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Vertices, Blocks, and Virtual Characters

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INTRODUCTION

M. Broué and L. Puig derived in [2] a rather remarkable construction for virtual characters of a finite group. Here we make an extension of their construction which says essentially that in their work the defect group may be replaced by a vertex. See Theorem 3.2 below. Along the way we also obtain a result (Corollary 2.7) relating vertices and blocks which is of independent interest and has consequences in strengthening Brauer's Second Main Theorem in a special case.

Our original results and proofs dealt only with modules and vertices. However, after reading an early version of this paper L. Puig pointed out to us that our arguments can easily be generalized to G-algebras and pointed groups. This allows one to prove the more general Theorem 2.5 rather than our original result, now called Corollary 2.7. Using pointed groups —especially using Puig's characterization of generalized decomposition numbers (Theorem 5.6 of [3])—also allows significant simplification of our original proof of Theorem 3.2. With his kind permission, we include this more general work here. Indeed, Puig has suggested even more generality, one version of which we discuss in Section 4.

1. NOTATION

In the sequel we assume the reader is familiar with both [1] and [3], whose notations and results we will use freely. Throughout, G is a fixed but arbitrary finite group. As in [3], $\mathcal{O}$ is to be a complete discrete valuation ring of characteristic 0 which contains a primitive $|G|$th root of 1 and whose residue field k is algebraically closed and of characteristic $p > 0$. Modules over $\mathcal{O}$-algebras considered here are to be unital and free and finitely generated over $\mathcal{O}$.

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As is common in this area, it is convenient to say that an element e of a ring is an idempotent if it satisfies $e^2 = e \neq 0$. That is, the zero element of a ring is not considered to be an idempotent. Apparently this differs from the approach in [3], but should cause no confusion to the reader.

Our point of view differs slightly from that of [1], in that we consider a subpair to be a pair (P, b) consisting of a p-subgroup P of G and a centrally primitive idempotent b of $\mathcal{O}[PC(P)]$. In [1] one uses instead centrally primitive idempotents of $k[PC(P)]$. Our hypotheses on $\mathcal{O}$ guarantee that reduction modulo the Jacobson radical of $\mathcal{O}$ yields a one-to-one correspondence between such idempotents, and results we need from [1] translate easily to our setting.

2. Vertex Subpairs

We fix $A = \mathcal{O}[G]$ and E an arbitrary interior G-algebra over $\mathcal{O}$. Let f be the canonical map from A to E. When P_γ is a local pointed group on A, we let b_γ be the block idempotent of $\mathcal{O}[PC(P)]$ associated with γ; that is, $b_\gamma i = i b_\gamma = i$ for some $i \in \gamma$.

We begin with a few elementary results relating pointed groups over A and E.

Lemma 2.1. *If $P_{\gamma'}$ is a local pointed group on E and $m_{\gamma'}^{\gamma}(f) \neq 0$ for some point γ of P on A then P_γ is a local pointed group on A.*

Proof. If γ is not local then

$$\gamma \subseteq \sum_{Q < P} A_Q^P$$

whence

$$f(\gamma) \subseteq \sum_{Q < P} E_Q^P,$$

which is an ideal of E^P. If $i' \in \gamma'$ satisfies $f(i)i' = i'f(i) = i'$ for some $i \in \gamma$, then i' is also in this ideal, so γ is not local, a contradiction.

Lemma 2.2. *Suppose $Q_{\delta'} \subseteq P_{\gamma'}$ are pointed groups on E and P_γ is a pointed group on A with $m_{\gamma'}^{\gamma}(f) \neq 0$. Then there is a pointed group Q_δ on A such that $m_{\delta'}^{\delta}(f) \neq 0$ and $Q_\delta \subseteq P_\delta$.*

Proof. We have

$$0 \neq m_{\delta'}^{\gamma}(f) = \sum_{\delta \in P_A(Q)} m_\delta^\gamma \, m_{\delta'}^{\delta}(f).$$

Since at least one of the terms in the sum must be non-zero, Q_δ exists as claimed.

LEMMA 2.3. *Suppose $Q_\delta \subseteq P_\gamma$ are pointed groups on A and $Q_{\delta'}$ is a pointed group on E with $m_{\delta'}^{\delta}(f) \neq 0$. Then there is a pointed group $P_{\gamma'}$ on E such that $m_{\gamma'}^{\gamma}(f) \neq 0$ and $Q_{\delta'} \subseteq P_{\gamma'}$.*

Proof. We have

$$0 \neq m_{\delta'}^{\gamma}(f) = \sum_{\gamma' \in P_E(P)} m_{\gamma'}^{\gamma}(f)\, m_{\delta'}^{\gamma'}.$$

Again, one of the terms in the sum must be non-zero, so $P_{\gamma'}$ exists as claimed.

