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Theoretical Computer Science 374 (2007) 214–228

Theoretical Computer Science

www.elsevier.com/locate/tcs

Characterizations of recognizable picture series☆

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Received 3 February 2006; received in revised form 21 December 2006; accepted 15 January 2007

Communicated by Z. Esik

Abstract

The theory of two-dimensional languages as a generalization of formal string languages was motivated by problems arising from image processing and pattern recognition, and also concerns models of parallel computing. Here we investigate power series on pictures. These are functions that map pictures to elements of a semiring and provide an extension of two-dimensional languages to a quantitative setting. We assign weights to different devices, ranging from picture automata to tiling systems. We will prove that, for commutative semirings, the behaviours of weighted picture automata are precisely alphabetic projections of series defined in terms of rational operations, and also coincide with the families of series characterized by weighted tiling or weighted domino systems.

Keywords: Picture series; Two-dimensional languages; Unambiguity; Automata

1. Introduction

In the literature, a variety of formal models to recognize or generate two-dimensional arrays of symbols, called pictures, have been proposed [3,13,17,21,31], and further properties of string languages have been formulated for two dimensions [4–6,19,23,24,33]. This research was motivated by problems arising from the area of image processing and pattern recognition [10,28], and also plays a role in frameworks concerning cellular automata and other models of parallel computing [22,32]. In the nineties, Restivo and Giammarresi defined the family REC of *recognizable picture languages* (cf. [11,13]). This family is very robust and has been characterized by many different devices, generalizing well-known equivalences of regular word languages. Several authors obtained an equivalence theorem for picture languages describing recognizable languages in terms of types of automata, projections of local sets of tiles, rational operations with projections or existential monadic second order logic [4,12,14,17,21]. It is the goal of this paper to generalize these equivalences to a quantitative setting.

We will investigate weighted picture automata (WPA) and their behaviours, cf. [25]. The interesting model of weighted (quadrapolic) automata was introduced by Bozapalidis and Grammatikopoulou [4] (for a very similar definition of Wang-systems see also [7]). WPA are automata operating in a natural way (the unweighted version

☆ Supported by the GK 446/3 of the German Research Foundation.

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0304-3975/$ - see front matter

doi:10.1016/j.tcs.2007.01.019

of a WPA characterizes precisely recognizable picture languages) on pictures and whose transitions carry weights; the weights are taken as elements from a given commutative semiring. The behaviour or the computation of a WPA is a function which maps pictures over a finite alphabet to elements of the semiring. We call such functions picture series. This generalization of picture languages enables us not only to describe the existence of certain patterns as some qualitative event, but to formulate now quantitative properties and attributes of two-dimensional objects. More precisely, these weighted picture devices can be used to model several examples, e.g. the intensity of light of a picture (interpreting the alphabet as different levels of gray) or the amplitude of a monochrome subpicture of a coloured picture.

Bozapalidis and Grammatikopoulou showed that picture series computed by WPA are closed under certain operations and projections on series. This was our starting point for raising the question whether the converse holds, i.e. whether the family of series that are behaviours of WPA and the family of projections of rational picture series coincide. We will prove this equivalence theorem for any alphabet and any commutative semiring in the first part of this paper. We will characterize the family of picture series recognized by WPA, also by using weighted tiling and weighted domino systems, and thus we obtain a robust definition of a class of recognizable picture series. Further characterizations, e.g. by weighted two-dimensional on-line tessellation automata (W2OTA) and in terms of weighted monadic second order logics are contained in [27,26]. These results extend the main findings of [13,14] to the weighted case; we get the results for languages by restricting the semiring to the Boolean semiring.

In the proofs, one has to be careful when arguing using an automaton which might have several successful runs for an input picture. If necessary, one has to consider or construct unambiguous picture automata, where every underlying input picture has at most one successful run, in order not to count weights twice. The notion of unambiguity for picture languages as injective projections of local languages was briefly introduced in [11]. There, Giammarresi and Restivo also posed the conjecture that unambiguous languages are properly included in the family of recognizable languages. This conjecture was solved very recently in [1], where the authors showed that unlike with words, there exist recognizable picture languages that are inherently ambiguous, i.e. not computable by unambiguous 2OTA; moreover, the problem whether a tiling system is unambiguous is undecidable. In [25–27] we proved an equivalence theorem for unambiguous picture languages in terms of unambiguous 2OTA and unambiguous tiling systems, unambiguously rational operations and also of unambiguous (quadrapolic) picture automata (the equivalence of unambiguous 2OTA and unambiguous tiling systems was independently derived in [1]).

Also, we will use ideas of constructions given in [13], but adapt them by involving weights and using the model of (quadrapolic) picture automata instead of 2-dimensional on-line tessellation automata (2OTA).

The paper is organized as follows. In Section 2, we give examples of pictures with weights and recall concepts of two-dimensional languages. In Section 3 we introduce picture series, rational operations on them and the concept of a weighted picture automaton computing a recognizable picture series. We start Section 3 by briefly recalling the necessary notation and background for formal power series on strings and weighted finite automata. Section 4 presents the main theorem on the coincidence of recognizable series with projections of rational series for commutative semirings. Finally, in Section 5, we compare new models of weighted tile systems and weighted domino systems as extensions of local and hv-local picture languages with the family of recognizable series; the different devices defining picture series are composed in Section 6.

2. Pictures and examples for pictures with weights

We recall notions and results of two-dimensional languages, required for this paper. For more details see [13,17,21].

Let $\mathbb{N} = \{0, 1, \ldots\}$ and let $\mathbb{R}$ be the set of real numbers, we set $\mathbb{N}_1 := \mathbb{N} \setminus \{0\}$. Let Σ be a finite non-empty set, called *alphabet*. Let $m, n \in \mathbb{N}_1$.

A *picture* of *size* (m, n) over Σ is a non-empty[1] $m \times n$-matrix over Σ, i.e. a mapping $p : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \Sigma$. We write $p(i, j)$ or $p_{i,j}$ for the component of p at position (i, j), where $1 \le i \le m$, $1 \le j \le n$. For a picture p of size (m, n), we define $l_v(p) = m$ and $l_h(p) = n$ (v stands for vertical, h for horizontal). A *picture language* is a set of pictures. We denote by Σ^{++} the set of all pictures over Σ. The set $\Sigma^{m \times n}$ comprises all pictures of size (m, n) over the alphabet Σ.

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1 We assume a picture to be non-empty for technical simplicity, as in [3,17,21].

Assume $p \in \Sigma^{++}$. We denote by $\hat{p}$ the picture that results from p by surrounding it with the (fresh) boundary symbol $\# \notin \Sigma$. If p has size (m, n), then $\hat{p}$ has size $(m + 2, n + 2)$. It marks the border elements of the picture needed for recognizing algorithms for sets of pictures.

As an example, we consider the picture p of size (4, 5) over the alphabet $\Sigma = \{0, 1, 2, 3\}$:

$$p = \boxed{\begin{array}{ccccc} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 3 & 2 \\ 3 & 1 & 0 & 0 & 2 \end{array}}$$

then

$$\hat{p} = \boxed{\begin{array}{ccccccc} \# & \# & \# & \# & \# & \# & \# \\ \# & 1 & 1 & 1 & 0 & 2 & \# \\ \# & 0 & 1 & 0 & 0 & 1 & \# \\ \# & 1 & 3 & 1 & 3 & 2 & \# \\ \# & 3 & 1 & 0 & 0 & 2 & \# \\ \# & \# & \# & \# & \# & \# & \# \end{array}}.$$

The theory of two-dimensional languages collects and characterizes properties of picture languages. In this paper, rather than considering picture languages, we will investigate characteristics of certain functions, ones mapping pictures over a fixed alphabet to some weights in order to describe quantitative predicates of pictures. In general, the set of weights will have the structure of a commutative semiring.

For motivation we give two examples of such functions $S : \Sigma^{++} \to \mathbb{R} \cup \{\infty\}$ and $T : \Sigma^{++} \to \mathbb{N}$.

