. The fact that $\equiv_\Theta$ is coarser than $\equiv$ (Proposition 5.11(ii)) may be equivalently restated as follows:

$$\langle p \rangle = \bigcup \{[q] \in L \mid q \equiv_\Theta p\}. \tag{18}$$

Therefore we can define the equivalence relation $\approx$ on L by the following stipulation

$$[p] \approx [q] \quad \text{iff} \quad \langle p \rangle = \langle q \rangle \text{ (i.e., iff } p \equiv_\Theta q), \tag{19}$$

for any $p, q \in S$. We can also define the (quotient) map α: $L/\approx \to S/\equiv_\Theta$ by

$$[p]/\approx \stackrel{\alpha}{\mapsto} \langle p \rangle, \tag{20}$$

for any $p \in S$, where $[p]/\approx = \{[q] \mid [q] \approx [p]\}$ is the $\approx$-equivalence class of $[p]$. Recall from 3.15 the definition of the congruence relation associated with an ideal. See 5.1 for the definition of the ideal I_Θ.

5.13. PROPOSITION. *For any theory $\Theta \subseteq S$, let R_Θ be the congruence relation on L associated with the ideal I_Θ. Let $\approx$ be the above equivalence relation. Then R_Θ coincides with $\approx$.*

Proof. If $\Theta = \varnothing$ then $I_\Theta = \{0\}$ and R_Θ is the equality relation on L, since $[p]\, R_\Theta [q]$ holds iff $[p]^* \cdot [q] \oplus [p] \cdot [q]^* = 0$ iff $[p] = [q]$, by [4, 3.14]. On the other hand, when $\Theta = \varnothing$, then Eq. (19) becomes $[p] \approx [q]$ iff $p \equiv q$ iff $[p] = [q]$, because $\equiv_\Theta$ then coincides with $\equiv$. Thus, $\approx = R_\Theta$. We now deal with the case $\Theta \neq \varnothing$. For arbitrary $p, q \in S$ we have

$$\begin{aligned}[p]\, R_\Theta [q] \quad &\text{iff} \quad [p]^* \cdot [q] \oplus [p] \cdot [q]^* \in I_\Theta \\ &\text{iff} \quad [p]^* \cdot [q] \in I_\Theta \text{ and } [p] \cdot [q]^* \in I_\Theta;\end{aligned}$$

indeed, the $\rightarrow$-direction holds because of the monotony of addition [4, 1.10], ideals being closed under minorants (3.15); the $\leftarrow$-direction holds because ideals are closed under addition. Letting now $F_\Theta = I_\Theta^*$ (**5.1.**), we can write

$$
\begin{aligned}
[p]\, R_\Theta [q] \ &\text{iff} \ ([p]^* \cdot [q])^* \in F_\Theta \text{ and } ([p] \cdot [q]^*)^* \in F_\Theta \\
&\text{iff} \ [p] \oplus [q]^* \in F_\Theta \text{ and } [p]^* \oplus [q] \in F_\Theta \qquad \text{by } Ax7,\ 6' \text{ in 2.1,} \\
&\text{iff} \ [Cpq] \in F_\Theta \text{ and } [Cqp] \in F_\Theta, \qquad \text{by 4.9(ii),} \\
&\text{iff} \ Cpq \in \tilde{\Theta} \text{ and } Cqp \in \tilde{\Theta}, \qquad \text{by Proposition 5.3,} \\
&\text{iff} \ p \equiv_\Theta q \\
&\text{iff} \ [p] \approx [q]. \ \blacksquare
\end{aligned}
$$

5.14. PROPOSITION. *Given a theory* $\Theta \subseteq S$ *define on* $S/\equiv_\Theta$ *operations* $\oplus, \cdot, {}^*$ *and elements* 0_Θ, 1_Θ *as follows*:

$$1_\Theta = \langle CXX \rangle$$

$$0_\Theta = \langle NCXX \rangle$$

$$\langle p \rangle \oplus \langle q \rangle = \langle CNpq \rangle$$

$$\langle p \rangle \cdot \langle q \rangle = \langle NCpNq \rangle$$

$$\langle p \rangle^* = \langle Np \rangle.$$

Then $(S/\equiv_\Theta, \oplus, \cdot, {}^*, 0_\Theta, 1_\Theta)$ *is an MV algebra.*

Proof. First note that $\oplus$, $\cdot$ and $*$ are well defined, by Proposition 5.11(iii), (iv). Also notice that replacement of CXX by any other valid sentence t would result in the same definition of 1_Θ, since $[t] = [CXX]$ (4.8), whence $\langle t \rangle = \langle CXX \rangle$ by 5.11(ii). To verify that $(S/\equiv_\Theta, \oplus, \ldots)$ is an MV algebra we have to check the axioms given in 2.1: we limit ourselves to $Ax2$, since no new ideas are used in checking the remaining axioms.

Claim. $\langle p \rangle \oplus (\langle q \rangle \oplus \langle r \rangle) = (\langle p \rangle \oplus \langle q \rangle) \oplus \langle r \rangle$. Indeed, $\langle p \rangle \oplus (\langle q \rangle \oplus \langle r \rangle) = \langle p \rangle \oplus \langle CNqr \rangle = \langle CNpCNqr \rangle$. On the other hand, $(\langle p \rangle \oplus \langle q \rangle) \oplus \langle r \rangle = \langle CNpq \rangle \oplus \langle r \rangle = \langle CNCNpqr \rangle$. Since $\equiv_\Theta$ is coarser than $\equiv$, it suffices now to prove that $[CNpCNqr] = [CNCNpqr]$, i.e., going backwards through the definition of L (4.7), we have to show that $[p] \oplus ([q] \oplus [r]) = ([p] + [q]) \oplus [r]$. But this is a consequence of L being an MV algebra (4.7). $\blacksquare$

5.15. THEOREM. *For every theory* $\Theta \subseteq S$, *the map* α *defined in Eq.* (20) *is an MV isomorphism of* L/I_Θ *onto* $(S/\equiv_\Theta, \oplus, \cdot, {}^*, 0_\Theta, 1_\Theta)$.

