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On simplicial commutative algebras with Noetherian homotopy ☆

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Abstract

In this paper, we introduce a strategy for studying simplicial commutative algebras over general commutative rings R. Given such a simplicial algebra A, this strategy involves replacing A with a connected simplicial commutative $k(\wp)$-algebra $A(\wp)$, for each $\wp \in \mathrm{Spec}(\pi_0 A)$, which we call the *connected component of* A *at* $\wp$. These components retain most of the André–Quillen homology of A when the coefficients are $k(\wp)$-modules ($k(\wp)$=residue field of $\wp$ in $\pi_0 A$). Thus, these components should carry quite a bit of the homotopy theoretic information for A. Our aim will be to apply this strategy to those simplicial algebras which possess *Noetherian homotopy*. This allows us to have sophisticated techniques from commutative algebra at our disposal. One consequence of our efforts will be to resolve a more general form of a conjecture of Quillen that was posed in Invent. Math. 142 (3) (2000) 547. © 2002 Elsevier Science B.V. All rights reserved.

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0. Overview

Our focus, in this paper, is to take the view that the study of Noetherian rings and algebras through homological methods is a special case of the study of simplicial commutative algebras having Noetherian homotopy type. Our goal is to show that such simplicial algebras can be given a suitably rigid structure in the homotopy category, which then allows us to bring in methods from commutative algebra. Such methods

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should enable more facile techniques from homological algebra to be ferried in for the purpose of elaborating the global structure of such simplicial algebras.

To begin, we define for a simplicial commutative algebra A to have *Noetherian homotopy* provided:

(1) $\pi_0 A$ is a Noetherian ring, and
(2) each $\pi_m A$ is a finite $\pi_0 A$-module.

If, more strongly, $\pi_* A$ is a finite graded $\pi_0 A$-module, we that A has *finite Noetherian homotopy*.

In order to achieve a more systematic study of simplicial algebras with Noetherian homotopy, particularly to allow us a straighter path to proving our main result, Theorem B below, we first seek to rigidify the action of π_0 from the homotopy groups to the simplicial algebra. This is accomplished by the following:

Theorem A. *Any simplicial commutative algebra A is weakly equivalent to a connected simplicial supplemented $\pi_0 A$-algebra.*

Theorem A provides the means to import in methods from commutative algebra, most notably localizations and completions. In particular, we use these methods as a means to provide a proof of a conjecture posed in [12] which generalizes a conjecture of Quillen regarding the vanishing of André–Quillen homology. Our larger interests lie in providing an understanding of the homotopy type of a simplicial commutative algebra A with Noetherian homotopy over a Noetherian ring R through its André–Quillen homology $D(A|R;-)$. Here we shall view this homology as a functor of $\pi_0 A$-modules. This enables us to be specific about the homology's rigidity properties.

Before stating our result, we first need a homotopy invariant notion of complete intersection. To obtain one, we first define a map $A \to B$ of simplicial commutative R-algebras, augmented over a field ℓ, to be *virtually acyclic* provided $D_{\geqslant 1}(B|A;\ell)=0$. Also, if W is a graded ℓ-module, define the simplicial ℓ-algebra $S_\bullet(W)$ by

$$S_\bullet(W) = \bigotimes_n S(W_n, n),$$

where $S(V,n)$ is the free commutative ℓ-algebra generated by the Eilenberg–MacLane space $K(V,n)$.

Define a simplicial commutative R-algebra A over ℓ to be a *homotopy n-intersection*, for $n \geqslant 1$, provided there is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta'} \\ A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ \ell & \xrightarrow{=} & \ell \end{array}$$

with the horizontal maps being virtually acyclic over ℓ and in the homotopy category there is an isomorphism.

$$A' \otimes^{\mathbf{L}}_{R'} \ell \cong S_{\bullet}(W)$$

with W a graded ℓ-module satisfying $W_{>n} = 0$. We call a general simplicial commutative R-algebra A a *locally homotopy n-intersection* if, for each $\bar{\wp} \in \mathrm{Spec}(\pi_0 A)$, A is a homotopy n-intersection over the residue field $k(\wp)$.