Example 2.1. Let $D \subset [0, 1]$ be a finite set of discrete and let $L \subseteq D^{++}$ be a recognizable picture language. Consider the function $S : D^{++} \to \mathbb{R} \cup \{\infty\}$ defined by

$$S(p) = \begin{cases} \sum_{i,j} p_{i,j} & p \in L, \\ \infty & \text{otherwise.} \end{cases}$$

One could interpret the values in D as different levels of gray, as in [8], where the authors introduce fuzzy association rules in order to analyse data sets including numerical attributes. Then, for each recognizable picture $p \in L$, the function S provides the total value $S(p)$ of light of p.

Example 2.2. Let C be a finite set, interpreted for instance as a set of colours or attributes and consider the function $T : C^{++} \to \mathbb{N}$, defined for $p \in C^{++}$ by

$$T(p) = \max\{l_v(q) \cdot l_h(q) \mid q \text{ is a monochrome subpicture of } p\}.$$

Then $T(p)$ returns the largest area of a monochrome rectangle, enclosed as a subpicture within p.

Functions S from Σ^{++} into $\mathbb{R} \cup \{\infty\}$ or, more generally, a semiring K, will be called picture series and abstractly defined in the next section. We will also provide tools to describe the functions S and T of the above examples as the behaviours of weighted picture automata over particular semirings.

In order to be able to define rational operations on picture languages and series, we need two different partial concatenations on pictures: the *column concatenation* $p \mathbin{⦶} q$ juxtaposes two pictures provided they have the same height, i.e. for $p \in \Sigma^{m\times k}$ and $q \in \Sigma^{m\times l}$ we have $r := p \mathbin{⦶} q \in \Sigma^{m\times(k+l)}$, where

$$\text{for } 1 \le i \le m, 1 \le j \le k + l;\ r(i, j) = \begin{cases} p(i, j) & j \le k \\ q(i, j - k) & j > k. \end{cases}$$

The *row concatenation* $p \ominus q$ of two pictures p and q is defined similarly for pictures having identical width; for $p \in \Sigma^{k\times n}, q \in \Sigma^{l\times n}$: $s := p \ominus q \in \Sigma^{(k+l)\times n}$, where

$$\text{for } 1 \le i \le k + l, 1 \le j \le n;\ \sigma(i, j) = \begin{cases} p(i, j) & i \le k \\ q(i - k, j) & i > k. \end{cases}$$

We depict the described situation as:

$$p \obar q = \begin{array}{|c|c|}\hline p & q \\ \hline \end{array}, \qquad p \ominus q = \begin{array}{|c|}\hline p \\ \hline q \\ \hline \end{array}.$$

For any set S, we denote by $\mathscr{P}(S)$ the set of all subsets of S. We introduce the symbol $\uplus$ to denote a union that is disjoint. As usual, concatenations on pictures can be extended pointwise to languages and then also be iterated, that is to say, for $L, L' \in \mathscr{P}(\Sigma^{++})$, $k \in \mathbb{N}_1$, we set

$$L \obar L' = \{p \obar p' \mid p \in L, p' \in L'\}$$

and

$$L^{\obar^1} := L, \quad L^{\obar^{k+1}} := L^{\obar^k} \obar L \quad \text{and} \quad L^{\obar^+} := \bigcup_{k \geq 1} L^{\obar^k}.$$

In an analogous way, for the vertical direction we define the operations $\ominus$, $\ominus^k$ (for $k \in \mathbb{N}_1$) and $\ominus^+$ for picture languages. Eventually, we have operations on languages $\obar$ (respectively $\ominus$) : $\left(\mathscr{P}(\Sigma^{++})\right)^2 \to \mathscr{P}(\Sigma^{++})$ and $\obar^+$ (respectively $\ominus^+$) : $\mathscr{P}(\Sigma^{++}) \to \mathscr{P}(\Sigma^{++})$, referred to as *column* (respectively *row*) *concatenation* and *column* (respectively *row*) *closure*.

For any two alphabets Σ and Γ, a mapping $\pi : \Gamma \to \Sigma$ is called *(alphabetic) projection*. It can be lifted pointwise to pictures and picture languages as usual. If not otherwise indicated, we do not distinguish between a word w and the picture having only row (or only column) w.

In the literature, there are many equivalent devices defining or recognizing picture languages in terms of projections of local languages (tiling systems) and rational expressions [11–13], domino systems [21], two-dimensional on-line tessellation automata (2OTA) [17,18], monadic second-order (MSO) logic [14] or recently quadrapolic picture automata [4]. These devices characterize *recognizable* picture languages, collected in the class $\mathrm{Rec}(\Sigma^{++})$.

3. Picture series and weighted automata

A *semiring* K is a structure $(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, $x \cdot 0 = 0 = 0 \cdot x$ for all elements $x \in K$, and $0 \neq 1$. In case the multiplication is commutative, K is called *commutative*. Examples of semirings useful to model problems in operation research and carrying quantitative properties for many devices include e.g.

- the *Boolean* semiring $\mathbb{B} = \left(\{0, 1\}, \vee, \wedge, 0, 1\right)$ where $1 \vee 1 = 1$,
- the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$,
- the *tropical* (or *min-plus*) semiring $\mathbb{T} = (\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$,
- the *arctical* (or *max-plus*) semiring $\mathrm{Arc} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$,
- the *language semiring* $\left(\mathscr{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*\right)$ (Σ^* denotes the set of all words over Σ),
- $\left([0, 1], \max, \cdot, 0, 1\right)$ (to capture probabilities).

For the algebraic theory of semirings and their applications, we refer the reader to [15] and to [16].

3.1. Formal power series on strings

We will first provide some notions and results from the theory of formal power series on words needed in the succeeding sections. For further details about definitions and basic results on series on words, as well as to Schützenberger's theorem, we refer to [2,9,20,29,30].

We fix K and Σ. A *formal power series* is a mapping $S : \Sigma^* \to K$. The support of S is $\mathrm{supp}(S) = \{w \in \Sigma^* \mid S(w) \neq 0\}$. A polynomial is a series with finite support. Characteristic series are defined, as usual. We let $K\langle\langle \Sigma^* \rangle\rangle$

comprise all formal power series over Σ. Now let $S, T \in K\langle\langle\Sigma^*\rangle\rangle$. The sum $S + T$, the Hadamard product $S \odot T$ and the (Cauchy) product $S \cdot T$ are defined for $w \in \Sigma^*$ by

$$(S+T, w) = (S, w) + (T, w); \quad (S \odot T, w) = (S, w) \cdot (T, w);$$
$$(S \cdot T, w) = \sum_{w=w_1 w_2} (S, w_1) \cdot (T, w_2).$$

The powers $S^n (n \geq 0)$ of S are defined in a natural way: we set $S^0 = \mathbb{1}_{\{\varepsilon\}}$, for $n \geq 1$ we have $S^{n+1} = S^n \cdot S$. The series S is proper if $(S, \varepsilon) = 0$. In this case, we set $S^* = \sum_{n \geq 0} S^n$, the star of S. Note that this sum is such that for every w, there are only finitely many summands different from 0.

The class of *rational* (string) series (denoted by $K^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$) can be constructed from polynomials by applying the operations $+$, $\cdot$ and $*$, where the star $*$ is restricted to proper series. Rational expressions and their series for words are defined is a straightforward way.

A *weighted finite automaton (WFA)* is a quadruple $T = (Q, E, I, F)$, where Q is a finite set of states, E is a finite subset of $Q \times \Sigma \times K \times Q$, and $I, F : Q \to K$. We call the tuples in E transitions and I respectively F the *initial* (respectively *final*) weight function.

