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A digital analogue of the Jordan curve theorem

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Abstract

We study certain closure operations on $\mathbb{Z}^2$, with the aim of showing that they can provide a suitable framework for solving problems of digital topology. The Khalimsky topology on $\mathbb{Z}^2$, which is commonly used as a basic structure in digital topology nowadays, can be obtained as a special case of the closure operations studied. By proving an analogy of the Jordan curve theorem for these closure operations, we show that they provide a convenient model of the real plane and can therefore be used for studying topological and geometric properties of digital images. We also discuss some advantages of the closure operations investigated over the Khalimsky topology.

Keywords: Digital topology; Khalimsky topology; *n*-ary relation; Closure operation; Jordan curve

0. Introduction

In the classical approach to digital topology, graph-theoretic methods are used to obtain a structure on $\mathbb{Z}^2$ suitable for studying topological and geometric properties of digital images see [7,8] and [10–12]. Such a structure is usually given by an adjacency relation between rectangular grid points (pixels) of $\mathbb{Z}^2$. The most frequently used adjacencies are 4- and 8-adjacency (given a point $z \in \mathbb{Z}^2$, the 4-adjacent points to z are just the four nearest points to z in the Euclidean metric restricted from $\mathbb{R}^2$ to $\mathbb{Z}^2$; analogously for 8-adjacency). However, neither 4- nor 8-adjacency provides a satisfactory model of the real (Euclidean) plane because neither of the two adjacencies fulfills an analogy of the Jordan curve theorem (recall that the classical Jordan curve theorem states that a simple closed curve separates the real plane into precisely two

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components). But it was shown by Rosenfeld [11] that such a model is obtained by using a combination of the two adjacencies (one adjacency for digital simple closed curves and the other for their complements). Most results in digital topology are based on this fact.

It was only in 1990 that Khalimsky et al. [4] proposed a new, purely topological approach to digital topology, which was then used also in [5] and [9]. They showed that there is a topology on $\mathbb{Z}^2$, the so-called Khalimsky topology, which provides a convenient model of the real plane and can therefore be used for solving problems of digital topology. In the present paper, we generalize the topological approach to digital topology from [4]. Our view is based on the idea that it is not necessary to have a topology in the usual sense on $\mathbb{Z}^2$ when we want to study topological properties of digital images. We will show that more general topological structures can be convenient too.

Topological structures which are more general than the usual topology occur in many branches of mathematics and are utilized in numerous applications. Particularly, closure operations fulfilling only some of the Kuratowski closure axioms proved to provide a suitable framework for various approaches to digital topology—see [16]. The closure operations employed in the present paper are obtained from the Kuratowski closure operations by omitting the axioms of idempotency and additivity (but retaining the axiom of monotony)—cf. [2]. We will take advantage of the fact that connectedness with respect to quotient maps behaves somewhat better in the class of closure spaces than in the class of topological spaces.

The paper is a continuation of [15] where closure operations associated with α-ary relations ($\alpha > 1$ an ordinal) were studied with a special emphasis upon those defined on the set of integers $\mathbb{Z}$. For the convenience of the reader, the relevant material from [15] is repeated without proofs, which makes our exposition self-contained. For each natural number $n > 1$, we define a closure operation on $\mathbb{Z} \times \mathbb{Z}$, which is obtained as a product of two copies of a closure operation on $\mathbb{Z}$ associated with a special n-ary relation on $\mathbb{Z}$. In the particular case when $n = 2$, we get the Khalimsky topology. The closure operations defined are studied and, as the main result, an analogue of the Jordan curve theorem is formulated and proved for them. It means that these closure operations can be used, as an alternative to the Khalimsky topology, for solving problems of computer graphics and computer image processing. We also demonstrate that using them has some advantages over using the Khalimsky topology.