Recall that the *flat dimension* of an R-module M to be the positive integer $\mathrm{fd}_R M$ such that

$$\mathrm{fd}_R M \leqslant m \Leftrightarrow \mathrm{Tor}_i^R(M, -) = 0 \quad \text{for } i > m. \tag{0.1}$$

Theorem B. *Let A be a simplicial commutative R-algebra with finite Neotherian homotopy*, $\mathrm{char}(\pi_0 A) \neq 0$, *and* $\mathrm{fd}_R(\pi_* \mathrm{A})$ *finite. Then $D_s(A|R; -) = 0$ for $s \gg 0$ if and only if A is a locally homotopy* 1-*intersection.*

This resolves a conjecture posed in [12] generalizing a conjecture of Quillen [10, 5.7].

Notes:

(1) Theorem B fails when $\mathrm{char}(\pi_0 A) = 0$, as shown in [12].
(2) Theorem B fails for general simplicial algebras having Noetherian homotopy. The case of a simplicial algebra $S(V, n)$ over a field of non-zero characteristic provides counterexample, by computations of Cartan [5].
(3) A homomorphism between Noetherian rings is a locally complete intersection if and only if it is a locally homotopy 1-intersection, as shown in [2,12].

Quillen further conjectured a more general result [10, 5.6] which drops the finite flat dimension condition. We would like to indicate a possible simplicial version of this conjecture. To formulate it, we first indicate a special vanishing result for André–Quillen homology that we will prove.

Theorem C. *Let A be a simplicial commutative R-algebra with Noetherian homotopy. Then $D_s(A|R; -) = 0$ for $s \geqslant 3$ if and only if A is a locally homotopy* 2-*intersection.*

This now leads us to pose the following:

Conjecture. *Let A have finite Noetherian homotopy with* $\mathrm{char}(\pi_0 A) \neq 0$. *Then $D_s(A|R; -) = 0$ for $s \gg 0$ implies that A is a locally homotopy* 2-*intersection.*

The strategy for proving Theorem B is to show that $D_s(A|R; k(\wp)) = 0$ for $s \geqslant 2$ for each $\wp \in \mathrm{Spec}(\pi_0 A)$. This is sufficient by a result of André [1, S.30]. Following a strategy of Avramov [2], we use Theorem A coupled with commutative algebra techniques developed in [3] to replace A with $A(\wp)$, its *connected component at* $\wp$, which has the following properties:

(1) $A(\wp)$ is a connected simplicial supplemented $k(\wp)$-algebra;
(2) $\mathrm{fd}_R(\pi_* A) < \infty$ implies that $A(\wp)$ has finite Noetherian homotopy; and
(3) $D_s(A|R;k(\wp)) \cong D_s(A(\wp)|k(\wp);k(\wp))$ for $s \geqslant 2$.

Theorem B now follows from the algebraic version of a theorem of Serre established in [12].

1. Postnikov systems and Theorem A

Throughout this paper, we fix a commutative ring with unit Λ and let $\mathscr{A}lg_\Lambda$ be the category of (unitary) commutative rings augmented over Λ. Finally, we denote by ${}_\Lambda\mathscr{A}lg_\Lambda$ the category of Λ-algebras in $\mathscr{A}lg_\Lambda$.

We will also be assuming the reader has an acquaintance with closed (simplicial) model category theory. Our main resource is [9]. We will further need specific results on the model category structure for simplicial commutative rings and algebras, our primary sources being [9,12,6].

1.1. Postnikov systems

Let A be an object in the category $s\mathscr{A}lg_\Lambda$ of simplicial commutative rings over Λ. We review the construction of a Postnikov tower for A derived from [4,7] which we will be use in the proof of Theorem A.