LEMMA 2.4. *Suppose 1_E is the unique idempotent in E^G and let $P_{\gamma'}$ be a maximal local pointed group on E. Then there is a unique block idempotent b of $\mathcal{O}[N(P_{\gamma'})]$ such that $f(b)i' = i'f(b) = i'$ for some $i' \in \gamma'$.*

Proof. Set $N = N(P_{\gamma'})/P$. According to Proposition 1.3 of [3] we have $S_{\gamma'}(E_P^G) = E(P_{\gamma'})_1^N$. Since $1_E \in E_P^G$, this means that $E(P_{\gamma'})_1^N = E(P_{\gamma'})^N$. Also, 1_E is the unique idempotent in E_P^G so $E(P_{\gamma'})^N$ likewise contains a unique idempotent. The choice of b above guarantees that $0 \neq S_{\gamma'}(f(b)) \in E(P_{\gamma'})^N$, whence $S_{\gamma'}(f(b))$ must be this unique idempotent. Since distinct block idempotents of $\mathcal{O}[N(P_{\gamma'})]$ are orthogonal, the uniqueness of b follows.

It is important that 1_E be the unique idempotent in E^G in the above. Of course, one could fix a primitive idempotent $e \in E^G$ and consider eEe. Since $(eEe)^G$ has a unique idempotent e, the result applies with E replaced by eEe. However, different choices of e can lead to different blocks b.

THEOREM 2.5. *Suppose 1_E is the unique idempotent in E^G. Let $P_{\gamma'}$ be a maximal local pointed group on E and let γ be any point of P on A with $m_{\gamma'}^{\gamma}(f) \neq 0$. If $Q_{\delta'}$ is any local pointed group on E and δ any local point of Q on A with $m_{\delta'}^{\delta}(f) \neq 0$ then there is an element $g \in G$ such that both $(Q_{\delta'})^g \subseteq P_{\gamma'}$ and $(Q, b_\delta)g \subseteq (P, b_\gamma)$.*

Proof. By Theorem 1.2 (iii) of [3] there is an element $x \in G$ with $(Q_{\delta'})^x \subseteq P_{\gamma'}$, so $|Q| \leqslant |P|$. We proceed by induction on the ratio $|P|/|Q|$. Set $H = N(Q_{\delta'})$.

First suppose $|Q| = |P|$. In this case we may assume $Q_{\delta'} = P_{\gamma'}$. Let b be the block idempotent of $\mathcal{O}[H]$ guaranteed by Lemma 2.4. Then both b_γ^H and b_δ^H are b so b_γ and b_δ are conjugate in H.

Now consider the case $|Q| < |P|$. Let β be a point of H on A such that $Q_\delta \subseteq H_\beta$. By Lemma 2.3 there is a point β' of H on E such that $m_{\beta'}^{\beta}(f) \neq 0$

and $Q_{\delta'} \subseteq H_{\beta'}$. According to Corollary 1.4 of [3]. $Q_{\delta'}$ is not maximal among local pointed groups contained in $H_{\beta'}$. Let $R_{\varepsilon'}$ be maximal among local pointed groups contained in $H_{\beta'}$ and containing $Q_{\delta'}$. By Lemma 2.2 there is a point ε of R on A such that $m_{\varepsilon'}^{\varepsilon}(f) \neq 0$ and $R_\varepsilon \subseteq H_\beta$. Since ε' is local, Lemma 2.1 guarantees that ε is local as well. Since $|Q| < |R|$ the induction hypothesis implies the existence of an element $g \in G$ such that both

$$(R_{\varepsilon'})^g \subseteq P_{\gamma'} \qquad \text{and} \qquad (R, b_\varepsilon)^g \subseteq (P, b_\gamma).$$

Now let S_φ be a maximal local pointed group contained in H_β and containing R_ε. Since all maximal local pointed groups contained in H_β are H-conjugate according to Theorem 1.2 (iii) of [3], there is an element $h \in H$ such that $(Q_\delta)^h \subseteq S_\varphi$. Thus, $(Q, b_\delta)^h$ and (R, b_ε) are both contained in (S, b_φ). Since $Q^h = Q \leqslant R$, uniqueness forces $(Q, b_\delta)^h \subseteq (R, b_\varepsilon)$. We therefore have

$$(Q_{\delta'})^{hg} \subseteq (R_{\varepsilon'})^g \subseteq P_{\gamma'} \qquad \text{and} \qquad (Q, b_\delta)^{hg} \subseteq (R, b_\varepsilon)^g \subseteq (P, b_\gamma),$$

as required.