Let $n \geq 1$. A path π of length n is a sequence

$$(q_0, a_0, k_0, q_1)\,(q_1, a_1, k_1, q_2) \cdots (q_{n-1}, a_{n-1}, k_{n-1}, q_n)$$

of transitions in E. The word $a_0 \ldots a_{n-1}$ is called the *label* of π. We say that π starts at q_0 and ends at q_n. We define the *weight* of π by $\mathrm{weight}(\pi) := k_0 \cdot k_1 \cdot \cdots \cdot k_{n-1}$. We assume that for every $q \in Q$ there is a path of length 0 which starts and ends at q, is labelled with ε and weighted with 1. For every $p, q \in Q$ and every $w \in \Sigma^*$, we denote by $p \overset{w}{\rightsquigarrow} q$ the set of all paths with label w which start at p and end at q. The WFA T computes a formal power series $\|T\| : \Sigma^* \to K$, defined for every $w \in \Sigma^*$, by

$$\|T\|(w) = \sum_{p \in I, q \in F, \pi \in p \overset{w}{\rightsquigarrow} q} I(p) \cdot \mathrm{weight}(\pi) \cdot F(q).$$

We call $\|T\|$ the *behaviour* of T. A formal power series $S \in K\langle\langle\Sigma^*\rangle\rangle$ is *recognizable* if S is the behaviour of some WFA. We let $K^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ comprise all recognizable word series over K and the alphabet Σ.

Schützenberger's theorem states the following equivalence between recognizable and rational formal power series.

Theorem 3.1 (*Schützenberger [30]*). *A formal power series is rational if and only if it is the behaviour of some weighted finite automaton.*

3.2. Series on pictures

Subsequently, K will always denote a commutative semiring. Let Σ and Γ be alphabets. We will now assign weights to pictures and define some notions for them quite similarly to the approach taken in the theory of formal power series on words (see Section 3.1). This provides a generalization of the theory of picture languages to series over pictures, cf. [4,25–27].

A *picture series* is a mapping $S : \Sigma^{++} \to K$. We let $K\langle\langle\Sigma^{++}\rangle\rangle$ comprise all picture series over Σ. We write (S, p) for $S(p)$; then a series S often is written as a formal sum $S = \sum_{p \in \Sigma^{++}} (S, p) \cdot p$. The set $\mathrm{supp}(S) = \{p \in \Sigma^{++} \mid (S, p) \neq 0\}$ is the *support* of S. Series having finite support are called *polynomials* and form the set $K\langle\Sigma^{++}\rangle$. We now define the *rational* operations on picture series

$$\oplus, \odot, \varobar, \ominus : \left(K\langle\langle\Sigma^{++}\rangle\rangle\right)^2 \to K\langle\langle\Sigma^{++}\rangle\rangle,$$

referred to as *sum, Hadamard product, horizontal multiplication* and *vertical multiplication*, respectively, and also $\varobar^+, \ominus^+ : K\langle\langle\Sigma^{++}\rangle\rangle \to K\langle\langle\Sigma^{++}\rangle\rangle$, the *horizontal star* and the *vertical star*, as follows. We fix $S, T \in K\langle\langle\Sigma^{++}\rangle\rangle$ and

$p \in \Sigma^{++}$. Then we set

$$(S \oplus T, p) := (S, p) + (T, p) \quad \text{and} \quad (S \odot T, p) := (S, p) \cdot (T, p)$$
$$(S \varobar T, p) := \sum_{p_1 \varobar p_2 = p} (S, p_1) \cdot (T, p_2) \quad \text{and} \quad (S \ominus T, p) := \sum_{p_1 \ominus p_2 = p} (S, p_1) \cdot (T, p_2)$$
$$(S^{\varobar^+}, p) := \sum_{\substack{p_1 \varobar \cdots \varobar p_n = p \\ n \geq 1}} (S, p_1) \cdot \cdots \cdot (S, p_n)$$
$$(S^{\ominus^+}, p) := \sum_{\substack{p_1 \ominus \cdots \ominus p_n = p \\ n \geq 1}} (S, p_1) \cdot \cdots \cdot (S, p_n).$$

The star operations are not partial, since every picture is nonempty. We define the (pointwise) *scalar multiplications* with elements of the semiring, that is, for $k \in K$, we put $(k \cdot S, p) = k \cdot (S, p)$. Then $k \cdot S$ defines a series in $K\langle\langle \Sigma^{++} \rangle\rangle$, as usual. For a language $L \subseteq \Sigma^{++}$, the *characteristic series* $\mathbb{1}_L : \Sigma^{++} \to K$ is defined for $p \in \Sigma^{++}$ by $(\mathbb{1}_L, p) = 1$ if $p \in L$, and $(\mathbb{1}_L, p) = 0$ otherwise. Note that $k \cdot S = (k \cdot \mathbb{1}_{\Sigma^{++}}) \odot S$.

Definition 3.2. A picture series $S \in K\langle\langle \Gamma^{++} \rangle\rangle$ is called *rational* if it is obtained from a finite set of polynomials by finitely many applications of the rational operations $\oplus, \odot, \varobar, \ominus, \varobar^+$ and $\ominus^+$.

We define the respective rational expressions and their languages in a straightforward way. The family of rational series over a commutative semiring K and an alphabet Γ will be denoted by $K^{\text{rat}}\langle\langle \Gamma^{++} \rangle\rangle$. Now, extending projections for languages to series, for $\pi : \Gamma \to \Sigma$ and $S' \in K\langle\langle \Gamma^{++} \rangle\rangle$, we set for each $p \in \Sigma^{++}$;

$$(\pi(S'), p) := \sum_{\pi(p') = p} (S', p').$$

It defines a series $\pi(S') \in K\langle\langle \Sigma^{++} \rangle\rangle$ which we call the *projection* of S' by π. We say S is a *projection of a rational series* if there exists an alphabet Γ, a series $S' \in K^{\text{rat}}\langle\langle \Gamma^{++} \rangle\rangle$ and a projection $\pi : \Gamma \to \Sigma$ with $S = \pi(S')$. We denote the family of series over Σ that are projections of rational series by $K^{\text{Prat}}\langle\langle \Sigma^{++} \rangle\rangle$. Next we define weighted picture automata. These devices were first introduced by Bozapalidis and Grammatikopoulou in [4] and named weighted quadrapolic picture automata.

Definition 3.3 (*[4]*). A *weighted (quadrapolic) picture automaton (WPA)* is a 6-tuple $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ consisting of a finite set Q of states, a finite set of rules $R \subseteq \Sigma \times K \times Q^4$, as well as four *poles of acceptance* $F_w, F_n, F_e, F_s \subseteq Q$ (w, n, e, s stand for the four points of the compass).

Given a rule $r = (a, k, q_w, q_n, q_e, q_s) \in R$, we denote by label$(r)$ its (input) label a, by weight(r) its weight k and corresponding to the four poles, and we let

$$\sigma_w(r) := q_w, \quad \sigma_n(r) := q_n, \quad \sigma_e(r) := q_e, \quad \sigma_s(r) := q_s.$$

We extend the functions label and weight pointwise to pictures by setting for a picture $c = (c_{i,j}) \in R^{++}$ over the set of rules:

$$\text{label}(c)_{i,j} := \text{label}(c_{i,j}) \quad \text{and} \quad \text{weight}(c) = \prod_{i,j} \text{weight}(c_{i,j}).$$

We call label(c) the *label* of c and weight(c) the *weight* of c.