Following [7, Section 5], define the nth Postnikov section of A as follows: for fixed k, let $I_{n,k} \to A_k$ be the kernel of the map

$$d : A_k \to \prod_{\phi : [m] \to [k]} A_n,$$

where ϕ runs over all injections in the ordinal number category with $m \leqslant n$, d is induced by the maps $\phi^* : A_k \to A_m$, and $\prod$ denotes the product in the category of algebras augmented over Λ. Define

$$A(n)_k = A_k / I_{n,k}. \tag{1.1}$$

Notice that there is a quotient map in $s\mathscr{A}lg_\Lambda$, $A \to A(n)$, and that if $k \leqslant n$, $A(n)_k = A_k$. There are also quotient maps

$$q_n : A(n) \to A(n-1) \tag{1.2}$$

and $A \cong \lim A(n)$. Let $F(n)$ be the fibre of q_n, i.e.

$$F(n) = \ker(q_n : A(n) \to A(n-1)). \tag{1.3}$$

Note that $F(n) \to A(n) \xrightarrow{q_n} A(n-1)$ forgets to a fibration sequence as simplicial abelian groups. As such, the following can be proved just as in [7, 5.5].

Lemma 1.1. *The homotopy groups of $F(n)$ are computed as follows*:

$$\pi_k F(n) = \begin{cases} \pi_n A & k = n; \\ 0 & k \neq n. \end{cases}$$

1.2. Eilenberg–MacLane objects

Following [4, Section 5], define an object A of $s\mathscr{A}lg_\Lambda$ to be *of type K_Λ* if $\pi_0 A \cong \Lambda$ and the higher homotopy groups of A are trivial. Suppose M is a Λ-module. We say that a map $A \to B$ is *of type $K_\Lambda(M,n)$* $n \geqslant 1$, if A is of type K_Λ, $\pi_0 B \cong \Lambda$, $\pi_n B \cong M$ (as a Λ-module), all other homotopy groups of B are trivial, and the map $A \to B$ is a π_0-isomorphism.

For a general map $f : A \to B$ in $s\mathscr{A}lg_\Lambda$, let C be the pushout of the diagram $B' \leftarrow A' \to A(0)'$ obtained by using a functorial construction to replace A by a cofibrant object and the two maps $A \to B$ and $A \to A(0)$ by cofibrations. There is then a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\sim}\uparrow & & \uparrow{\sim} \\ A' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ A(0)' & \xrightarrow{\Delta_n(f)} & C(n+1) \end{array} \tag{1.4}$$

The bottom map $\Delta_n(f)$ is called the *difference construction of f*. The following can be proved just as in [4, 6.3].

Proposition 1.2. *Suppose that $A \to B$ is a map of simplicial commutative algebras which is a π_0-isomorphism and whose homotopy fibre F is $(n-1)$-connected. Let $M = \pi_n F$. Then M is naturally a Λ-module for $\Lambda = \pi_0 B$ and $\Delta_n(f)$ is a map of type $K_\Lambda(M, n+1)$. If $\pi_k F$ vanishes except for $k = n$, then the right-hand square in* 1.4 *is a homotopy fibre square.*

1.3. Differentials functor

For an object A in $\mathscr{A}lg_\Lambda$, define its *Λ-differentials* to be the Λ-module

$$D_\Lambda A = J/J^2 \otimes_A \Lambda,$$

where J is the kernel of the product $A \otimes A \to A$. As a functor to the category of Λ-modules, D_Λ posseses a right adjoint—the functor

$$(-)_+ : \mathrm{Mod}_\Lambda \to \mathscr{A}lg_\Lambda$$

defined by $M_+ = M \oplus \Lambda$ with the usual twisted product

$$(x, a) \cdot (y, b) = (bx + ay, ab).$$

An equivalent identification of the differentials functor

$$D_\Lambda \cong I/I^2 \otimes_A \Lambda, \tag{1.5}$$

where I is the augmentation ideal of A, which can be seen to follow from Yoneda's lemma.

The next proposition is proved in [9, Section II.5].

Proposition 1.3. *The prolonged adjoint pair of functors*

$$D_\Lambda : s\mathscr{A}lg_\Lambda \Leftrightarrow s\,\mathrm{Mod}_\Lambda : (-)_+$$

induces an adjoint pair on the homotopy categories

$$\mathbf{L}D_\Lambda : \ \mathrm{Ho}(s\mathscr{A}lg_\Lambda) \Leftrightarrow \mathrm{Ho}(s\,\mathrm{Mod}_\Lambda) : \mathbf{R}(-)_+.$$

Finally, the following useful property of the derived functor of differentials follows from [11, 7.3].