1. Preliminaries

Given a set X, we denote by $\exp X$ its power set, i.e., the set of all subsets of X. By a *closure operation* u on a set X we mean a map u: $\exp X \to \exp X$ fulfilling $u\emptyset = \emptyset$, $A \subseteq X \Rightarrow A \subseteq uA$, and $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$. Such closure operations were studied by Čech in [1] (who called them topologies). A pair (X, u), where X is a set and u is a closure operation on X, is called a *closure space*. Given a pair u, v of closure operations on a set X, we put $u \leqslant v$ if $uA \subseteq vA$ for each $A \subseteq X$. Clearly, $\leqslant$ is a partial order on the set of all closure operations on X. A closure operation u on a set X is called *additive* (respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A closure operation u on a set X which is both additive and idempotent is called a *Kuratowski closure operation* or briefly a *topology* and the pair (X,u) is called a *topological space*. According to [13], given a cardinal $n>1$, a closure operation u on a set X and the closure space (X,u) are called an S_n*-closure operation* and an S_n*-space*, respectively, if the following condition is satisfied:

$$A \subseteq X \quad \Rightarrow \quad uA = \bigcup \{uB;\ B \subseteq A,\ \operatorname{card} B < n\}.$$

S_2-closure operations and S_2-spaces are called *quasi-discrete* in [1]. S_2-topological spaces are often called *Alexandroff spaces*—see e.g. [6]. Of course, any S_2-closure operation is additive, and any S_m-closure operation is an S_n-closure operation whenever m,n are cardinals with $1<m<n$. Since any closure operation on a set X is obviously an S_n-closure operation for each cardinal n with $n>\operatorname{card} X$, there exists a least cardinal n such that u is an S_n-closure operation. Such a cardinal is then an important invariant of the closure operation u. Evidently, if $n \leqslant \aleph_0$, then any additive S_n-closure operation is an S_2-closure operation.

We will work with some basic topological concepts naturally extended from topological spaces to closure ones. Given a closure space (X,u), a subset $A \subseteq X$ is called *closed* if $uA=A$, and it is called *open* if $X-A$ is closed. A closure space (X,u) is said to be a *subspace* of a closure space (Y,v) if $X \subseteq Y$ and $uA = vA \cap X$ for each subset $A \subseteq X$. We will speak briefly about a subspace X of (Y,v). A closure space (X,u) is said to be *connected* if $\emptyset$ and X are the only subsets of X which are both closed and open. A subset $X \subseteq Y$ is considered to be connected in a closure space (Y,v) if the subspace X of (Y,v) is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected sets and components in topological spaces (see e.g. [3]) are preserved also in closure spaces. A closure space (X,u) is said to be a T_0*-space* if for any points $x,y \in X$ from $x \in u\{y\}$ and $y \in u\{x\}$ it follows that $x=y$, and it is called a $T_{1/2}$*-space* if each singleton subset of X is closed or open. Given closure spaces (X,u) and (Y,v), a map $\varphi: X \to Y$ is said to be a *continuous map* of (X,u) into (Y,v) if $f(uA) \subseteq vf(A)$ for each subset $A \subseteq X$. If, moreover, φ is a bijection and $\varphi^{-1}: Y \to X$ is a continuous map of (Y,v) into (X,u), then φ is called a *homeomorphism* of (X,u) onto (Y,v). We say that closure spaces (X,u) and (Y,v) (and the closure operations u and v) are *homeomorphic* if there exists a homeomorphism of (X,u) onto (Y,v).

If (X_j,u_j), $j \in J$, is a system of closure spaces, then the closure operation v on $\prod_{j\in J} X_j$ generated by the projections $pr_j: \prod_{j\in J} X_j \to X_j$, $j \in J$ (i.e., the greatest—with respect to $\leqslant$—closure operation v on $\prod_{j\in J} X_j$ such that all projections $pr_j: (\prod_{j\in J} X_j, v) \to (X_j,u_j)$, $j\in J$, are continuous) is given by $vA = \prod_{j\in J} u_j\, pr_j(A)$ whenever $A \subseteq \prod_{j\in J} X_j$.