DEFINITION 2.6. In the situation of Theorem 2.5, we will call the subpair (P, b_γ) a *vertex subpair for E*. It is obvious from Theorem 2.5 that vertex subpairs for E are uniquely determined up to G-conjugacy. In case V is an indecomposable A-module, the G-algebra $E = \mathrm{End}\mathcal{O}(V)$ satisfies the conditions on E and a vertex subpair for E will also be called a vertex subpair for V in this case.

We state the following important special case of Theorem 2.5.

COROLLARY 2.7. *Let V be an indecomposable A-module and (P, b) a vertex subpair for V. Suppose Q is a p-subgroup of G and* $\mathrm{Res}_{QC(Q)}(V)$ *has an indecomosable summand W with vertex containing Q. If b_Q is the block of $\mathcal{O}[QC(Q)]$ containing W then there is an element $g = G$ such that*

$$(Q, b_Q)^g \subseteq (P, b).$$

Proof. Let R be a vertex of the $\mathcal{O}[QC(Q)]$-module W, so that $Q \leqslant R \leqslant RC(R) \leqslant QC(Q)$. Let U be a summand of $\mathrm{Res}_{RC(R)}(W)$ with vertex R and let b_R be the block of $\mathcal{O}[RC(R)]$ containing U. According to Nagao's Lemma, b_R corresponds to b_Q via the Brauer correspondence in $QC(Q)$. That is, as subpairs, $(Q, b_Q) \subseteq (R, b_R)$. Let $E = \mathrm{End}_{\mathcal{O}}(V)$. Then there is a local pointed group $R_{\varepsilon'}$ on E such that Vi' is a summand of $\mathrm{Res}_R(U)$ for some $i' \in \varepsilon'$. Since multiplication by b_R induces the identity transformation on U, we have $f(b_R)i' = i'f(b_R) = i'$. That is, $f(b_R)$ may be expressed as a sum of orthogonal primitive idempotents of E^R with i' being one of the summands. Thus, there is a pointed group R_ε on A such that $b_R = b_\varepsilon$ and $m^\varepsilon_{\varepsilon'}(f) \neq 0$. Theorem 2.5 now guarantees that there is an element $g \in G$ such that $(R, b_\varepsilon)^g \subseteq (P, b)$ so we have

$$(Q, b_Q)^g \subseteq (R, b_R)^g = (R, b_\varepsilon)^g \subseteq (P, b),$$

as claimed.

Corollary 2.7 improves Brauer's Second Main Theorem in a special case. Let χ be an irreducible character of G, u a p-element of G, and φ an irreducible Brauer character of $C(u)$. Let b_φ be the block of $\mathcal{O}[C(u)]$ containing φ. Corollary 2.7 implies that the generalized decomposition number associated with χ, u, and φ is zero unless (u, b_φ) is contained in a vertex subpair for an $\mathcal{O}[G]$-module V affording χ. When V has vertex properly contained in a defect group of the block B, this may be slightly stronger information than Brauer's Second Main Theorem normally supplies.

The corollary also shows how to identify a vertex subpair for V. Namely, if P is a vertex of V and W is an indecomposable summand of $\mathrm{Res}_{PC(P)}(V)$ with vertex P then (P, b) is a vertex subpair for V when b is the block of $\mathcal{O}[PC(P)]$ containing W.

3. The Broué–Puig *-Product

We use the results of the previous section to derive an extension of the *-product introduced in [2].

Definition 3.1. Let P be a p-subgroup of G and (P, b) a subpair. We say an $\mathcal{O}$-valued virtual character η of P is *G-stable with respect to* (P, b) provided $\eta(u) = \eta(u^x)$ whenever $u \in P$, (u, e) is a Brauer element, $x \in G$, and both (u, e) and $(u, e)^x$ are contained in (P, b). In this case we define an $\mathcal{O}$-valued function $\hat{\eta}$ on Brauer elements (u, e) of $\mathcal{O}[G]$ via

$$\hat{\eta}(u, e) = \begin{cases} \eta(u^x) & \text{if } (u, e)^x \in (P, b) \text{ for some } x \in G \\ 0 & \text{if there is no such } x \in G. \end{cases}$$

If χ is an $\mathcal{O}$-valued class function on G, we also define a class function $\eta * \chi$ from G to $\mathcal{O}$ via

$$(\eta * \chi)(us) = \sum_e \hat{\eta}(u, e)\, \chi(eus),$$

where u is a p-element of G, s a p'-element of $C(u)$, and the sum is over all blocks e of $\mathcal{O}[C(u)]$. Our main result is the following.