Proof. If $\Theta = \emptyset$ than $\equiv_\Theta = \equiv$ and we have nothing to prove. Assume $\Theta \neq \emptyset$. As in [4, p. 484] we shall use L/I_Θ and L/R_Θ interchangeably, where R_Θ is the congruence relation associated with I_Θ (3.15). For the same of readability we shall write R instead of R_Θ in the rest of this proof. By Proposition 5.13, R coincides with the equivalence relation $\approx$ defined in Eq. (19). Elements of L/I_Θ will be denoted $[p]/R$, where p ranges over sentences. Thus, $[p]/R = \{[q] \in L \mid [q] \approx [p]\} = \{[q] \in L \mid q \equiv_\Theta p\}$. By definition, the map α sends $[p]/R$ into $\langle p \rangle = \bigcup \{[q] \in L \mid q \equiv_\Theta p\}$. This shows in particular that α maps L/I_Θ one–one onto $S/\equiv_\Theta$, recalling that $\equiv_\Theta$ is coarser than $\equiv$ (5.11(ii)). The quotient MV algebra $(L/I_\Theta, \hat{\oplus}, \hat{\cdot}, \hat{*}, \hat{0}, \hat{1})$ is defined as follows [4, 4.3]:

$$\hat{1} = [CXX]/R, \text{ where } [CXX] \text{ is the unit in } L,$$
$$\hat{0} = [NCXX]/R$$
$$[p]/R \mathbin{\hat{\otimes}} [q]/R = ([p] \oplus [q])/R = [CNpq]/R$$
$$[p]/R \mathbin{\hat{\cdot}} [q]/R = ([p] \cdot [q])/R = [NCpNq]/R$$
$$([p]/R)^{\hat{*}} = [p]^*/R = [Np]/R.$$

The proof that this is indeed an MV algebra is in [4, 4.3]. The proof that $(S/\equiv_\Theta, ...)$ is an MV algebra is in Proposition 5.14. The proof that α is an MV isomorphism of L/I_Θ onto $S/\equiv_\Theta$ is now a particular instance of the second isomorphism theorem in universal algebra [16]. However, we can easily give a self-contained proof. Let us show, for example, that α preserves addition:

$$\begin{aligned} \alpha([p]/R \mathbin{\hat{\oplus}} [q]/R) &= \alpha([CNpq]/R) && \text{by definition of } \hat{\oplus}, \\ &= \langle CNpq \rangle, && \text{by definition of } \alpha, \text{ Eq. (20)}, \\ &= \langle p \rangle \oplus \langle q \rangle && \text{by definition of } S/\equiv_\Theta \end{aligned}$$

in Proposition 5.14. A similar proof shows that α preserves the other operations and distinguished elements in L/I_Θ. The proof of the theorem is now complete. ▮

6. Applications: Incompleteness, Axiomatizability, Simplicity

6.1. Theorem. *Let $\mathfrak{A}$ be an* AF *C^*-algebra with lattice ordered K_0. Assume there exists a theory $\Theta \in \theta(\mathfrak{A})$ such that the set $\tilde{\Theta}$ of consequences of Θ is recursively enumerable but not recursive. Then $\mathfrak{A}$ is not simple.*

Before proving the theorem we shall characterize those theories Ψ such that $\tilde{\Psi}$ is recursively enumerable, by an adaptation of Craig's well-known result [7] for the 2-valued case:

6.2. THEOREM. *For every theory $\Psi \subseteq S$ the following are equivalent:*

(i) *$\tilde{\Psi}$ is recursively enumerable;*

(ii) *there is a recursive theory $\Phi \subseteq S$ such that $\tilde{\Phi} = \tilde{\Psi}$ (stated otherwise, Ψ is recursively axiomatizable).*

Proof of 6.2. (ii) → (i). Let $P(p)$ be the predicate "$p \in \tilde{\Psi}$." Then $P(p)$ holds iff $\exists q_1 \cdots q_n$ ($q_1,..., q_n \in \Phi$ and $Cq_1 \cdots Cq_n p$ is valid). By Theorem 4.5 the predicate "$Cq_1 \cdots Cq_n p$ is valid" is recursively enumerable (indeed, it is recursive). Therefore, the predicate $P(p)$ is recursively enumerable.

(i) → (ii) In case $\Psi = \emptyset$, then by Definition 5.1, letting $\Phi = \emptyset$ we are done. Assume now $\Psi \neq \emptyset$. By hypothesis there is a recursive predicate $W(n, p)$ $(n \in \omega, p \in S)$ such that $p \in \tilde{\Psi}$ iff $\exists n\, W(n, p)$. Let $\Phi \subseteq S$ be given by the following definition:

$$q \in \Phi \quad \text{iff} \quad \overset{\|q\|}{\underset{0}{\exists}}\, n \in \omega \ \overset{\|q\|}{\underset{0}{\exists}}\, m \in \omega \ \overset{\|r\| \leqslant \|q\|}{\exists}\, r \in S(W(n, r) \text{ and } q = \underbrace{NN\,NN \cdots NN}_{\leftarrow 2m \quad N\text{'s} \rightarrow} r). \tag{21}$$

Note that $\Phi \neq \emptyset$, since $\tilde{\Psi} \neq \emptyset$. Now by (21) we have: $q \in \Phi$ implies $q = NN\,NN \cdots NNr$ for some $r \in \tilde{\Psi}$, i.e., for some r such that $Ct_1 \cdots Ct_k r$ is valid (for suitable $t_1,..., t_k \in \Psi$), which implies $Ct_1 \cdots Ct_k q$ is valid, by Corollary 4.12(i), whence $q \in \tilde{\Psi}$. Thus, $\Phi \subseteq \tilde{\Psi}$.

Conversely, if $r \in \tilde{\Psi}$ then $W(n, r)$ holds, for some $n \in \omega$ by definition of W. Consider now the following sequence of sentences:

$$r,\ NNr,\ NNNNr, ...,\ \underbrace{NN\,NN \cdots NN}_{\leftarrow 2i \quad N\text{'s} \rightarrow} r, ...$$

By Corollary 4.12(i), each sentence in the above list belongs to $\tilde{\Psi}$. For suitably large $i \in \omega$ we have

$$\|\underbrace{NN \cdots NN}_{\leftarrow 2i \quad N\text{'s} \rightarrow} r\| \geqslant n.$$