Let (X,u), (Y,v) be closure spaces and let $f:(X,u) \to (Y,v)$ be a surjective map. Then f is a quotient map (i.e., v is the least—with respect to $\leqslant$—closure operation on Y such that the map $f:(X,u)\to(Y,v)$ is continuous) if and only if $vB = f(uf^{-1}(B))$ for any $B \subseteq Y$.

Clearly, if (X,u), (Y,v) are closure spaces, $f:(X,u)\to(Y,v)$ is a continuous map and B is a closed subset of (Y,v), then $f^{-1}(B)$ is closed in (X,u). For quotient maps the following stronger statement obviously holds.

Lemma 1.1. *Let* (X,u), (Y,v) *be closure spaces*, $f:(X,u)\to(Y,v)$ *a quotient map and* $B\subseteq Y$ *a subset. Then* B *is closed in* (Y,v) *if and only if* $f^{-1}(B)$ *is closed in* (X,u).

We will need the following:

Lemma 1.2. *Let* (X,u), (Y,v) *be closure spaces, let* v *be idempotent and let* $f:(X,u)\to(Y,v)$ *be a continuous surjection. Then* f *is a quotient map if and only if* $f(uf^{-1}(B))$ *is closed in* (Y,v) *for each* $B\subseteq Y$.

Proof. If $f:(X,u)\to(Y,v)$ is a quotient map, then $f(uf^{-1}(B))=vB$ and, as v is idempotent, $f(uf^{-1}(B))$ is closed in (Y,v) for each $B\subseteq Y$. Conversely, let $f(uf^{-1}(B))$ be closed in (Y,v) whenever $B\subseteq Y$. Since $B\subseteq f(uf^{-1}(B))$, we get $vB\subseteq f(uf^{-1}(B))$. As the inverse inclusion clearly holds (because f is continuous), we have $f(uf^{-1}(B))=vB$. Therefore, $f:(X,u)\to(Y,v)$ is a quotient map. □

Corollary 1.3. *Let* (X,u), (Y,v) *be closure spaces, let* v *be idempotent and let* $f:(X,u)\to(Y,v)$ *be a quotient map. Then the restriction* $f\,|\,f^{-1}(B):f^{-1}(B)\to B$ *is a quotient map for each subset* B *of* (Y,v).

Proof. Let B be a subset of (Y,v). Clearly, $f\,|\,f^{-1}(B):f^{-1}(B)\to B$ is a continuous surjection. Let $C\subseteq B$ be an arbitrary subset. As $f\,|\,f^{-1}(C)=vC$ and v is idempotent, $f(uf^{-1}(C))$ is closed in (Y,v) and we clearly have $f(uf^{-1}(C)\cap f^{-1}(B))=f(uf^{-1}(C))\cap B$. Consequently, $f(uf^{-1}(C)\cap f^{-1}(B))$ is closed in the subspace B of (Y,v) (because, given a subspace T of a closure space (Z,w) and a subset $A\subseteq Z$, $A\cap T$ is closed in the subspace T whenever A is closed in (Z,w)). Thus, by Lemma 1.2, $f\,|\,f^{-1}(B)$ is a quotient map. □

As usual, given a map $f:X\to Y$, a *fibre* of f is any set $f^{-1}(y)$ with $y\in Y$. If all fibres of f are connected, then we say that f has connected fibres. For closure spaces, the following analogy of a theorem known for topological spaces is valid.

Proposition 1.4. *Let* (X,u), (Y,v) *be closure spaces, let* v *be idempotent and let* $f:(X,u)\to(Y,v)$ *be a quotient map having connected fibres. Then* (X,u) *is connected if and only if* (Y,v) *is connected.*