A *run* (or *computation*) c in $\mathfrak{A}$ is an element in R^{++} satisfying natural compatibility properties, more precisely, for $c = (c_{i,j}) \in R^{m \times n}$ we have

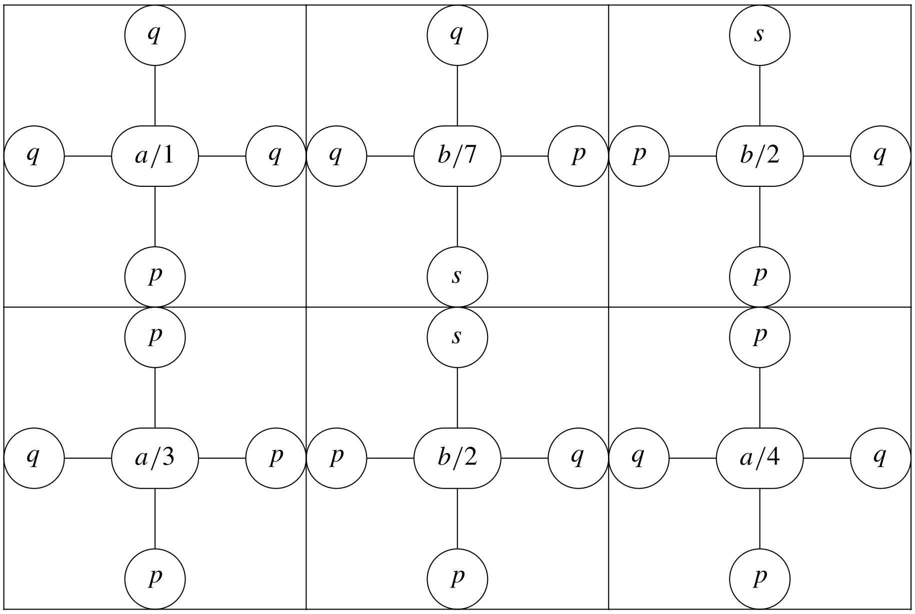
$$\forall i \leq m-1, j \leq n : \sigma_s(c_{i,j}) = \sigma_n(c_{i+1,j}), \quad \forall i \leq m, j \leq n-1 : \sigma_e(c_{i,j}) = \sigma_w(c_{i,j+1}).$$

The run c *reads* the picture label(c). A run c is *successful* if it has its (outer) pole states in the respective poles of acceptance, that is to say:

$$\forall i \leq m, j \leq n : \sigma_w(c_{i,1}) \in F_w, \sigma_n(c_{1,j}) \in F_n, \sigma_e(c_{i,n}) \in F_e, \sigma_s(c_{m,j}) \in F_s. \tag{1}$$

We denote the set of successful runs c with label$(c) = p$ by Succ(p).

For instance, a run in a weighted picture automaton over $\mathbb{N}$ and the alphabet $\{a, b\}$ where s, p, q are states could be depicted as



In case it holds that $q \in F_w$; $q, s \in F_n$; $q \in F_e$ and $p \in F_s$, then this run would be successful.

We define a picture series $\|\mathfrak{A}\| : \Sigma^{++} \to K$, as follows. Let $p \in \Sigma^{++}$. If $\text{Succ}(p) = \emptyset$, then $\|\mathfrak{A}\|$ sends p to 0. Otherwise, we let

$$(\|\mathfrak{A}\|, p) = \sum_{c \in \text{Succ}(p)} \text{weight}(c).$$

That is, the *weight* of a picture p is the sum of the weights of all successful runs with label p. We call $\|\mathfrak{A}\|$ the *behaviour* of $\mathfrak{A}$ and also say that $\mathfrak{A}$ *computes* (or *recognizes*) $\|\mathfrak{A}\|$.

The family of picture series computed by weighted picture automata over Σ will be denoted by $K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$, elements of which are referred to as *recognizable* series.

We give an example of a weighted picture automaton.

Example 3.4. Let $k \in K$. The WPA $\mathfrak{A} = \big(\{p, q\}, R, \{p\}, \{p\}, \{q\}, \{q\}\big)$ defined by

$$R = \bigcup_{a \in \Sigma} \{(a, k, p, p, q, q), (a, 1, p, q, q, q), (a, 1, q, p, q, q), (a, 1, q, q, q, q)\}$$

computes $\|\mathfrak{A}\| = k \cdot \mathbb{1}_{\Sigma^{++}}$.

Considering the unweighted case of Definition 3.3, where $R \subseteq \Sigma \times Q^4$ or equivalently, restricting the semiring K to $\mathbb{B}$, we get the correct definition of a *(quadrapolic) picture automaton (PA)*, introduced in [4] (a very similar approach is taken in [7], the devices there are called Wang systems). Devices of PA over an alphabet Σ in a natural way define picture languages, and were shown to compute precisely the family of recognizable picture languages $(\text{Rec}(\Sigma^{++}))$ [4]. For $K = \mathbb{B}$, the correspondence $L \mapsto \mathbb{1}_L$ gives a natural bijection between languages over Σ and series in $\mathbb{B}\langle\langle \Sigma^{++} \rangle\rangle$. Results done in the next section are also true for PA.

Let us consider again Examples 2.1 and 2.2 of the previous section. By simulating an unweighted picture automaton A recognizing L, and assigning the weight $p_{i,j}$ to every rule with label $p_{i,j}$ in A, we get a WPA $\mathfrak{A}$ over the tropical semiring. One can prove that $\mathfrak{A}$ computes the function S, i.e. $S \in \mathbb{T}^{\text{rec}}\langle\langle D^{++}, WPA \rangle\rangle$. Also, there is an automaton over the max-plus semiring Arc computing T. Here, for a picture p, the automaton provides one successful computation for every different monochrome subpicture q of p. We weight the rules overlaying on q with 1. Then the weight of this particular computation is $l_v(q) \cdot l_h(q)$. Since we get the behaviour by adding the weights for successful runs reading p, in Arc, the maximal size is extracted.

4. A Kleene–Schützenberger theorem for picture series

In this section we will prove the equivalence of the families of projections of rational series and series recognized by WPA. We will start by showing that projections of rational picture series are behaviours of weighted picture automata.

4.1. Projections of rational series are recognizable

We will give the initialization of a structural induction and use the results in [4], where the authors proved closure properties of picture series computed by weighted picture automata. We obtain the closure of the family of recognizable picture series under rational operations and projections.

Clearly, the *monomials*, i.e. series with supports as singletons, are recognizable:

Lemma 4.1. *Let $p \in \Sigma^{++}$ and let $k \in K$. Then $k \cdot \mathbb{1}_{\{p\}},\ k \cdot \mathbb{1}_{\Sigma^{++}} \in K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$.*

Proof. Let $p \in \Sigma^{m\times n}$ and let $k \in K$. The weighted picture automaton $\mathfrak{A} = \big(\{0, \ldots, \max\{m, n\}\}, R, \{0\}, \{0\}, \{n\}, \{m\}\big)$ defined by

$$R = \big\{(p_{i+1,j+1}, c, j, i, j+1, i+1) \mid 0 \le i \le m, 0 \le j \le n\big\}$$

such that $c = k$ if $(i, j) = (0, 0)$ and $c = 1$ otherwise, computes $\|\mathfrak{A}\| = k \cdot \mathbb{1}_{\{p\}}$. Similarly one could specify an automaton for $k \cdot \mathbb{1}_{\Sigma^{++}}$. $\square$

Lemma 4.2 ([4]). *The family $K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$ of recognizable picture series is closed under operations $\oplus, \odot, \varobar, \ominus, \varobar^+$ and $\ominus^+$.*

Note, that using both, Lemmas 4.1 and 4.2, the family $K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$ is also closed under scalar multiplication, since for $k \in K$ and $S \in K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$, we get $k \cdot S = S \odot (k \cdot \mathbb{1}_{\Sigma^{++}})$.

Lemma 4.3 ([4]). *Let $\pi : \Gamma \to \Sigma$ be a projection and let $T \in K^{\text{rec}}\langle\langle \Gamma^{++} \rangle\rangle$. Then $\pi(T) \in K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$.*

Now the following theorem is immediate by Lemmas 4.1–4.3.

Theorem 4.4 ([4]). *We have $K^{\text{Prat}}\langle\langle \Sigma^{++} \rangle\rangle \subseteq K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$.*

4.2. Recognizable series are projections of rational series

The idea for the proof in the other direction of a Kleene–Schützenberger Theorem for picture series is to convert the automaton into some "deterministic" device of a certain type via a projection. The behaviour of this deterministic device will then be proved rational by using Schützenberger's Theorem for recognizable and rational formal power series on words (Theorem 3.1).

There is a natural correspondence between formal power series reading words and picture series reading only rows or only columns. We can consider a picture having only one row (respectively one column) also as word over Σ, and later on, we will not distinguish between the notations of these two cases. In this sense, picture series over Σ fulfilling the condition that the support is a subset of Σ^* can be viewed as word series.