Proposition 1.4. *If $f : A \to B$ is a $\pi_{\leqslant n}$-isomorphism, then $\mathbf{L}D_\Lambda(f)$ is a $\pi_{\leqslant n}$-isomorphism.*

1.4. Characterizing $K_\Lambda(M,n)$-type

Fix a Λ-module M. In $s\,\mathrm{Mod}_\Lambda$, the fibration $p_n : E(M,n) \to K(M,n)$ is determined by the Dold–Kan correspondence to correspond to the map of normalized chain complexes $\{M \xrightarrow{1} M\} \to \{M\}$ with the source concentrated in degrees n and $n-1$, the target concentrated in degree n, and the map being the identity in degree n and trivial otherwise.

Applying $(-)_+$ to p_n gives a $K_\Lambda(M,n)$-type fibration in $s\mathscr{A}lg_\Lambda$

$$(p_n)_+ : E_\Lambda(M,n) \to K_\Lambda(M,n),$$

which we call the *canonical map of type $K_\Lambda(M,n)$*.

Proposition 1.5. *Let $A \to B$ be of type $K_\Lambda(M,n)$ between cofibrant objects in $s\mathscr{A}lg_\Lambda$. Then there is a commuting diagram in $s\mathscr{A}lg_\Lambda$*

$$\begin{array}{ccc} A & \xrightarrow{\sim} & E_\Lambda(M,n) \\ \downarrow & & \downarrow{\scriptstyle p_n} \\ B & \xrightarrow{\sim} & K_\Lambda(M,n) \end{array}$$

with the horizontal maps being weak equivalences.

Proof. To begin, note that the canonical map $B \to \Lambda$ is $(n-1)$-connected. Thus the induced map $D_\Lambda B \to 0$ is $(n-1)$-connected by Proposition 1.4. Let $I = \ker(B \to \Lambda)$. Filtering B by powers of I we note that B cofibrant implies that

$$I^q/I^{q+1} = S_q^\Lambda(I/I^2) \cong S_q^\Lambda(D_\Lambda B),$$

where the last identity always holds when the augmentation is surjective, by (1.5). Thus there is a convergent spectral sequence

$$E^1_{p,q} = H_{p+q}[S_q^\Lambda(D_\Lambda B)] \Rightarrow \pi_{p+q}B.$$

From the connectivity indicated above and [11, 7.40], $E^1_{p,q} = 0$ for $0 < p+q \leqslant 2(q-2)+n$. Thus we obtain

$$M \cong \pi_n B \cong \pi_n D_\Lambda B.$$

Thus there is an n-connected map $D_\Lambda B \to K(M,n)$ and its adjoint $B \to K_\Lambda(M,n)$ will be a weak equivalence by the computations above and the assumption that $A \to B$ is of type $K_\Lambda(M,n)$.

Finally, $A \to \Lambda$ is a weak equivalence, hence $D_\Lambda A \to 0$ is a weak equivalence by Proposition 1.4. Since A, and hence $D_\Lambda A$, are cofibrant, the composite $D_\Lambda A \to D_\Lambda B \to K(M,n)$ lifts to a map $D_\Lambda A \to E(M,n)$, whose adjoint $A \to E_\Lambda(M,n)$ is necessarily a weak equivalence. □

1.5. Proof of Theorem A

Fix an object A in $s\mathscr{A}lg_\Lambda$. We will show, by induction, that there is a map $X \to Y$ in $s_\Lambda\mathscr{A}lg_\Lambda$ and a commutative diagram in $\mathrm{Ho}(s\mathscr{A}lg_\Lambda)$

$$\begin{array}{ccc} A(n) & \xrightarrow{\sim} & X \\ {\scriptstyle q_n}\downarrow & & \downarrow \\ A(n-1) & \xrightarrow{\sim} & Y \end{array} \tag{1.6}$$

with the horizontal maps being equivalences. It is clear for $n = 0$ as $A(0) \to \Lambda$ is a weak equivalence.

Using 1.4, some closed model category theory and induction, we may assume that there is a trivial fibration $\sigma : A(n-1)' \to Y$ with the target Y a cofibrant object in $s_\Lambda\mathscr{A}lg_\Lambda$.