Proof. If (X,u) is connected, then (Y,v) is connected because f is continuous. Let (Y,v) be connected. Let $X=X_1\cup X_2$ where X_1, X_2 are disjoint closed subsets of (X,u). Put $Y_i=f(X_i)$ for $i=1,2$. As f has connected fibres, $Y_1\cap Y_2=\emptyset$ (because for each connected subset $C\subseteq X$ we clearly have $C\cap X_1=\emptyset$ or $C\cap X_2=\emptyset$) and thus $X_i=f^{-1}(Y_i)$ for $i=1,2$. Since f is a quotient map, we have $Y_1\cup Y_2=Y$ and, by Lemma 1.1, Y_1 and Y_2 are closed. Therefore, Y_1 or Y_2 is empty, hence X_1 or X_2 is empty. Consequently, (X,u) is connected. □

Corollary 1.5. *Let (X,u), (Y,v) be closure spaces, let v be idempotent and let $f:(X,u)\to(Y,v)$ be a quotient map having connected fibres. Then a subset $B\subseteq Y$ is connected in (Y,v) if and only if $f^{-1}(B)$ is connected in (X,u).*

Proof. The statement follows from Corollary 1.3 and Proposition 1.4. □

Remark 1.6. Recall that for topological spaces (and topological quotient maps) Proposition 1.4 and Corollary 1.5 are not valid in general. The Corollary is valid provided that B is open or closed (see e.g. [3]).

From now on, n is understood to be a natural number with $n>1$. As natural numbers are understand to be finite cardinals, we always have $n=\{0,1,\ldots,n-1\}$. Given a set X, we denote by X^n the set of all maps of n into X, i.e., ordered n-tuples $(x_i|\ i<n)$ consisting of elements of X. Ordered n-tuples will be sometimes referred to as finite sequences. Any subset $R\subseteq X^n$ is called an *n-ary relation* on X and the pair (X,R) is called an *n-ary relational system*. An n-ary relation on a set X is said to be *reflexive* if it contains all constant n-tuples consisting of elements of the set X. Given n-ary relational systems (X,R) and (Y,S), a map $\varphi:X\to Y$ is called a *homomorphism* of (X,R) into (Y,S) if the implication $(x_i|\ i<n)\in R\Rightarrow(\varphi(x_i)|\ i<n)\in S$ is valid. If R_j is an n-ary relation on a set X_j for each $j\in J$, then the *product* of the system R_j, $j\in J$, is the n-ary relation $\prod_{j\in J}R_j$ on the Cartesian product $\prod_{j\in J}X_j$ generated by the projections $pr_j:\prod_{i\in J}X_i\to X_j$, $j\in J$ (i.e., the greatest—with respect to the set inclusion—n-ary relation R on $\prod_{j\in J}X_j$ such that all projections $pr_j:(\prod_{i\in J}X_i,R)\to(X_j,R_j)$, $j\in J$, are homomorphisms). Clearly, $\prod_{j\in J}R_j=\{(x_i|\ i<n)\in(\prod_{j\in J}X_j)^n;\ (pr_j(x_i)|\ i<n)\in R_j$ for each $j\in J\}$. Analogous to the Cartesian product of sets, also the product of a pair R, S of n-ary relations will be usually denoted by $R\times S$.

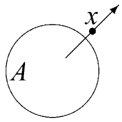
Let (X,R), (Y,S) be n-ary relational systems and $f:(X,R)\to(Y,S)$ a surjective map. Then f is a quotient map (i.e., S is the least—with respect to the set inclusion—n-ary relation on Y such that $f:(X,R)\to(Y,S)$ is a homomorphism) if and only if, whenever $(y_i|\ i<n)\in Y^n$, we have $(y_i|\ i<n)\in S\Leftrightarrow\forall i<n\ \exists x_i\in f^{-1}(y_i)$: $(x_i|\ i<n)\in R$. It is obvious that relational systems are quotient-productive, i.e., if (X_i,R_i), (Y_i,S_i) are n-ary relational systems and $f_i:(X_i,R_i)\to(Y,S_i)$ is a quotient map for each $i\in I$, then also the map $\prod_{i\in I}f_i:\prod_{i\in I}(X_i,R_i)\to\prod_{i\in I}(Y_i,S_i)$ is quotient.

2. Closure operations associated with relations

Most results and definitions of this paragraph are taken from [14] and [15] (where all proofs not presented here can be found).