Lemma 4.5. *Let $S : \Sigma^* \to K$ be a rational formal power series over words. There exist rational picture series $S_h, S_v \in K^{\text{rat}}\langle\langle \Sigma^{++} \rangle\rangle$ such that for all $p \in \Sigma^{++}$, we have*

$$S_h(p) = \begin{cases} S(p) & p \in \Sigma^{1\times n}\ (n \in \mathbb{N}_1), \\ 0 & \textit{otherwise}. \end{cases} \qquad S_v(p) = \begin{cases} S(p) & p \in \Sigma^{n\times 1}\ (n \in \mathbb{N}_1), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since the class of rational (string) series is closed under the Hadamard product [9], the series $R = S \odot \mathbb{1}_{\Sigma^* \setminus \{\varepsilon\}}$ is rational and proper. We consider the structure of a rational expression for R. We can naturally embed the polynomials of $K^{\text{rat}}\langle\langle \Sigma^* \rangle\rangle$ into $K\langle \Sigma^{++} \rangle$ having their supports in $\Sigma^{1\times n}$ (respectively $\Sigma^{n\times 1}$) where $n \in \mathbb{N}_1$; the operations $+, \cdot, *$ are simulated by $\oplus$, $\varobar(\ominus)$, $\varobar^+(\ominus^+)$. □

We do the following definition.

Definition 4.6. A weighted picture automaton $\mathfrak{A}$ is called *rule deterministic* if for every input label a of the alphabet, there is at most one rule with label a in the rule set of $\mathfrak{A}$.

Clearly, rule deterministic (W)PA satisfy that every underlying input picture has at most one successful run in the automaton. Such automata are called unambiguous.

Proposition 4.7. *Let $S \in K^{\text{rec}}\langle\langle \Gamma^{++} \rangle\rangle$ be a series computed by a rule deterministic WPA. Then S is a rational picture series.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be a rule deterministic WPA over Γ computing S. We group the proof into 3 steps and show that $\mathfrak{A}$ computes a rational picture series. For $a \in \Gamma$, we set $r(a) = (a, k, q_1, q_2, q_3, q_4)$ if $(a, k, q_1, q_2, q_3, q_4) \in R$.

Step 1 We use the horizontal direction of the rules in R to define a WFA $A_h = (Q, E_h, I_h, F_h)$ over words, as follows. Let $E_h \subseteq Q \times \Gamma \times K \times Q$ be the set of transitions, defined by

$$(q_1, a, k, q_3) \in E_h \Leftrightarrow \exists r = (a, k, q_1, q_2, q_3, q_4) \in R,$$

and put

$$I_h(q) = \begin{cases} 1 & q \in F_w, \\ 0 & \text{otherwise}, \end{cases} \qquad F_h(q) = \begin{cases} 1 & q \in F_e, \\ 0 & \text{otherwise} \end{cases}$$

as initial and final weight functions.

Then A_h is a WFA having successful computations for all words corresponding to rows which have a run in $\mathfrak{A}$ leading from F_w to F_e. For such a row $w = a_1 a_2 \cdots a_n$ $(a_i \in \Gamma)$, since $\mathfrak{A}$ is rule deterministic, we have

$$(\|A_h\|, w) = 1 \cdot \left(\prod_{1 \le i \le n} \text{weight}(r(a_i)) \right) \cdot 1.$$

The series $\|A_h\|$ maps all other words in $\Gamma^* \setminus \{\varepsilon\}$ to 0.

Using Theorem 3.1 and Lemma 4.5 we conclude that there exists a rational picture series S_h such that for all $n \in \mathbb{N}_1$, $p \in \Gamma^{1\times n}$ we have $(S_h, p) = (\|A_h\|, p)$, and S_h maps elements having more than one row to 0.

Step 2 Similarly, we use the vertical direction of rules in R for the definition of transitions of a WFA $A_v = (Q, E_v, I_v, F_v)$ over the Boolean semiring where $E_v \subseteq Q \times \Gamma \times \{0, 1\} \times Q$ is the set of transitions, defined by

$$(q_2, a, 1, q_4) \in E_v \Leftrightarrow \exists r = (a, k, q_1, q_2, q_3, q_4) \in R,$$

and

$$I_v(q) = \begin{cases} 1 & q \in F_n, \\ 0 & \text{otherwise}, \end{cases} \qquad F_v(q) = \begin{cases} 1 & q \in F_s, \\ 0 & \text{otherwise} \end{cases}$$

as weight functions.

Then A_v is an automaton having successful computations for all words corresponding to columns which have a run in $\mathfrak{A}$ leading from F_n to F_s. Such a column $w = a_1 a_2 \cdots a_m$ $(a_i \in \Gamma)$, again, since $\mathfrak{A}$ is rule deterministic, satisfies $(\|A_v\|, w) = 1$. All other words in $\Gamma^* \setminus \{\varepsilon\}$ are mapped to 0 by $\|A_v\|$.

Now, as before, Theorem 3.1 and Lemma 4.5 provide a rational picture series S_v over $\mathbb{B}$ such that for all $n \in \mathbb{N}_1$, $p \in \Gamma^{n\times 1} : (S_v, p) = (\|A_v\|, p)$.

Step 3 **(C) Claim:** $\forall x \in \Gamma^{++} : \big(\|\mathfrak{A}\|, x\big) = \big(S_h{}^{\ominus^+}, x\big) \cdot \big(S_v{}^{\oplus^+}, x\big)$.

For pictures $x = (x_{i,j})$ $(1 \le i \le m, 1 \le j \le n)$ where every row has a successful run in A_h, the picture series $S_h{}^{\ominus^+}$ is a rational series that maps x to the product of the weights of the composed rules for pixels of x in $\mathfrak{A}$. The series $S_h{}^{\ominus^+}$ maps all other pictures to 0. We get

$$\big(S_h{}^{\ominus^+}, x\big) = \prod_{i \le m, j \le n} \text{weight}(r(x_{i,j})). \tag{2}$$

Analogously, for a pictures $y = (y_{i,j})$ where every column has a successful run in A_v, we get

$$\big(S_v{}^{\oplus^+}, y\big) = 1. \tag{3}$$

$S_v{}^{\oplus^+}$ maps all other pictures to 0.

Now, to prove (C), let $x = (x_{i,j}) \in \Gamma^{++}$ $(1 \le i \le m, 1 \le j \le n)$, we consider three cases. First, assume $x \in \Gamma^{m\times n}$ such that there exists an $i \in \{1, \ldots, m\}$ and $(x_{i,1}x_{i,2}\cdots x_{i,n}) \in \Gamma^{1\times n}$ has no run in $\mathfrak{A}$ satisfying $\sigma_w(r(x_{i,1})) \in F_w, \sigma_e(r(x_{i,n})) \in F_e$. With the definition of $\|\mathfrak{A}\|$ and (2) we conclude $\big(\|\mathfrak{A}\|, x\big) = 0 = \big(S_h{}^{\ominus^+}, x\big)$, hence: (C).

Now, let $x \in \Gamma^{m\times n}$ such that there exists an $j \in \{1, \ldots, n\}$ with $(x_{1,j}x_{2,j}\cdots x_{m,j})^T \in \Gamma^{m\times 1}$ having no run in $\mathfrak{A}$ satisfying $\sigma_n(r(x_{1,j})) \in F_n, \sigma_s(r(x_{m,j})) \in F_s$. Then using (3), we get $\big(\|\mathfrak{A}\|, x\big) = 0 = \big(S_v{}^{\oplus^+}, x\big)$, hence (C).