Let m be the least such $i \in \omega$. Then the sentence

$$q = \underbrace{NN\,NN \cdots NN}_{\leftarrow 2m \quad N\text{'s} \rightarrow} r$$

has the following properties: $\|q\| \geqslant m$, $\|q\| \geqslant n$, and $\|q\| \geqslant \|r\|$. Since $W(n, r)$

holds then by (21), $q \in \Phi$. Since Cqq is valid by Proposition 4.2, then Cqr is valid by Corollary 4.12(i), hence $r \in \tilde{\Phi}$, by definition of $\tilde{\Phi}$, since $q \in \Phi$. We have thus proved that $\tilde{\Psi} \subseteq \tilde{\Phi}$, which completes the proof of Theorem 6.2. ▮

6.3. *Proof of* 6.1. Assume $\mathfrak{A}$ to be simple, and $\Theta \in \theta(\mathfrak{A})$ to be a theory such that $\tilde{\Theta}$ is recursively enumerable (r.e.). We shall prove that $\tilde{\Theta}$ is recursive. In the light of Theorem 4.5 and definition of $\tilde{\varnothing}$ (5.1), we may safely limit attention to the case $\Theta \neq \varnothing$. Recalling the well-known correspondence between ideals of $\mathfrak{A}$ and ideals of $(G, u) = (K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$, [9, p. 22] we see that (G, u) is simple, whence every $g \in G$ with $g > 0$ is an order unit for G [10, p. 389; 14, p.196]. Let $\Gamma(G, u) = (A, \oplus, \cdot, {}^*, 0, 1)$ as in Definition 2.4. Then for every $x \in A$ we have

$$x > 0 \qquad \text{iff} \quad \exists n(n > 0 \text{ and } \underbrace{x \oplus \cdots \oplus x}_{\leftarrow n \quad x\text{'s} \rightarrow} = 1). \tag{22}$$

Indeed, the ←-direction is trivial; the →-direction follows from 2.9, since $x > 0$ is an order unit for G. As a consequence, for each $y \in A$ exactly one of the following cases may occur: either

$$y = 1 \text{ or } y^* > 0, \qquad \text{i.e., } \exists n \in \omega(n > 0 \text{ and } \underbrace{y^* \oplus \cdots \oplus y^*}_{\leftarrow n \quad y\text{'s} \rightarrow} = 1). \tag{23}$$

By the definition of $\theta(\mathfrak{A})$, which is made possible by Theorem 4.7 and Corollary 5.6, we can identify $A = \tilde{\Gamma}(\mathfrak{A})$ with the Lindenbaum algebra L/I_Θ; by Theorem 5.15 we can identify the latter with the MV algebra $(S/\equiv_\Theta, \oplus, \cdot, {}^*, 0_\Theta, 1_\Theta)$. We are now in a position to equivalently state (23) as follows: for each $p \in S$,

$$\langle p \rangle \neq 1_\Theta \qquad \text{iff } \exists n \in \omega \ (n > 0 \text{ and } 1_\Theta = \underbrace{\langle p \rangle^* \oplus \cdots \oplus \langle p \rangle^*}_{\leftarrow n \quad p\text{'s} \rightarrow}). \tag{24}$$

Claim. For every sentence $r \in S$, $\langle r \rangle = 1_\Theta$ iff $r \in \tilde{\Theta}$. Indeed,

$\langle r \rangle = 1_\Theta$ iff $r \equiv_\Theta CXX$,

iff $CrCXX \in \tilde{\Theta}$ and $CCXXr \in \tilde{\Theta}$, by definition of $\equiv_\Theta$,

iff $CCXXr \in \tilde{\Theta}$, since $CrCXX$ is valid (4.2), hence it belongs to $\tilde{\Theta}$, by 5.2,

iff $\exists w_1 \cdots w_k \in \Theta$ such that $Cw_1 \cdots Cw_k CCXXr$ is valid,

iff $\exists w_1 \cdots w_k \in \Theta$ such that $Cw_1 \cdots Cw_k r$ is valid (4.12(iii)),

iff $r \in \tilde{\Theta}$.

This settles our claim.

The predicate $P(p)$ defined "$p \in \tilde{\Theta}$" is r.e., by hypothesis. To prove that $P(p)$ is recursive it suffices to show that its negation $\neg P(p)$ is r.e. To this purpose, note that $\neg P(p)$ holds iff p is a word over the alphabet Σ such that $p \notin \tilde{\Theta}$. Since the set of sentences S is a recursive subset of Σ^* (4.1), then it is enough to prove the recursive enumerability of the following predicate

$$\text{"}p \text{ is a sentence not belonging to } \tilde{\Theta}.\text{"} \tag{25}$$

By our claim, (25) is equivalent to $\langle p \rangle \neq 1_\Theta$; the latter, by (24) is equivalent to

$$\exists n \in \omega\ (n > 0 \text{ and } 1_\Theta = \underbrace{\langle p \rangle^* \oplus \cdots \oplus \langle p \rangle^*}_{\leftarrow n \quad p\text{'s} \rightarrow}), \text{ i.e., equivalent to} \tag{26}$$

$$\exists n \in \omega\ (n > 0 \text{ and } 1_\Theta = \langle \underbrace{CNNpCNNp \cdots CNNpNp}_{\leftarrow n \quad p\text{'s} \rightarrow} \rangle), \tag{27}$$

as can be seen in the light of Proposition 5.14. One more application of our claim shows that (27) is equivalent to

$$\exists n \in \omega\ (n > 0 \text{ and } \underbrace{CNNpCNNp \cdots CNNpNp}_{\leftarrow n \quad p\text{'s} \rightarrow} \in \tilde{\Theta}). \tag{28}$$

Since the predicate $Q(p)$ defined in (28) is r.e., then so is the predicate defined in (25), as well as the predicate "$p \notin \tilde{\Theta}$." Therefore $\tilde{\Theta}$ is a recursive set of sentences. This completes the proof of our theorem. ∎

6.4. Example. Let X_n be short for $X|\cdots|_n$. Let the theory $\Theta_B \subseteq S$ be defined by

$$\Theta_B = \{CCNX_nX_nX_n,\ CX_nCNX_nX_n \mid n \in \omega\}.$$

Intuitively, the theory states that each variable X_n is $\{0, 1\}$-valued. More precisely, in the MV algebra $L_B = L/I_{\Theta_B}$ we have $\langle X_n \rangle = \langle X_n \rangle \oplus \langle X_n \rangle$, and hence, by [4, 1.7] the operations $\oplus$ and $\cdot$ collapse to $\vee$ and $\wedge$, respectively. Moreover, $(L_B, \vee, \wedge, {}^*, 0, 1)$ is the free boolean algebra over denumerably many free generators. We can identify $\tilde{\Theta}_B$ with the set of tautologies in the 2-valued sentential calculus. Let γ be a Turing computable bijection onto ω of the set of sentences of first-order logic in the language of Peano arithmetic (PA, for short), and let $\Theta_{PA} \subseteq S$ be given by