Let X be a set and R an n-ary relation on X. Then, we define a map $u_R:\exp X\to\exp X$ as follows:

$$u_RA=A\cup\{x\in X;\ \text{there exist } (x_i|\ i<n)\in R \text{ and } i_0,\ 0<i_0<n,\ \text{such that } x=x_{i_0} \text{ and } x_i\in A \text{ for all } i<i_0\}.$$



Clearly, u_R is a closure operation on X.

It can be shown that u_R is idempotent if and only if (X, u_R) is an Alexandroff topological space. Of course, u_R is not additive in general, but, on the other hand, the union of a system of closed subsets of (X, u_R) is a closed subset of (X, u_R).

Obviously, we have

Proposition 2.1. *For any n-ary relation R on a set X*, (X, u_R) *is an* S_n*-space.*

We will need the following assertion.

Proposition 2.2. *Let* (X,R), (Y,S) *be n-ary relational systems and let* $f:(X,R) \to (Y,S)$ *be a quotient map. Then* $f:(X,u_R) \to (Y,u_S)$ *is a quotient map.*

Proof. Let $B \subseteq Y$, $y \in Y$ and suppose that $y \in u_S B$. Then there exist $(y_i|\ i<n) \in S$ and a natural number i_0, $0 < i_0 < n$, such that $y = y_{i_0}$ and $y_i \in B$ for all $i < i_0$. Further, for each $i < n$ there exists $x_i \in f^{-1}(y_i)$ such that $(x_i|\ i<n) \in R$. Now we have $y = f(x_{i_0})$ and $x_{i_0} \in u_R\{x_i|\ i < i_0\} \subseteq u_R f^{-1}(B)$. Hence $y \in f(u_R f^{-1}(B))$.

Conversely, suppose that $y \in f(u_R f^{-1}(B))$. Then there is $x \in u_R f^{-1}(B)$ such that $f(x) = y$. Next, there exist $(x_i|\ i<n) \in R$ and a natural number i_0, $0 < i_0 < n$, such that $x = x_{i_0}$ and $x_i \in f^{-1}(B)$ for each $i < i_0$. Since $(f(x_i)|\ i<n) \in S$, we have $y = f(x_{i_0}) \in u_S\{f(x_i);\ i < i_0\} \subseteq u_S B$ (because $\{f(x_i);\ i<i_0\} \subseteq B$). Thus, $vB = f(uf^{-1}(B))$ and the proof is complete. □

Definition 2.3. An n-ary relation R on a set X is said to be *terse* provided that it is reflexive and fulfills the following condition: If $(x_i|\ i<n) \in R$, $(y_i|\ i<n) \in R$, and there are natural numbers $i_0, i_1 < n$, $i_0 \neq i_1$, such that $x_0 = y_{i_0}$ and $x_1 = y_{i_1}$, then $(x_i|\ i<n) = (y_i|\ i<n)$.

Thus, an n-ary relation R is terse if and only if R is reflexive and any nonconstant n-tuple $(x_i|\ i<n) \in R$ is injective (as a map) and is the only n-tuple belonging to R which contains the elements x_0 and x_1. So, a binary relation is terse if and only if it is reflexive and antisymmetric.

Proposition 2.4. *Let R be a terse n-ary relation on a set X. Then* (X, u_R) *is a* T_0*-space and R is a minimal element* (*with respect to the set inclusion*) *of the set of all reflexive n-ary relations S on X fulfilling* $u_R = u_S$.

Proposition 2.5. *For terse n-ary relations the correspondence* $R \mapsto u_R$ *is one-to-one.*

Definition 2.6. A closure operation u on a set X and the closure space (X, u) are called an S_n^*-*closure operation* and an S_n^*-*space*, respectively, if there is a terse n-ary relation R on X such that $u = u_R$. This (unique) relation R will be denoted by R_u.