THEOREM 3.2. *Let V be an indecomposable $\mathcal{O}[G]$-module with character χ and let (P, b) be a vertex subpair for V. If η is an $\mathcal{O}$-valued character of P which is G-stable with respect to (P, b) then $\eta^*\chi$ is a virtual character of G.*

Proof. We apply Theorem 5.6 of [3]. Thus, it suffices to show that $(\eta^*\chi)^\delta$ is a virtual character of Q whenever Q_δ is a local pointed group on $A = \mathcal{O}[G]$. Now, for $u \in Q$,

$$(\eta^*\chi)^\delta(u) = \sum_{\varepsilon \in LP_A(u)} (\eta^*\chi)^\varepsilon(u)\, m_\varepsilon^\delta = \sum_{\varepsilon \in LP_A(u)} \hat{\eta}(u, b_\varepsilon)\, \chi^\varepsilon(u)\, m_\varepsilon^\delta.$$

Clearly $m_\varepsilon^\delta \neq 0$ implies $(u, b_\varepsilon) \in (Q, b_\delta)$, so the block b_ε appearing in the non-zero terms here is unique; call it e_u. As before, let $E = \mathrm{End}_{\mathcal{O}}(V)$ and f the canonical homomorphism from A to E. Then we have

$$\begin{aligned}(\eta^*\chi)^\delta(u) &= \hat{\eta}(u, e_u) \sum_{\varepsilon \in LP_A(u)} \chi^\varepsilon(u)\, m_\varepsilon^\delta \\ &= \hat{\eta}(u, e_u)\, \chi^\delta(u) \\ &= \hat{\eta}(u, e_u) \sum_{\delta' \in P_E(Q)} \chi^{\delta'}(u)\, m(f)_{\delta'}^\delta,\end{aligned}$$

so it suffices to show that the function $u \to \hat{\eta}(u, e_u)\chi^{\delta'}(u)$ is a virtual character of Q whenever $\delta' \in P_E(Q)$ satisfies $m(f)_{\delta'}^\delta \neq 0$. Let $R_{\nu'}$ be a maximal local pointed group on E contained in $Q_{\delta'}$. According to Lemma 5.5 of [3] we have $\chi^{\delta'} = \mathrm{Ind}_R^Q(\chi^{\nu'})$. Since $\hat{\eta}$ is Q-stable, it suffices to show that the function $u \to \hat{\eta}(u, e_u)\chi^{\nu'}(u)$ is a virtual character of R. By Lemma 2.2, there is a point ν of R on A such that $m_{\nu'}^\nu(f) \neq 0$ and $R_\nu \subseteq Q_\delta$. Furthermore, Lemma 2.1 ensures that ν is local. Thus, $(R, b_\nu) \subseteq (Q, b_\delta)$. By uniqueness we must have $(u, e_u) \in (R, b_\nu)$. According to Theorem 2.5 there is an element $g \in G$ such that $(R, b_\nu)^g \subseteq (P, b)$, so also $(u, e_u)^g \in (P, b)$. Since g is independent of u, the function

$$u \to \hat{\eta}(u, e_u)\chi^{\nu'}(u) = \eta(u^g)\chi^{\nu'}(u)$$

is a product of virtual characters of R. This completes the proof.

Note that $\eta^*\chi$ is like the *-product in [2] except that there one uses a Sylow subpair instead of a vertex subpair. However, Corollary 2.7 implies that $\chi(eus) = 0$ unless (u, e) is contained in a vertex subpair, so there is no point in using a subpair larger than a vertex subpair. Furthermore, here we require only that η be a virtual character of a vertex rather than of a defect group.

Corollary 5.3 of [3] gives another generalization of the *-product, apparently different from ours. Puig has described to us a common generalization of both.

4. (b, G)-Brauer Pairs

One can generalize Theorem 2.5 to use the (b, G)-Brauer pairs of [2], Definition 1.6, rather than just subpairs. For this, let A be a p-permutation interior G-algebra and suppose there is a homomorphism F of interior G-algebras from A to E. Let P be a p-subgroup of G. The restriction $F^P : A^P \to E^P$ induces a homomorphism $F(P){:}A(P) \to E(P)$ and so homomorphisms $F_{\gamma'} : A(P) \to E(P_{\gamma'})$ satisfying

$$F_{\gamma'}(Br_P^A(a)) = S_{\gamma'}(F(a)), \qquad \text{for all} \quad a \in A^P,$$

whenever γ' is a local point of P on E. These $F_{\gamma'}$ can be used to give information similar to what we get out of the homomorphism f in Section 2, and all results there generalize quite easily to this setting. The reader may complete the details. Beware that one apparently needs the uniqueness of containment of pairs, Theorem 1.8 (1) of [2].

Puig has suggested further generalizations of all this as well (including a generalization of Theorem 5.2 of [3]), but since we are already straying from our own original results, we will leave the rest to him.

References

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