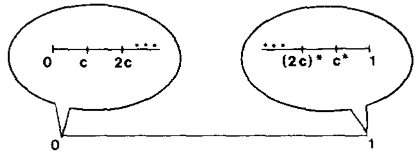
$\Theta_{\mathrm{PA}} = \{X_n \mid \gamma^{-1}(n)$ is a theorem of PA$\} \cup \{NX_n \mid$ the negation of $\gamma^{-1}(n)$ is a theorem of PA$\}$. Also let $\Theta = \Theta_{\mathrm{B}} \cup \Theta_{\mathrm{PA}}$. Since the set of theorems of PA is r.e., then Θ_{PA} is r.e. Therefore, $\tilde{\Theta}$ is r.e. We now *claim* that for every $n \in \omega$ we have

$$X_n \in \Theta_{\mathrm{PA}} \quad \text{iff} \quad X_n \in \tilde{\Theta}. \tag{29}$$

As a matter of fact, assume $X_n \in \tilde{\Theta}$, i.e., $Cp_1 Cp_2 \cdots Cp_r X_n$ is valid in the $\aleph_0$-valued sentential logic, for suitable $p_1, \ldots, p_r \in \Theta$. Then there are $q_1, \ldots, q_t \in \Theta_{\mathrm{PA}}$ such that $Cq_1 Cq_2 \cdots Cq_t X_n$ is valid in the 2-valued sentential logic, i.e., $(q_1 \wedge \cdots \wedge q_t) \rightarrow X_n$ is a tautology in this latter logic. Since for each $i = 1, \ldots, t$, q_i is either a sentential variable or a negated sentential variable, an application of Craig's interpolation theorem [6, 1.2.7, p. 17] shows that $X_n = q_j$ for some $j = 1, \ldots, t$—unless PA turns out to be inconsistent, in which case replace PA by some other r.e. nonrecursive set of sentences, e.g., the valid sentences of first-order logic. It follows that $X_n \in \Theta_{\mathrm{PA}}$. Since the converse implication in (29) is trivial, our claim is settled.

We now note that $\tilde{\Theta}$ is not recursive, for otherwise the set $\tilde{\Theta} \cap \{$sentential variables$\}$ is recursive, whence by (29) so is the set $\Theta_{\mathrm{PA}} \cap \{$sentential variables$\}$, thus contradicting the Gödel undecidability theorem for PA [26, 16.1]. Let the AF C^*-algebra $\mathfrak{A}_{\mathrm{PA}}$ we defined by $\Theta \in \theta(\mathfrak{A}_{\mathrm{PA}})$. In the terminology of the Introduction of this paper, $\mathfrak{A}_{\mathrm{PA}}$ is Gödel incomplete. We shall now see that this incompleteness is irreparable. Since $\tilde{\Theta}$ is r.e. and not recursive, by Theorem 6.1 $\mathfrak{A}_{\mathrm{PA}}$ is not simple. In view of the commutativity of $\mathfrak{A}_{\mathrm{PA}}$, the effect of Theorem 6.1 is that the boolean space of maximal ideals of $\mathfrak{A}_{\mathrm{PA}}$ is not a singleton. Moving through our gödelization γ, we can find a sentence ψ in the language of Peano arithmetic such that neither ψ, nor not-ψ is a theorem of PA: this is Gödel's incompleteness theorem for PA [26, 16.2]. Let now $\mathfrak{B}$ be an arbitrary simple quotient of $\mathfrak{A}_{\mathrm{PA}}$. Thus, $\mathfrak{B} \cong \mathbb{C}$. Trivially, there are infinitely many r.e. theories $\Phi \in \theta(\mathfrak{B})$ with Φ not containing Θ, and there are infinitely many non-r.e. theories $\Lambda \in \theta(\mathfrak{B})$ with Λ containing Θ. However, we *claim* that there is no r.e. theory $\Psi \in \theta(\mathfrak{B})$ with $\Psi \supseteq \Theta$. For otherwise, if Ψ were a counterexample, then by 6.1, Ψ would be recursive, and hence $\Psi \supseteq \Theta \supseteq \Theta_{\mathrm{PA}}$ would be a counterexample to the inseparability of PA [26, 16.1]. Our second claim is settled. Intuitively, any completion process $\Theta \in \theta(\mathfrak{A}_{\mathrm{PA}}) \hookrightarrow \Psi \in \theta(\mathfrak{B})$ paralleling the ideal-elimination process $\mathfrak{A}_{\mathrm{PA}} \mapsto \mathfrak{B}$ does not preserve recursive enumerability.

6.5. EXAMPLE. We shall describe here a primitive, nonsimple, Gödel complete AF C^*-algebra. Let C be the MV algebra defined in [4, p. 474], whose picture is



with obvious MV operations: thus for example, $3c \oplus 5c = 8c$, $(3c)^* \oplus (5c)^* = 1$, $3c \oplus (5c)^* = (2c)^*$, $5c \oplus (3c)^* = 1$. For all $m, n \in \omega$, we have $nc < (mc)^*$. Define the AF C^*-algebra $\mathfrak{B}$ by $\tilde{\Gamma}(\mathfrak{B}) \cong C$. Then $\mathfrak{B}$ is primitive and is not simple. We *claim* that $\mathfrak{B}$ is Gödel complete. As a matter of fact, let $\Theta \in \theta(\mathfrak{B})$. Let $q \in S$ be such that $\langle q \rangle = c$ in the isomorphism $C \cong L/I_\Theta \cong \tilde{\Gamma}(\mathfrak{B})$. For every $p \in S$ exactly one of the following alternatives holds:

— either Np is a consequence of Θ,

— or Cqp is a consequence of Θ.

For, if $\langle p \rangle \neq 0_\Theta$ then $\langle p \rangle \geqslant c = \langle q \rangle$, i.e., Cqp is a consequence of Θ. If in particular $\tilde{\Theta}$ is r.e., then the above argument yields a decision procedure for the predicate "$\langle p \rangle = 0_\Theta$," and hence, for the predicate "$\langle r \rangle = 1_\Theta$." Therefore, $\tilde{\Theta}$ is recursive, and $\mathfrak{B}$ is Gödel complete, as claimed.