For the remaining case, again, let $x \in \Gamma^{m\times n}$. For every row in x, there exists a unique computation leading in $\mathfrak{A}$ from F_w to F_e, that is, for all $1 \le i \le m$ and all $1 \le j \le (n-1)$:

$$\sigma_e(r(x_{i,j})) = \sigma_w(r(x_{i,j+1})), \sigma_w(r(x_{i,1})) \in F_w, \sigma_e(r(x_{i,n})) \in F_e. \tag{4}$$

On the other hand, for every column in x, there exists a unique computation in $\mathfrak{A}$ having the northern state in F_n and the southern state in F_s, i.e., for all $1 \le i \le m-1$ and all $1 \le j \le n$:

$$\sigma_s(r(x_{i,j})) = \sigma_n(r(x_{i+1,j})), \sigma_n(r(x_{1,j})) \in F_n, \sigma_s(r(x_{m,j})) \in F_s. \tag{5}$$

With (4) and (5), $c := \big(r(x_{i,j})\big)_{i,j}$ forms a successful computation for x in $\mathfrak{A}$. Since $\mathfrak{A}$ is rule deterministic, there is at most one computation for x. We obtain

$$\begin{aligned}\big(\|\mathfrak{A}\|, x\big) &= \prod_{i \le m, j \le n} \text{weight}(r(x_{i,j})) \overset{(2)}{=} \big(S_h{}^{\ominus^+}, x\big) \cdot 1 \\ &\overset{(3)}{=} \big(S_h{}^{\ominus^+}, x\big) \cdot \big(S_v{}^{\oplus^+}, x\big).\end{aligned}$$

Therefore, claim (C) holds and thus (using Lemma 4.2)

$$\|\mathfrak{A}\| = S_h{}^{\ominus^+} \odot S_v{}^{\oplus^+} \in K^{\text{rat}}\langle\langle \Gamma^{++} \rangle\rangle. \quad \square$$

Next we show that every recognizable series is the projection of a series computed by a rule deterministic automaton. The idea is to encode the rules of the given automaton into the new alphabet. Then we will prove that this encoding can be reversed by a projection.

Proposition 4.8. *Let $\mathfrak{A}$ be a WPA over Σ. There exists a rule deterministic WPA $\mathfrak{A}'$ over an alphabet Γ and a projection $\pi : \Gamma \to \Sigma$ satisfying $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be a WPA over Σ and K. We put $\Gamma := R$ and define a rule deterministic WPA over Γ as $\mathfrak{A}' = (Q, R', F_w, F_n, F_e, F_s)$ with

$$R' := \{((a, k, q_1, q_2, q_3, q_4), k, q_1, q_2, q_3, q_4) \mid (a, k, q_1, q_2, q_3, q_4) \in R\}.$$

For every input label $(a, k, q_1, q_2, q_3, q_4) \in \Gamma$, there is at most one rule with label $(a, k, q_1, q_2, q_3, q_4)$ in $\mathfrak{A}'$. We define a projection $\pi : \Gamma \to \Sigma$ by mapping pixels $(a, k, q_1, q_2, q_3, q_4)$ to a. We claim that

$$\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|). \tag{6}$$

Let $x \in \Sigma^{m\times n}$. If there was no successful run of x in $\mathfrak{A}$, then there is no picture in Γ^{++} with a successful run in $\mathfrak{A}'$ which is mapped to x by π, so (6) holds. For the other case, let $\{c_1, c_2, \ldots, c_s\} \subseteq R^{++}$ be the set of successful runs for x in $\mathfrak{A}$. These runs belong to successful runs $\{c'_1, c'_2, \ldots, c'_s\} \subseteq R'^{++}$ in $\mathfrak{A}'$ such that

$$\forall 1 \le i \le s : \pi(l(c'_i)) = x, \quad \sum_{1\le i\le s} \mathrm{weight}\big(c_i\big) = \sum_{1\le i\le s} \mathrm{weight}\big(c'_i\big).$$

Since there cannot be other successful runs in $\mathfrak{A}'$ mapped by the projection π to x, we conclude (6):

$$(\|\mathfrak{A}\|, x) = \sum_{1\le i\le s} \mathrm{weight}\big(c_i\big) = \sum_{\pi(x')=x} (\|\mathfrak{A}'\|, x') = (\pi(\|\mathfrak{A}'\|), x). \quad \square$$

Theorem 4.9. *We have* $K^{\mathrm{rec}}\langle\langle \Sigma^{++}\rangle\rangle \subseteq K^{\mathrm{Prat}}\langle\langle \Sigma^{++}\rangle\rangle$.

Proof. Immediate by Propositions 4.7 and 4.8. $\square$

As a consequence of Theorems 4.4 and 4.9, we obtain the following Kleene–Schützenberger-like theorem for picture series:

Theorem 4.10. *Let K be a commutative semiring and Σ an alphabet. Then*

$$K^{\mathrm{rec}}\langle\langle \Sigma^{++}\rangle\rangle = K^{\mathrm{Prat}}\langle\langle \Sigma^{++}\rangle\rangle.$$

Remark 4.11. From the above theorem, the theorem on the coincidence of projections of rational picture languages and recognizable picture languages [13] follows by considering the Boolean semiring and noting that a language $L \subseteq \Sigma^{++}$ is recognizable if and only if $\mathbb{1}_L \in \mathbb{B}\langle\langle \Sigma^{++}\rangle\rangle$ is recognizable [4]; and a language $L \subseteq \Sigma^{++}$ is a projection of a rational language if and only if $\mathbb{1}_L \in \mathbb{B}\langle\langle \Sigma^{++}\rangle\rangle$ is a projection of a rational series (finite languages correspond to polynomials and rational operations on languages are simulated by the respective operation of the corresponding characteristic series).

Now, having this relation, as in the case of picture languages [13], for the definition of the class of rational (respectively recognizable) picture series, the operations and projections used are necessary. For instance, defining $L = \{x \in \{a\}^{++} \mid l_v(x) = l_h(x)\}$, using the relationship between languages and characteristic series over $\mathbb{B}$, the series $\mathbb{1}_L$ clearly is recognizable over $\mathbb{B}$, but not in $\mathbb{B}^{\mathrm{rat}}\langle\langle \Sigma^{++}\rangle\rangle$.

5. Tile-local and Hv-local Series

Local sets play an important role in the theory of recognizable string languages. Several authors generalized this notion to picture languages [13,21]. We will now assign weights to these local and hv-local picture devices using tiles or dominoes. This yields, via a projection, a very simple local definition and characterization of recognizable picture series.

Let Σ and Γ be alphabets. *Tiles* are pictures of size $(2, 2)$ and *dominoes* have size $(1, 2)$ or $(2, 1)$. For a picture p, we denote by $T(p)$ (respectively $D(p)$) the set of all sub-tiles (respectively sub-dominoes) of p. A language $L \subseteq \Gamma^{++}$ is *local* (respectively *hv-local*) if there exists a set Θ of tiles (respectively dominoes) over $\Gamma \cup \{\#\}$, such that $L = \{p \in \Gamma^{++} \mid T(\hat{p}) \subseteq \Theta\}$ (respectively $L = \{p \in \Gamma^{++} \mid D(\hat{p}) \subseteq \Theta\}$). Then (Γ, Θ) *characterizes* L. We write $L = \mathcal{L}(\Theta)$.

For a picture $p \in \Sigma^{m\times n}$, we will consider sub-tiles (sub-dominoes) at certain positions of $\hat{p}$. For tiles, we define

$$\forall 1 \le i \le m+1 \ \ \forall 1 \le j \le n+1 : \ t(\hat{p}_{i,j}) := \boxed{\begin{matrix} \hat{p}_{i,j} & \hat{p}_{i,j+1} \\ \hat{p}_{i+1,j} & \hat{p}_{i+1,j+1} \end{matrix}}.$$

Also, we consider the sub-dominoes in horizontal or vertical direction distinguished by their positions in $\hat{p}$:

$$\forall 1 \le i \le m+2 \ \ \forall 1 \le j \le n+1 : \ d^h(\hat{p}_{i,j}) := \boxed{\begin{matrix} \hat{p}_{i,j} & \hat{p}_{i,j+1} \end{matrix}}$$

$$\forall 1 \le i \le m+1 \ \ \forall 1 \le j \le n+2 : \ d^v(\hat{p}_{i,j}) := \boxed{\begin{matrix} \hat{p}_{i,j} \\ \hat{p}_{i+1,j} \end{matrix}}.$$

We give the following definitions.