Lemma 1.6. *Let $M = \pi_n A$. Then there is a commuting diagram in* $\mathrm{Ho}(s\mathscr{A}lg_\Lambda)$ *of the form*

$$\begin{array}{ccc} A(n-1)' & \longrightarrow & C(n+1) \\ \sim\downarrow{\scriptstyle\sigma} & & \downarrow\sim \\ Y & \longrightarrow & K_\Lambda(M,n+1) \end{array}$$

with the top arrow from 1.4.

Proof. First, note that since $\sigma: A(n-1)' \to Y$ is a trivial fibration between suitably cofibrant objects (see above) it follows from that and from 1.5 that

$$D_A\sigma : D_A A(n-1)' \to D_A Y$$

is a trivial fibration between cofibrant objects in $s\,\mathrm{Mod}_A$. By [9, I.1.7], $D_A\sigma$ has a homotopy left inverse i ($i \circ D_A\sigma \simeq \mathrm{Id}_{D_A A(n-1)}$).

Next, utilizing Lemma 1.6, let $t: A(n-1)' \to K_A(M, n+1)$ be the composite of $A(n-1)' \to C(n+1) \to K_A(M, n+1)$. Let $w: D_A Y \to K(M, n+1)$ be the composite $(D_A t) \circ i$. Then $w \circ D_A\sigma \simeq D_A t$ and the result now follows from Proposition 1.3. □

From the previous lemma, we may form the homotopy pullback diagram in $s_A\mathscr{A}lg_A$

$$\begin{array}{ccc} X & \longrightarrow & E_A(M, n+1) \\ \big\downarrow & & \big\downarrow {\scriptstyle (p_n)_+} \\ Y & \longrightarrow & K_A(M, n+1). \end{array} \tag{1.7}$$

By Proposition 1.2, the diagram below is also a homotopy pullback in $s\mathscr{A}lg_A$

$$\begin{array}{ccc} A(n)' & \longrightarrow & A(0)' \\ {\scriptstyle q_n'}\big\downarrow & & \big\downarrow {\scriptstyle \Delta[q_n]} \\ A(n-1)' & \longrightarrow & C(n+1). \end{array} \tag{1.8}$$

By Proposition 1.5 and Lemma 1.6, there is an induced map of diagrams (1.8) to (1.7) in the category $\mathrm{Ho}(s\mathscr{A}lg_A)$. Since fibrations and pullbacks in $s\mathscr{A}lg_A$ are fibrations and pullbacks as simplicial groups, a computation of homotopy groups can be performed utilizing Lemma 1.1 to show that the induced map $A(n)' \to X$ is a weak equivalence. This completes the induction step.

2. André–Quillen homology and Theorems B and C

2.1. Base change property of André–Quillen homology

Recall that the *cotangent complex* of a simplicial R-algebra A is defined to be the object of $\mathrm{Ho}(\mathrm{Mod}_A)$

$$\mathscr{L}(A|R) := \Omega_{P|R} \otimes_P A, \tag{2.9}$$

where the T-module $\Omega_{T|S} = J/J^2$, $J = \ker(T \otimes_S T \to T)$, denotes the *Kahler differentials* of an S-algebra T, and $P \to A$ is a cofibrant replacement of A as a simplicial R-algebra.

Note: As in Section 1.3, $\Omega_{T|S}$ is left adjoint to the functor $M \mapsto M \oplus T$ where the image has a T-algebra structure with $M^2 = 0$.

Also recall that given another simplicial R-algebra B, the *derived tensor product* of A and B to be the object of $\mathrm{Ho}(s\,\mathrm{Mod}_\mathrm{R})$

$$A \otimes_R^{\mathbf{L}} B := P \otimes_R Q,$$

where $Q \to B$ is a cofibrant replacement of B.

We now derive a base change property for the cotangent complex following [11].

Lemma 2.1. *If* $\mathrm{Tor}_q^R(A_k, B_k) = 0$ *for all* $k \geqslant 0$ *and all* $q > 0$ *then* $A \otimes_R^{\mathbf{L}} B \simeq A \otimes_R B$.

Proof. This follows immediately from the spectral sequence [9, Section II.6]

$$E^2_{p,q} = \pi_p \mathrm{Tor}_q^R(A, B) \Rightarrow \pi_{p+q}(A \otimes_R^{\mathbf{L}} B). \qquad \square$$

Lemma 2.2. $\Omega_{A \otimes_R B|B} \cong \Omega_{A|R} \otimes_R B$.