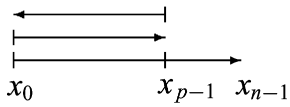
Clearly, the (relational) product of a system of terse n-ary relations is a terse n-ary relation. This fact, together with Proposition 2.5, enables us to define:

Definition 2.7. Let (X_j, u_j), $j \in J$, be a system of S_n^*-spaces. By the *product* of this system we understand the S_n^*-space $(\prod_{j\in J} X_j, u_R)$ where $R = \prod_{j\in J} R_{u_j}$.

The product of a pair (X,u), (Y,v) of S_n^*-spaces will be denoted by $(X,u)\times(Y,v)$.

Remark 2.8. Let (X_j, u_j), $j \in J$, be a system of S_n^*-spaces and let v be the closure operation on $\prod_{j\in J} X_j$ generated by the projections $pr_j : \prod_{j\in J} X_j \to X_j$, $j\in J$. If $(\prod_{j\in J} X_j, u)$ is the product of the system defined in 2.7, then it can easily be seen that $u \leqslant v$. The equality $u = v$ is valid, in general, only for $n = 2$.

Definition 2.9. Let R be an n-ary relation on a set X and let p be a natural number with $1 < p \leqslant n$. An ordered p-tuple $(y_i|\ i<p)$ of points of X is called a *connected element* in (X,u_R) if there is an n-tuple $(x_i|\ i<n)\in R$ such that $y_i = x_i$ for all $i<p$ or $y_i = x_{p-1-i}$ for all $i<p$.

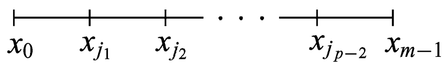


Clearly, each connected element is a connected set. We will need the following observation which immediately follows from Definition 2.9.

Lemma 2.10. *Let $(y_i|\ i<p)$ be a connected element in (X,u_R) and let $(x_i|\ i<n)\in R$ be an n-tuple with $y_i = x_i$ for all $i<p$ or $y_i = x_{p-1-i}$ for all $i<p$. If $i_0 < p$ is a natural number, then either $(y_{i_0-i}|\ i\leqslant i_0)$ or $(y_i|\ i_0\leqslant i<p)$ is a connected element in (X,u_R) with the first member y_{i_0} and the last one x_0.*

Definition 2.11. Let R be an n-ary relation on a set X. A finite nonempty sequence $C=(x_i|\ i<m)$ of points of X is called a *path* in (X,u_R) if, whenever $m>1$, there is a finite increasing sequence $(j_k|\ k<p)$ of natural numbers with $j_0=0$ and $j_{p-1}=m-1$ such that $(x_j|\ j_k\leqslant j\leqslant j_{k+1})$ is a connected element in (X,u_R) for each natural number $k<p-1$.

Clearly, each path is a connected set and each connected element is a path. If $(x_i|\ i<m)$ is a path, then also its *inversion*, i.e., the sequence $(y_i|\ i<m)$ where $y_i = x_{m-1-i}$ for all $i<m$, is a path. Further, if $(x_i|\ i<m)$, $(y_i|\ i<p)$ are paths such that $x_{m-1}=y_0$, then also their *union*, i.e., the sequence $(z_i|\ i<m+p-1)$ where $z_i=x_i$ for all $i<n$ and $z_i = y_{i-m+1}$ for all i with $m\leqslant i<p$, is a path.



Theorem 2.12. *Let R be an n-ary relation on a set X and $A \subseteq X$ be a subset. Then A is connected in (X, u_R) if and only if any two points of A can be joined by a path in (X, u_R) contained in A.*

Lemma 2.13. *Let (X_0, R_0), (X_1, R_1) be reflexive n-ary relational systems and let $(y_i^j|\ i < p_j)$ be a connected element in (X_j, u_{R_j}) for each $j = 0, 1$. Then $\{y_i^0;\ i < p_0\} \times \{y_i^1;\ i < p_1\}$ is a connected set in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$.*