Definition 5.1. We call $\mathcal{T} = (\Sigma, T)$, where $T : (\Sigma \cup \{\#\})^{2\times 2} \to K$ is a function mapping tiles over Σ to K, a *(weighted) tile-system*. It computes the picture series $\|\mathcal{T}\| : \Sigma^{++} \to K$, defined by

$$\forall p \in \Sigma^{++} : \|\mathcal{T}\|(p) := \prod_{\substack{1\le i\le l_v(p)+1\\ 1\le j\le l_h(p)+1}} T\big(t(\hat{p}_{i,j})\big).$$

We call $S : \Sigma^{++} \to K$ *tile-local* if there exists a tile-system $\mathcal{T}$ satisfying $\|\mathcal{T}\| = S$.

Similarly, for dominoes we have:

Definition 5.2. A pair $\mathcal{D} = (\Sigma, D)$, where $D : (\Sigma \cup \{\#\})^{2\times 1, 1\times 2} \to K$ maps dominoes over Σ to K, is a *(weighted) domino-system*. It computes the series $\|\mathcal{D}\| : \Sigma^{++} \to K$, defined by

$$\forall p \in \Sigma^{++} : \|\mathcal{D}\|(p) := \prod_{\substack{1\le i\le l_v(p)+2\\ 1\le j\le l_h(p)+1}} D\big(d^h(\hat{p}_{i,j})\big) \cdot \prod_{\substack{1\le i\le l_v(p)+1\\ 1\le j\le l_h(p)+2}} D\big(d^v(\hat{p}_{i,j})\big).$$

A picture series $S : \Sigma^{++} \to K$ is called *hv-local* if there exists a domino-system $\mathcal{D}$ satisfying $\|\mathcal{D}\| = S$. We denote the families of tile-local and hv-local series by $K^{\mathrm{loc}}\langle\langle\Sigma^{++}\rangle\rangle$ and $K^{hv}\langle\langle\Sigma^{++}\rangle\rangle$, respectively. We call the functions T (respectively D) *tile (respectively domino)-function*. For a picture p, tile-systems (respectively domino-systems) then compute the product of these functions ranging over the *(canonical) tile (respectively domino)-covering* of $\hat{p}$. As usual, one defines projections of tile-local and hv-local series. We denote the families of series that are projections of tile-local (respectively hv-local) series by $K^{\mathrm{Ploc}}\langle\langle\Sigma^{++}\rangle\rangle$ (respectively $K^{\mathrm{Phv}}\langle\langle\Sigma^{++}\rangle\rangle$).

The following proposition holds, indicating that the given devices generalize the notion of local and hv-local picture languages.

Proposition 5.3. *A picture language $L \subseteq \Gamma^{++}$ is local (hv-local respectively) if and only if its characteristic series $\mathbb{1}_L \in \mathbb{B}\langle\langle\Gamma^{++}\rangle\rangle$ is tile-local (hv-local respectively).*

Proof. If (Γ, Θ) characterizes the local language $L \subseteq \Gamma^{++}$, we define a weighted tile-system (Γ, T) by setting, for $t \in (\Gamma \cup \{\#\})^{2\times 2}$; $T(t) = 1$, if $t \in \Theta$ and $T(t) = 0$, otherwise. Then, for all $p \in \Gamma^{++}$, we have: $\|(\Gamma, T)\|(p) = 1 \iff p \in L$. Similarly, we define a weighted domino-system in the domino-case. For the opposite implication, the set of tiles (respectively dominoes) for the definition of the local (respectively hv-local) language comprises those that are mapped to 1 under the tile (respectively domino) function. □

We will show that series computed by WPA are presentable as projections of hv-local series. For this, we define a domino-system in such a way that for a picture p the domino function (taken over the canonical domino-covering of $\hat{p}$) coincides with the weight of the unique computation (in case it exists) for p in a rule deterministic automaton.

Proposition 5.4. *We have* $K^{\mathrm{rec}}\langle\langle\Sigma^{++}\rangle\rangle \subseteq K^{\mathrm{Phv}}\langle\langle\Sigma^{++}\rangle\rangle$.

Proof. We restrict ourselves to rule deterministic automata, using a projection (Proposition 4.8). Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be rule deterministic, computing $\|\mathfrak{A}\| = S$. We may use the notations developed for the proof of Proposition 4.7 and succeeding Definition 3.3. For $a, b \in \Sigma$, in case the occurring rules exist, we define a domino-function $D : (\Sigma \cup \{\#\})^{2\times 1, 1\times 2} \to K$ as follows:

$\boxed{\#\ \#} \mapsto 1$	$\boxed{\begin{array}{c}\#\\ a\end{array}} \mapsto 1,\ \text{if } \sigma_n(r(a)) \in F_n$
$\boxed{\#\ a} \mapsto \text{weight}(r(a)),\ \text{if } \sigma_w(r(a)) \in F_w$	$\boxed{\begin{array}{c}a\\ \#\end{array}} \mapsto 1,\ \text{if } \sigma_s(r(a)) \in F_s$
$\boxed{a\ \#} \mapsto 1,\ \text{if } \sigma_e(r(a)) \in F_e$	$\boxed{\begin{array}{c}a\\ b\end{array}} \mapsto 1,\ \text{if } \sigma_s(r(a)) = \sigma_n(r(b))$
$\boxed{a\ b} \mapsto \text{weight}(r(b)),\ \text{if } \sigma_e(r(a)) = \sigma_w(r(b))$	$\boxed{\begin{array}{c}\#\\ \#\end{array}} \mapsto 1.$

D maps all other dominoes to 0. Then $\mathcal{D} := (\Sigma, D)$ is a domino-system. For a picture p with (unique) successful computation $c \in R^{++}$ in $\mathfrak{A}$, the product of values of D (taken over the canonical domino-covering of $\hat{p}$) coincides with weight(c). This is because for every position $p_{i,j}$ of p there exists precisely one factor weight($r(p_{i,j})$) in the product for $\|\mathcal{D}\|(p)$. On the other hand, if p has no successful computation in $\mathfrak{A}$, then clearly the definition of D gives $\|\mathcal{D}\|(p) = 0$. Thus $\|\mathcal{D}\| = S$. □

Every hv-local language is local [13,21]. The analogous result for picture series provides the following proposition. In the proof here, we have to define the tile-function using the weights of the given domino-function such that respective products of the canonical coverings for a picture coincide.

Proposition 5.5. *Every hv-local series is tile-local.*

Proof. Let $S : \Gamma^{++} \to K$ be an hv-local series over an alphabet Γ, computed by $\mathcal{D} = (\Gamma, D)$. We define $\mathcal{T} = (\Gamma, T)$ as a tile-system computing S such that $T : (\Gamma \cup \{\#\})^{2\times 2} \to K$ denotes the tile-function. For arbitrary $a \in \Gamma$ and $b, c, d \in \Gamma \cup \{\#\}$, we put (here ulc, ue, le, m stand for "upper left corner", "upper edge", "left edge", "middle", respectively):

- ulc: $T\left(\boxed{\begin{smallmatrix}\# & \#\\ \# & a\end{smallmatrix}}\right) = D\left(\boxed{\begin{smallmatrix}\# & \#\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}\# & a\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}\#\\ \#\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}\#\\ a\end{smallmatrix}}\right)$
- ue : $T\left(\boxed{\begin{smallmatrix}\# & \#\\ a & b\end{smallmatrix}}\right) = D\left(\boxed{\begin{smallmatrix}\# & \#\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}a & b\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}\#\\ b\end{smallmatrix}}\right)$
- le: $T\left(\boxed{\begin{smallmatrix}\# & a\\ \# & b\end{smallmatrix}}\right) = D\left(\boxed{\begin{smallmatrix}\#\\ \#\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}\# & b\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}a\\ b\end{smallmatrix}}\right)$
- m: $T\left(\boxed{\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}}\right) = D\left(\boxed{\begin{smallmatrix}c & d\end{smallmatrix}}\right) \cdot D\left(\boxed{\begin{smallmatrix}b\\ d\end{smallmatrix}}\right)$.