Proof. Let $A' = A \otimes_R B$ and fix an A'-module M. Then

$$\begin{aligned}\hom_{A'}(\Omega_{A'|B}, M) &\cong \hom_{_B\mathrm{Alg}_{A'}}(A', M \oplus A')\\ &\cong \hom_{_R\mathrm{Alg}_{A}}(A, M \oplus A)\\ &\cong \hom_A(\Omega_{A|R}, M)\\ &\cong \hom_{A'}(\Omega_{A|R} \otimes_R B, M).\end{aligned}$$

The result now follows from Yoneda's lemma. $\square$

Proposition 2.3. $\mathscr{L}(A \otimes_R^{\mathbf{L}} B|B) \simeq \mathscr{L}(A|R) \otimes_R^{\mathbf{L}} B$.

Proof. Fix cofibrant replacements P and Q for A and B, respectively. Then

$$\mathscr{L}(A \otimes_R^{\mathbf{L}} B|B) = \Omega_{P \otimes_R Q|Q} \cong \Omega_{P|R} \otimes_R Q \qquad (2.10)$$

by Lemma 2.2. Since P is projective as a simplicial R-module then $\Omega_{P|R}$ is a projective P-module. Thus, by Lemma 2.1, the map $\Omega_{P|R} \xrightarrow{\sim} \Omega_{P|R} \otimes_P A$ is a weak equivalence. Since Q is projective, Lemma 2.1 further tells us that

$$\Omega_{P|R} \otimes_R Q \xrightarrow{\sim} (\Omega_{P|R} \otimes_P A) \otimes_R Q \cong \mathscr{L}(A|R) \otimes_R^{\mathbf{L}} B \qquad (2.11)$$

is a weak equivalence. The result now follows by combining 2.10 with 2.11. $\square$

Corollary 2.4. *As a functor of* $A \otimes_R B$*-modules,* $D_*(A \otimes_R^{\mathbf{L}} B|B; -) \cong D_*(A|R; -)$.

Proof. This follows from Proposition 2.3 and the identity $D_*(T|S; M) := \pi_*[\mathscr{L}(T|S) \otimes_T M]$. $\square$

2.2. Proof of Theorem B

We first recall the main result of [12].

Theorem 2.5. *Let* A *be a homotopy connected simplicial supplemented commutative algebra over a field* ℓ *of non-zero characteristic. Then* $D_s(A|\ell; \ell) = 0$ *for* $s \gg 0$ *implies that there is an equivalence* $S_\ell(D_1(A|\ell; \ell), 1) \cong A$ *in the homotopy category.*

We now begin by establishing a special case of Theorem A. To that end let A be a simplicial commutative R-algebra and assume that the unit $R \to \pi_0 A = A$ is a surjection. For $\wp \in \operatorname{Spec} \Lambda$, define the *connected component of A at* $\wp$ to be the connected simplicial supplemented $k(\wp)$-algebra

$$A(\wp) = A \otimes_R^{\mathbf{L}} k(\wp).$$

Lemma 2.6. *Let A be as above. Then*

(1) $D_*(A|R; k(\wp)) \cong D_*(A(\wp)|k(\wp); k(\wp))$, *and*
(2) *if A also has finite Noetherian homotopy and* $\mathrm{fd}_R(\pi_* A) < \infty$ *it follows that* $A(\wp)$ *has finite Noetherian homotopy.*

Proof. (1) follows from Corollary 2.4. For (2), [9, Section II.6] gives a spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^R_s(\pi_t A, k(\wp)) \Rightarrow \pi_{s+t}(A \otimes^{\mathbf{L}}_R k(\wp)).$$

From the finiteness conditions, each $E^2_{s,t}$ is a finite $k(\wp)$-module and vanishes for $s,t \gg 0$. Thus $A \otimes^{\mathbf{L}}_R k(\wp))$ has finite Noetherian homotopy. □

Corollary 2.7. *Let A be as in* Lemma 2.6(2) *and further assume that* $\mathrm{char}(k(\wp)) \neq 0$. *Then* $D_s(A|R; k(\wp)) = 0$ *for* $s \gg 0$ *implies that* $D_s(A|R; k(\wp)) = 0$ *for* $s \geqslant 2$.