7. Example: Axiomatizing the CAR Algebra

7.1. Theorem. *Let $\mathfrak{A}$ be the canonical anticommutation relation* (CAR) *algebra defined in* [3, p. 227]. *Let $\Theta \subseteq S$ be the following set of sentences*:

$$CNXX$$

$CXCNX\| \; X\|$	$CCNX\| \; X\| \; X$
$CX\| \; CNX\|\| \; X\|\|$	$CCNX\|\| \; X\|\| \; X\|$
$CX\|\| \; CNX\|\|\| \; X\|\|\|$	$CCNX\|\|\| \; X\|\|\| \; X\|\|$
... ...	... ...

$$CXNX$$

$CNX\| \; CXX\|$	$CCXX\| \; NX\|$
$CNX\|\| \; CX\| \; X\|\|$	$CCX\| \; X\|\| \; NX\|\|$
$CNX\|\|\| \; CX\|\| \; X\|\|\|$	$CCX\|\| \; X\|\|\| \; NX\|\|\|$
... ...	... ...

Then $\Theta \in \theta(\mathfrak{A})$.

Proof. Adopting the abbreviations X_n for $X|\cdots|_{n\text{ strokes}}$, and Bpq for $CNpq$, we may equivalently write $\Theta = \Theta_0 \cup \Theta_1 \cup \Theta_2$, where

$$\Theta_0 = \{CNXX, CXNX\}$$
$$\Theta_1 = \{CX_n BX_{n+1} X_{n+1}, CBX_{n+1} X_{n+1} X_n \mid n \in \omega\}$$
$$\Theta_2 = \{CNX_{n+1} BX_{n+1} NX_n, CBX_{n+1} NX_n NX_{n+1} \mid n \in \omega\}.$$

As shown in [9], $(K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$ is the group D of dyadic rationals with addition and natural order, and with 1 as an order unit. By Theorem 5.9(i) our theorem amounts to proving that $\Gamma(D, 1) \cong L/I_\Theta$. By Definition 2.4, $\Gamma(D, 1)$ is the MV algebra $(A, \oplus, \cdot, *, 0, 1)$ given by

$$A = \text{dyadic rationals in } [0, 1]$$
$$x^* = 1 - x$$
$$x \oplus y = \min(1, x + y)$$
$$x \cdot y = \max(0, x + y - 1).$$

By Theorem 5.15 we may identify L/I_Θ with the MV algebra $(S/\equiv_\Theta, ...)$ defined in Proposition 5.14. Elements of $S/\equiv_\Theta$ have the form $\langle p \rangle$, for $p \in S$, where $\langle p \rangle = \{q \in S \mid q \equiv_\Theta p\}$. To prove the theorem we prepare a number of lemmas. The following holds in every MV algebra:

7.2. LEMMA. *Write nx instead of $x \oplus \cdots \oplus x$ (n times). Let $i, j, m \in \omega \backslash \{0\}$, with $i + j = m + 1$. If $x^* = mx$ then $(ix)^* = jx$.*

Proof. Since $x \oplus x^* = 1$ then by hypothesis $x \oplus mx = 1$, whence $1 = (m+1)x = (i+j)x = (ix)^{**} \oplus jx$. By [4, 1.13] we obtain

$$(ix)^* \leqslant jx. \tag{30}$$

With the help of (30) we prove the lemma by induction on i: *Basis*: $i = 1$. Trivial.

Induction step:

$x^* = mx = (i-1)x \oplus jx,$	by hypothesis,
$\geqslant (ix)^* \oplus (i-1)x,$	by (30) and [4, 1.10],
$= ((x^*)^* \oplus (i-1)x)^* \oplus (i-1)x$	
$= (x^* \oplus ((i-1)x)^*)^* \oplus x^*,$	by Lemma 2.6 (P8),
$\geqslant x^*,$	by [4, 1.10.]

Then we obtain in particular,

$$jx \oplus (i-1)x = (ix)^* \oplus (i-1)x. \tag{31}$$

Since $ix \geqslant (i-1)x$ then an application of [4, 1.4(vi)] yields

$$(ix)^* \leqslant ((i-1)x)^*. \tag{32}$$

By the induction hypothesis we have

$$((i-1)x)^* = (m+1-(i-1))x = (j+1)x \geqslant jx. \tag{33}$$

Applying [4, 1.14] to (31) in the light of (32) and (33) we can finally write $jx = (ix)^*$, as required.

7.3. Lemma. *Let $p \in S$ and $p \not\equiv_\Theta NCXX$. Then for some $b, n \in \omega$ with $b \geqslant 2$ we have*

$$p \equiv_\Theta BX_n BX_n \cdots BX_n X_n \qquad (b \text{ many } X_n\text{'s}).$$

Proof. If $p = X_n$, then by suitably choosing axioms from Θ_1 we can easily see that $X_n \equiv_\Theta BX_{n+1} X_{n+1}$, and we are done.

To deal with $p \neq X_n$ we proceed by induction on $\|p\|$, by cases:

Case 1. $p = Bqr$. Then by induction hypothesis, for suitable c, t, d, $u \in \omega$ we have

$$q \equiv_\Theta BX_c BX_c \cdots BX_c X_c \qquad (t \text{ many } X_c\text{'s})$$

and (34)

$$r \equiv_\Theta BX_d BX_d \cdots BX_d X_d \ (u \text{ many } X_d\text{'s})$$

(unless either q or r is $\equiv_\Theta$-equivalent to $NCXX$, in which case the proof becomes trivial). Assuming without loss of generality that $c \leqslant d$, by suitably choosing axioms from Θ_1 we obtain, for some $v \in \omega$,

$$X_c \equiv_\Theta BX_d BX_d \cdots BX_d X_d \qquad (v \text{ many } X_d\text{'s}). \tag{35}$$

Here, and in the rest of this paper, we shall use without explicit mention the fact that $\equiv_\Theta$ preserves N and C (5.11), whence $\equiv_\Theta$ preserves B. We are also using such well known facts [31] as the commutativity and associativity of B ($BxByz \equiv BBxyz$, hence, a fortiori, $BxByz \equiv_\Theta BBxyz$, by 5.11).

In the light of (34) and (35) we have, for suitable $w \in \omega$: $Bqr \equiv_\Theta BX_d BX_d \cdots BX_d X_d$ (w many X_d's), as required to complete the proof of this case.