Furthermore, we set $T\left(\boxed{\begin{smallmatrix}\# & \#\\ \# & \#\end{smallmatrix}}\right) = 0$. The values of D over the domino-covering of a picture p are distributed with T over the tile-covering for p. Every $d^h(\hat{p}_{i,j})$ and every $d^v(\hat{p}_{i,j})$ occurs precisely once in the product for $\|\mathcal{T}\|(p)$. For $p \in \Gamma^{++}$ we get $\|\mathcal{T}\|(p) = \|\mathcal{D}\|(p) = (S, p)$. □

In fact, to finish our argument for an equivalence theorem, we will show that projections of these tile-local series are recognizable. Since the image of a picture is composed by the weights of the contained tiles, the idea is to encode the tiles into the states of the rules analogous to a construction in [13]. But there, the authors considered the model of a 2-dimensional on-line tessellation automaton. Here we will derive a WPA that simulates the constructed underlying on-line tessellation automaton by defining rules that identify their southern and eastern poles. Also, since we now have weights, we will construct an unambiguous automaton in order not to add outputs over several runs reading identical pictures.

Proposition 5.6. *We have* $K^{\mathrm{Ploc}}\langle\langle \Sigma^{++}\rangle\rangle \subseteq K^{\mathrm{rec}}\langle\langle \Sigma^{++}\rangle\rangle$.

Proof. It suffices to prove the result for a tile-local series (Lemma 4.3). Let $S : \Sigma^{++} \to K$ be tile local, computed by $\mathcal{T} = (\Sigma, T)$ with tile-function $T : (\Sigma \cup \{\#\})^{2\times 2} \to K$. We define $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ as a WPA over Σ computing S by putting $Q = (\Sigma \cup \{\#\})^{2\times 2}$ and

- $F_w = \left\{\boxed{\begin{smallmatrix}\# & a\\ \# & b\end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}$, $F_n = \left\{\boxed{\begin{smallmatrix}\# & \#\\ a & b\end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}$
- $F_e = \left\{\boxed{\begin{smallmatrix}a & \#\\ b & \#\end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}$, $F_s = \left\{\boxed{\begin{smallmatrix}a & b\\ \# & \#\end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}$.

We set $R = R_{ulc} \cup R_{ue} \cup R_{le} \cup R_m \subseteq \Sigma \times K \times Q^4$ with arbitrary $a, b, c, d, f, g, h, t, x, y, z \in \Sigma \cup \{\#\}$:

- $R_{ulc} = \left\{\left(a, w(e), \boxed{\begin{smallmatrix}\# & a\\ \# & c\end{smallmatrix}}, \boxed{\begin{smallmatrix}\# & \#\\ a & b\end{smallmatrix}}, \boxed{\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}}, \boxed{\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}}\right) \mid a \in \Sigma\right\}$
  where $w(e) = T\left(\boxed{\begin{smallmatrix}\# & \#\\ \# & a\end{smallmatrix}}\right) \cdot T\left(\boxed{\begin{smallmatrix}\# & \#\\ a & b\end{smallmatrix}}\right) \cdot T\left(\boxed{\begin{smallmatrix}\# & a\\ \# & c\end{smallmatrix}}\right) \cdot T\left(\boxed{\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}}\right)$

- $R_{ue} = \left\{ \left(b, w(e), \boxed{\begin{smallmatrix} a & b \\ h & d \end{smallmatrix}}, \boxed{\begin{smallmatrix} \# & \# \\ b & c \end{smallmatrix}}, \boxed{\begin{smallmatrix} b & c \\ d & f \end{smallmatrix}}, \boxed{\begin{smallmatrix} b & c \\ d & f \end{smallmatrix}} \right) \mid a, b \in \Sigma \right\}$
  where $w(e) = T\left(\boxed{\begin{smallmatrix} \# & \# \\ b & c \end{smallmatrix}} \right) \cdot T\left(\boxed{\begin{smallmatrix} b & c \\ d & f \end{smallmatrix}} \right)$
- $R_{le} = \left\{ \left(c, w(e), \boxed{\begin{smallmatrix} \# & c \\ \# & g \end{smallmatrix}}, \boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}}, \boxed{\begin{smallmatrix} c & d \\ g & h \end{smallmatrix}}, \boxed{\begin{smallmatrix} c & d \\ g & h \end{smallmatrix}} \right) \mid a, c \in \Sigma \right\}$
  where $w(e) = T\left(\boxed{\begin{smallmatrix} \# & c \\ \# & g \end{smallmatrix}} \right) \cdot T\left(\boxed{\begin{smallmatrix} c & d \\ g & h \end{smallmatrix}} \right)$
- $R_{m} = \left\{ \left(a, w(e), \boxed{\begin{smallmatrix} z & a \\ t & c \end{smallmatrix}}, \boxed{\begin{smallmatrix} x & y \\ a & b \end{smallmatrix}}, \boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}}, \boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}} \right) \mid a, x, z \in \Sigma \right\}$
  where $w(e) = T\left(\boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}} \right)$.

To prove $\|\mathfrak{A}\| = S$, we observe the following. The given construction with its accepting condition defines a weighted picture automaton which has precisely one successful run for every element in Σ^{++}. This run simulates the distribution of weights along the canonical tile-covering of the picture. Assume we are given $p \in \Sigma^{++}$ with successful computation $c \in R^{++}$ in $\mathfrak{A}$. In the composition of weight$(c) = \prod_{i,j}$ weight$(c_{i,j})$, the image of the tile-function of the respective tile at every position of the canonical tile-covering of $\hat{p}$ occurs exactly once in the multiplication. Furthermore, the definition of the values of $w(e)$ simulate the covering in such a way that no other weights occur. For $p \in \Sigma^{++}$ we have

$$\|\mathfrak{A}\|(p) = \prod_{\substack{1 \le i \le l_v(p)+1 \\ 1 \le j \le l_h(p)+1}} T(t(\hat{p}_{i,j})) = \|\mathcal{T}\|(p) = (S, p). \quad \square$$

Now, we can prove a result originally stated by S. Bozapalidis (private communication):

Theorem 5.7. *Let K be any commutative semiring. Then the following is valid:*

$$K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle = K^{\text{Phv}}\langle\langle \Sigma^{++} \rangle\rangle = K^{\text{Ploc}}\langle\langle \Sigma^{++} \rangle\rangle.$$

Proof. Immediate by Propositions 5.4–5.6 $\square$

There is also a direct proof for the inclusion from the first to the third class, similar to the construction for the inclusion from left to right in Lemma 6.8 of [26], where we proved that unambiguous (unweighted) picture automata are equivalent to unambiguous tiling systems.

6. Comparing all families

We introduced different devices to characterize recognizable picture series. The theorem below shows that the definition of a recognizable picture series and hence the determination of some term of "recognizability" for picture series is robust.

Theorem 6.1. *Let Σ be an alphabet, K a commutative semiring and let $S : \Sigma^{++} \to K$ be a picture series. The following assertions are equivalent.*

1. *S is the behaviour of a weighted picture automaton.*
2. *S is the projection of a rational picture series.*
3. *S is the projection of a tile-local series.*
4. *S is the projection of an hv-local series.*

Proof. It immediately follows from Theorems 4.10 and 5.7. $\square$

This extends the main equivalences for characterizing recognizable picture languages to the weighted case of picture series over arbitrary commutative semirings. In [27], we established a notion of a weighted MSO logics over pictures, and we proved that the class of picture series defined by sentences of the weighted logics coincides with the family $K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$. We generalized the results of [13,14] to the quantitative setting of series; the respective results for languages follow by considering the Boolean semiring and using Remark 4.11 and Proposition 5.3.

Acknowledgements

The author would like to thank Manfred Droste and Paul Gastin for their helpful discussions and comments on earlier versions of this paper, as well as the work of the unknown referees which resulted in improvements.

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