Proof. This follows from Lemma 2.6 and Theorem 2.5 □

Now assume that the simplicial algebra A in question is a homotopy connected simplicial supplemented Λ-algebra, by Theorem A. We further assume that A has Noetherian homotopy.

Fix $\wp \in \operatorname{Spec}\Lambda$ and let $\widehat{(-)}$ denote the completion functor on R-modules at $\wp$. Define the homotopy connected simplicial supplemented $\hat{\Lambda}$-algebra A' by

$$A' = A \otimes^{\mathbf{L}}_{\Lambda} \hat{\Lambda}.$$

Proposition 2.8. *Suppose A is a simplicial commutative R-algebra, with R a Noetherian ring. Then* $\pi_* A' \cong \widehat{\pi_* A}$ *and there exists a (complete) Noetherian* R' *that fits into the following commutative diagram in* $\mathrm{Ho}(s_R\mathscr{A}lg)$

$$\begin{array}{ccc} R & \xrightarrow{\eta} & A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ R' & \xrightarrow{\eta'} & A' \end{array}$$

with the following properties:

(1) ϕ *is a flat map and its closed fibre* $R'/\wp R'$ *is weakly regular*;
(2) ψ *is a* $D_*(-|R; k(\wp))$-*isomorphism*;
(3) η' *induces a surjection* $\eta'_*: R' \to \pi_0 A'$;
(4) $\mathrm{fd}_R(\pi_* A)$ *finite implies that* $\mathrm{fd}_{R'}(\pi_* A')$ *is finite.*

Proof. First, Quillen's spectral sequence [9, II.6] $\mathrm{Tor}_*^{\Lambda}(\pi_*A, \hat{\Lambda}) \Rightarrow \pi_*A'$ collapses to give the first result since $\hat{\Lambda}$ is flat over Λ and each $\pi_m A$ is finite over Λ [8, 8.7 and 8.8].

Next, by [3, 1.1], the unit ring homomorphism $R \to \hat{\Lambda}$ factors as $R \xrightarrow{\phi} R' \xrightarrow{\eta'_*} \hat{\Lambda}$ with ϕ having the properties described in (1) and η'_* is a surjection. Thus the induced map $\eta' : R' \to A'$ induces a surjection on π_0, giving (3) and the desired diagram commutes.

Now, by the transitivity sequence [11, 4.12] applied to $R \to A \to A'$, (2) follows from the isomorphism

$$D_*(A'|A; k(\wp)) \cong D_*(\hat{\Lambda}|\Lambda; k(\wp)) \cong 0$$

which follows from Corollary 2.4.

Finally, (4) follows from [3, 3.2], as A has Noetherian homotopy. □

Now, let A have finite Noetherian homotopy with $D_s(A|R; -) = 0$ for $s \geqslant 0$. From Proposition 2.8, Theorem 2.5, Corollary 2.7, and [1, Section S.30], if $\mathrm{fd}_R(\pi_*A) < \infty$ then $A(\wp) \cong S_{k(\wp)}(D_1(A|R; k(\wp), 1)$, for each $\wp \in \mathrm{Spec}(\pi_0 A)$, if and only if $D_2(A|R; -) = 0$. Thus Theorem B follows from the definition of locally homotopy complete intersection (see introduction) and a transitivity sequence argument.

2.3. *Proof of Theorem C*

Let A be a simplicial commutative R-algebra with Noetherian homotopy. It follows from Lemma 2.6(1), Proposition 2.8, and [1, Section S.30], that $D_{\geqslant 3}(A|R; -) = 0$ if and only if $D_{\geqslant 3}(A(\wp)|k(\wp); k(\wp)) = 0$, for all $\wp \in \mathrm{Spec}(\pi_0 A)$. From the definition of locally virtual homotopy complete intersection (see introduction), Theorem C will follow if we can show that, for each prime ideal $\wp$, $A(\wp) \cong S_\bullet(D_{\leqslant 2}(A|R; k(\wp)))$ in the homotopy category. But this in turn follows from [12, (2.2)].

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