Case 2. $p = Nq$. By induction hypothesis we have, for suitable $c, t \in \omega$,

$$q \equiv_\Theta BX_c X_c \cdots BX_c X_c \qquad (t \text{ many } X_c\text{'s})$$

(unless $q \equiv_\Theta NCXX$, in which case the proof is trivial). By suitably choosing axioms from Θ_0 and Θ_2 we have

$$NX_c \equiv_\Theta BX_c BX_{c-1} \cdots BX_1 X.$$

At this point, by suitably choosing axioms from Θ_1 we obtain, for some $u \in \omega$,

$$NX_c \equiv_\Theta BX_c BX_c \cdots BX_c X_c \qquad (u \text{ many } X_c\text{'s}),$$

i.e., $\langle X_c \rangle^* = \langle X_c \rangle \oplus \cdots \oplus \langle X_c \rangle$ (u times). Applying now Lemma 7.2 to the MV algebra $(S/\equiv_\Theta, ...)$ we conclude that

$$\begin{aligned} p &\equiv_\Theta N(BX_c BX_c \cdots BX_c X_c) && (t \text{ many } X_c\text{'s}) \\ &\equiv_\Theta BX_c BX_c \cdots BX_c X_c && (v \text{ many } X_c\text{'s}) \end{aligned}$$

for suitable $v \in \omega$, as required to complete the proof of the lemma.

After the proof of Lemma 7.3 we define the map $\rho: A \to S/\equiv_\Theta$ by stipulating that for all $x \in A$,

$$\begin{aligned} \rho(x) &= \langle CXX \rangle, && \text{if } x = 1, \\ &= \langle NCXX \rangle, && \text{if } x = 0, \\ &= \langle X_n \rangle, && \text{if } x = 1/2^{n+1} (n \in \omega), \\ &= \langle BX_n BX_n \cdots BX_n X_n \rangle \ (b \text{ many } X_n\text{'s}), && \text{if } x = b/2^{n+1},\ 1 < b < 2^{n+1}; \\ & && n, b \in \omega. \end{aligned}$$

7.4. LEMMA. *ρ is well defined, i.e., if $x = b/2^{n+1} = c/2^{m+1}$ then $BX_n \cdots BX_n X_n$ (b many X_n's) $\equiv_\Theta BX_m \cdots BX_m X_m$ (c many X_m's).*

Proof. Assuming without loss of generality $m \leqslant n$, by suitably choosing axioms from Θ_1, we have $X_m \equiv_\Theta BX_{m+1} X_{m+1}$. Iterating this till n is reached, and recalling that B is preserved under $\equiv_\Theta$ and is commutative and associative, we obtain the desired conclusion by just noting that $b/c = 2^{n-m}$.

7.5. LEMMA. *ρ is 1–1.*

Proof. In the light of Lemma 7.3 and 7.4 it suffices to settle the following:

> if $1 < b < c \leqslant 2^{n+1}$ then $p_b \not\equiv_\Theta p_c$, where $p_b = BX_n BX_n \cdots BX_n X_n$ (b many X_n's), and $p_c = BX_n BX_n \cdots BX_n X_n$ (c many X_n's). (36)

To this purpose, let $k: \omega \to [0, 1]$ be the assignment (4.1) defined by $k(n) = 2^{-(n+1)}$, for all $n \in \omega$.

Claim 1. For all $p \in \Theta$, $p(k) = 1$.

This is a straightforward verification. We limit ourselves to verifying the claim for $CNX_{n+1}CX_nX_{n+1}$. Indeed, $CNX_{n+1}CX_nX_{n+1}(k) = X_{n+1}(k) \oplus (NX_n(k) \oplus X_{n+1}(k)) = 2^{-(n+2)} \oplus (1 - 2^{-(n+1)}) \oplus 2^{-(n+2)} = 2 \cdot 2^{-(n+2)} + 1 - 2^{-(n+1)} = 1$.

Claim 2. $b \cdot 2^{-(n+1)} = p_b(k) < p_c(k) = c \cdot 2^{-(n+1)}$.

This claim is verified by a straightforward computation.

Assume now $p_b \equiv_\Theta p_c$ (absurdum hypothesis). It follows that there are $w_1, ..., w_s \in \Theta$ such that $Cw_1 \cdots Cw_s Cp_c p_b$ is valid. By 4.10 we have $w_1(k) \leqslant Cw_2 \cdots Cw_s Cp_c p_b(k)$. Iterating this for s times and using Claim 1 we obtain $1 \leqslant Cp_c p_b(k)$, i.e., by 4.10, $p_c(k) \leqslant p_b(k)$, which contradicts Claim 2. Having thus settled (36), we have also completed the proof of Lemma 7.5.

After proving that ρ maps A one–one into $S/\equiv_\Theta$ we immediately see that ρ is onto $S/\equiv_\Theta$, by Lemma 7.3. To finally prove that ρ is an MV isomorphism, we assume $\rho(x) = \langle p \rangle$ and $\rho(y) = \langle q \rangle$; using Lemmas 7.2 and 7.3 by an easy computation we obtain that $\langle p \rangle^* = \langle Np \rangle = \rho(x^*)$, and $\langle p \rangle \oplus \langle q \rangle = \langle Bpq \rangle = \rho(x \oplus y)$. The proof of 7.1 is complete. ∎

7.6. *Remark*. By a quirk of fate, the two axioms in Θ_0 above state that the sentential variable X is equivalent to its negation. This is not a contradiction in Łukasiewicz logic, and may give an idea of the conceptual differences between classical and nonclassical physical systems.

8. The AF C^*-Algebra $\mathfrak{M}$ Corresponding to $(M, 1)$ and L

We refer to [9, Sects. 8, 9; 30; 2] for all the unexplained notions used in this section. Recall from Proposition 4.14 the definition of $(M, 1)$. By Theorem 1.2(ii), up to isomorphism there is a unique AF C^*-algebra $\mathfrak{M}$ such that $(K_0(\mathfrak{M}), [1_{\mathfrak{M}}]) \cong (M, 1)$. Given any ideal $\mathfrak{J}$ in $\mathfrak{M}$, upon identifying $K_0(\mathfrak{J})$ with the image of $\mathfrak{J}$ in M, the map $\mathfrak{J} \to K_0(\mathfrak{J})$ is an isomorphism of the lattice of ideals of $\mathfrak{M}$ onto the lattice of order-ideals (directed convex subgroups) of M. Since in any l-group order-ideals coincide with l-ideals (1.4), it follows that under this isomorphism, primitive ideals of $\mathfrak{M}$ correspond to proper prime l-ideals of M, and essential ideals of $\mathfrak{M}$ correspond to large ideals in M, i.e., those l-ideals having nonzero intersection with every nonzero l-ideal in M. Moreover, the space Prim($\mathfrak{M}$) of primitive ideals of $\mathfrak{M}$ with the Jacobson topology is homeomorphic to the space

Spec(M) of proper prime l-ideals of M equipped with the spectral topology of the zero-ring associated with M [2, Sect. 10]. Let Maxprim($\mathfrak{M}$) $\subseteq$ Prim($\mathfrak{M}$) be the space of maximal ideals of $\mathfrak{M}$ with the subspace topology.

8.1. LEMMA. Maxprim($\mathfrak{M}$) *is homeomorphic to the Hilbert cube* $[0, 1]^{\omega}$.

Proof. Let Maxspec(M) $\subseteq$ Spec(M) be the space of maximal l-ideals of M; then Maxspec(M) is homeomorphic to Maxprim($\mathfrak{M}$). Let λ assign to each $h \in [0, 1]^{\omega}$ the l-ideal $J_h = \{f \in M \mid f(h) = 0\}$. We shall prove that λ is a homeomorphism of $[0, 1]^{\omega}$ onto Maxspec(M). Note that since 1 is an order unit in M, then every proper l-ideal can be extended to a maximal l-ideal. The separation property given by Proposition 4.17, together with the fact that each member of the l-group M is a continuous function over $[0, 1]^{\omega}$, are to the effect that λ is a bijecton onto Maxspec(M). For every closed $X \subseteq$ Spec(M) there is an l-ideal J of M such that $X = H(J) = \{I \in \mathrm{Spec}(M) \mid I \supseteq J\}$, by [2, 10.1.7]. Accordingly, every closed set $Y \subseteq$ Maxspec(M) can be written as $Y = H(J) \cap \mathrm{Maxspec}(M)$ for some l-ideal J of M. The set $\lambda^{-1}(Y) = \{h \in [0, 1]^{\omega} \mid J_h \supseteq J\} = \bigcap \{f^{-1}(0) \mid f \in J\}$ is closed. Thus, λ is a continuous bijection from $[0, 1]^{\omega}$ onto the Hausdorff space [2, 10.1.11] Maxspec(M), whence λ is a homeomorphism. ∎

8.2. *Remark.* As an alternative proof of Lemma 8.1, note that by [15, 3.2] in the archimedean l-group M the ordering is determined by a compact set of pure states, namely, the point states in $[0, 1]^{\omega}$. Now [1, II 2.1] together with Proposition 4.17 yields a homeomorphic embedding of $[0, 1]^{\omega}$ onto the pure state space of M.

8.3. COROLLARY. (i) $\bigcap$ Maxprim($\mathfrak{M}$) $= \{0\}$.

(ii) Maxprim($\mathfrak{M}$) *is dense in* Prim($\mathfrak{M}$).

Proof. (i) is an immediate consequence of Lemma 8.1.

(ii) With reference to the proof of Lemma 8.1, it is sufficient to show that Maxspec(M) is dense in Spec(M). Let $X \subseteq$ Spec(M) be an open non-void subspace. By [2, 10.1.4] there exists $f \in M$ with $f \neq 0$ such that $X \supseteq \{J \in \mathrm{Spec}(M) \mid f \notin J\}$. Let $h \in [0, 1]^{\omega}$ be such that $f(h) \neq 0$, i.e., $f \notin J_h$. Then $J_h \in \mathrm{Maxspec}(M) \cap X$. ∎

8.4. THEOREM. *Every primitive ideal in* $\mathfrak{M}$ *is essential.*

For the proof we prepare

8.5. LEMMA. *Let* $f \in M$ *and* J *be an l-ideal of* M. *Let*

$$V_J = \{h \in [0, 1]^{\omega} \mid J \subseteq J_h\}.$$

If $f \mid U = 0$ *for some open set* U *with* $V_J \subseteq U \subseteq [0, 1]^{\omega}$, *then* $f \in J$.

Proof of Lemma 8.5. If $J = M$ we are done. If not, then 1 is not in J and by Zorn's lemma there is a maximal l-ideal containing J, whence $V_J \neq \emptyset$. Assume $f \notin J$. By [2, 8.4.6] we have

$$J = \bigcap \{I \in \mathrm{Spec}(M) \mid I \supseteq J\},$$

upon identifying, if necessary, M with the zero-ring M_0 associated with M. Thus, $f \notin I$ for some $I \in \mathrm{Spec}(M)$ with $I \supseteq J$. Let now $P \in \mathrm{Maxspec}(M)$ be the only maximal l-ideal of M containing I: existence of P follows from Zorn's lemma, since M has an order unit; uniqueness follows from [2, 2.4.1(6)]. By Lemma 8.1 we can write $P = J_h$ for a unique $h \in [0, 1]^{\omega}$. Since $P \in \mathrm{Spec}(M)$ and $J \subseteq I \subseteq P$, then a fortiori $h \in V_J$. Since every prime l-ideal contains a minimal prime l-ideal, from [2, 10.5.3] together with the fact that $f \notin I \subseteq P$, and I, $P \in \mathrm{Spec}(M)$, we obtain

(1) $f \notin v_P = \bigcap \{Q \in \mathrm{Spec}(M) \mid Q \subseteq P\}$, where v_P is the *germinal* l-ideal associated with P. By [2, 10.5.3(i)] we also have $v_P = \{g \in M \mid H(g)$ is a neighborhood of $P\}$, where, as usual, $H(g) = \{R \in \mathrm{Spec}(M) \mid g \in R\}$. Thus from (1) we infer that $H(f)$ is not a neighborhood of P, i.e.,

(2) for any open set V in $\mathrm{Spec}(M)$ with $P \in V$ there is $Q \in \mathrm{Spec}(M)$ such that $Q \in V \backslash H(f)$.

By 8.3(ii) $\mathrm{Maxspec}(M)$ is dense in $\mathrm{Spec}(M)$, and by [2, 10.1.4] the set $V \backslash H(f) = V \cap S(f)$ is open in $\mathrm{Spec}(M)$, where $S(f) = \{R \in \mathrm{Spec}(M) \mid f \notin R\}$. From (2) it follows that

(3) for any open set V in $\mathrm{Spec}(M)$ with $P \in V$ there is $P' \in \mathrm{Maxspec}(M)$ such that $P' \in V$ and $f \notin P'$, that is, by definition of subspace topology,

(4) for any open set W in $\mathrm{Maxspec}(M)$ with $P \in W$ there is $P' \in \mathrm{Maxspec}(M)$ such that $P' \in W$ and $f \notin P'$.

Recalling that $P = J_h$ and using the homeomorphism given by Lemma 8.1, we can reformulate (4) as follows:

(5) for any open set $U \subseteq [0, 1]^{\omega}$ with $h \in U$ there is $h' \in [0, 1]^{\omega}$ such that $h' \in U$ and $f(h') \neq 0$.

Since $h \in V_J$ we conclude from (5) that there is no open set $U \subseteq [0, 1]^{\omega}$ containing V_J and such that $f(h') = 0$ for all $h' \in U$. This completes the proof of Lemma 8.5.

8.6. *End of Proof of Theorem* 8.4. Recalling the introductory remarks in this section, it is sufficient to prove that every prime l-ideal I of M has nonzero intersection with each nonzero l-ideal J of M. Given any such I and J, by Lemma 8.1 there is exactly one maximal l-ideal $P \supseteq I$, and we can write

$P = J_h$ for a unique $h \in [0, 1]^\omega$. Moreover, the closed set $Y = H(J) \cap \text{Maxspec}(M)$ is mapped by the homeomorphism λ^{-1} (λ as in the proof of Lemma 8.1) one–one onto the closed set $V_J = \{k \in [0, 1]^\omega \mid J \subseteq J_k\}$, in the notation of Lemma 8.5. If V_J were equal to $[0, 1]^\omega$ then $J_k \supseteq J$ for all k, whence by 8.3(i) it would follow that $\{0\} \neq J \subseteq \bigcap \{J_k \mid k \in [0, 1]^\omega\} = \{0\}$, a contradiction. Therefore, $V_J \neq [0, 1]^\omega$. Thus the set $S = [0, 1]^\omega \setminus V_J$ is open and nonempty. Since the closed singleton $\{h\}$ is not an open set in the Hilbert cube, then there is a point $h' \neq h$ with $h' \in S$. By Hausdorff separation, there are open sets $V, W \subseteq [0, 1]^\omega$ such that

(6) $h' \in V \cap S,\ h \in W,\ V \cap W = \varnothing,\ V \cap V_J = \varnothing$.

By regularity of the Hilbert cube there is an open set $B \subseteq [0, 1]^\omega$ with the following properties:

(7) $h' \in B \subseteq \bar{B} \subseteq V$ (where $\bar{B}$ is the closure of B).

By Proposition 4.17 there is $f \in M$ such that $f(h') = 0$ and $f(k) = 1$ for all $k \notin B$. The function $g = 1 - f$ has the following properties:

(8) $g(h') = 1$, and $g(k) = 0$ for all $k \in [0, 1]^\omega \setminus \bar{B}$.

Let $U = [0, 1]^\omega \setminus \bar{B}$. Then by (6) and (7) we have

(9) $U \supseteq W \cup V_J$ with U open.

We now observe that letting $V_I = \{k \in [0, 1]^\omega \mid J_k \supseteq I\}$, then $V_I = \{h\}$, since P is the only maximal l-ideal containing I. Now $g(k) = 0$ for all $k \in U$, by (8), and U is open and contains W and V_I by (6) and (9). Therefore, $g \in I$ by Lemma 8.5. Similarly, since the open set U contains V_J by (9), and $g(k) = 0$ in U by (8), it follows that $g \in J$, by Lemma 8.5. To complete the proof of the theorem, we have only to note that $0 \neq g$ by (8). ▌

Following [12] we say that an AF C^*-algebra $\mathfrak{A}$ has *comparability of projections* (in the sense of Murray and von Neumann) iff given any two projections in $\mathfrak{A}$, one of them is the support of a partial isometry whose range is contained in the other. This property is equivalent to $K_0(\mathfrak{A})$ being totally ordered.

8.7. COROLLARY. *For every unital* AF C^**-algebra* $\mathfrak{A}$ *with comparability of projections there is a primitive essential ideal* $\mathfrak{I}$ *in* $\mathfrak{M}$ *such that* $\mathfrak{A} \cong \mathfrak{M}/\mathfrak{I}$.

Proof. Let $(G, u) = (K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$. Then G is totally ordered. An application of Corollary 4.16 together with [2, 2.4.3] yields a prime l-ideal J of M such that $(G, u) \cong (M/J, 1/J)$. The lattice isomorphism between ideals of $\mathfrak{M}$ and l-ideals of M discussed at the beginning of this section, now yields a primitive ideal $\mathfrak{I}$ of $\mathfrak{M}$ such that $\mathfrak{A} \cong \mathfrak{M}/\mathfrak{I}$. By Theorem 8.4, $\mathfrak{I}$ is essential. ▌

For the application of essential ideals in extensions of AF C^*-algebras, see [19]. The following is a characterization of unital AF C^*-algebras with totally ordered K_0:

8.8. COROLLARY. *For every C^*-algebra $\mathfrak{B}$ the following are equivalent:*

(i) $\mathfrak{B} \cong \mathfrak{M}/\mathfrak{I}$ *for some primitive ideal* $\mathfrak{I}$ *of* $\mathfrak{M}$.

(ii) $\mathfrak{B} \cong \mathfrak{M}/\mathfrak{J}$ *for some primitive essential ideal* $\mathfrak{J}$ *of* $\mathfrak{M}$.

(iii) $\mathfrak{B}$ *is a unital* AF *C^*-algebra with comparability of projections.*

Proof. The only implication still to be proved, namely (i) → (iii), is well known [30, 3.13.2, 5.4.9]. ∎

8.9. *Remark.* If we drop the comparability assumption, recalling Corollary 4.16(iii), Theorem 1.3, and the nice behaviour of K_0 on quotient C^*-algebras [9, 19], we can still conclude that every (possibly nonunital) AF C^*-algebra $\mathfrak{A}$ is isomorphic to a C^*-subalgebra of $\mathfrak{M}/\mathfrak{I}$ for some ideal $\mathfrak{I}$ of $\mathfrak{M}$.

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