Proof. For each $j = 0, 1$, there is an n-tuple $(x_i^j|\ i < n) \in R_j$ such that $y_i^j = x_i^j$ for all $i < p_j$ or $y_i^j = x_{p_j-1-i}^j$ for all $i < p_j$. Let $y \in \{y_i^0;\ i < p_0\} \times \{y_i^1;\ i < p_1\}$ be an arbitrary element. Then there are natural numbers i_0, i_1 with $i_0 < p_0$, $i_1 < p_1$ such that $y = (y_{i_0}^0, y_{i_1}^1)$. By Lemma 2.10, $(y_{i_0-i}^0|\ i \leqslant i_0)$ or $(y_i^0|\ i_0 \leqslant i < p_0)$ is a connected element in (X_0, u_{R_0}) with the first member $y_{i_0}^0$ and the last one x_0^0. Denote this connected element by $(z_i^0|\ i < q_0)$ and put $C_0 = ((z_i^0, y_{i_1}^1)|\ i < q_0)$. As (X_1, R_1) is reflexive, C_0 is a connected element in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ whose last member equals $(x_0^0, y_{i_1}^1)$. Further, by Lemma 2.10, $(y_{i_1-i}^1|\ i \leqslant i_1)$ or $(y_i^1|\ i_1 \leqslant i < p_1)$ is a connected element in (X_1, u_{R_1}) with the first member y_0^1 and the last one x_0^1. Denote this connected element by $(z_i^1|\ i < q_1)$ and put $C_1 = ((z_{q_0-1}^0, z_i^1)|\ i < q_1)$. As (X_0, R_0) is reflexive, C_1 is a connected element in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ whose first member equals $(x_0^0, y_{i_1}^1)$. Now, the union of the connected elements C_0 and C_1 is a connected element in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ with the first member $(y_{i_0}^0, y_{i_1}^1) = y$ and the last one (x_0^0, x_1^1). We have shown that any point $y \in \{y_i^0;\ i < p_0\} \times \{y_i^1;\ i < p_1\}$ can be joined with the point $(x_0^0, x_0^1) \in \{y_i^0;\ i < p_0\} \times \{y_i^1;\ i < p_1\}$ by a path contained in $\{y_i^0;\ i < p_0\} \times \{y_i^1;\ i < p_1\}$. Now the statement follows from Theorem 2.12. $\square$

Lemma 2.14. *Let (X_0, R_0), (X_1, R_1) be reflexive n-ary relational systems and let $C_j = (x_i^j|\ i < p_j)$ be a path in (X_j, u_{R_j}) for each $j = 0, 1$. Then $\{x_i^0;\ i < p_0\} \times \{x_i^1;\ i < p_1\}$ is a connected set in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$.*

Proof. If C_0 or C_1 contains only one member, then the assertion is trivial because (X_0, R_0) and (X_1, R_1) are reflexive. So, suppose that both C_0 and C_1 contain more than one member. By Definition 2.11, for each $j = 0, 1$ there is a finite increasing sequence $(i_k^j|\ k < q_j)$ of natural numbers with $i_0^j = 0$ and $i_{q_j-1}^j = p_j - 1$ such that $(x_i^j|\ i_k^j \leqslant i \leqslant i_{k+1}^j)$ is a connected element in (X_j, u_{R_j}) for each natural number $k < q_j - 1$. For each $j = 0, 1$, putting $C_k^j = \{x_i^j;\ i_k^j \leqslant i \leqslant i_{k+1}^j\}$, we get $\{x_i^j;\ i < p_j\} = \bigcup_{k < q_j - 1} C_k^j$. Thus, $\{x_i^0;\ i < p_0\} \times \{x_i^1;\ i < p_1\} = \bigcup_{k_0 < q_0 - 1} \bigcup_{k_1 < q_1 - 1} (C_{k_0}^0 \times C_{k_1}^1)$ where $C_{k_0}^0 \times C_{k_1}^1$ is connected in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ for any $k_0 < q_0 - 1$ and any $k_1 < q_1 - 1$ by Lemma 2.13. Thus, whenever $k_0 < q_0 - 1$, $(C_{k_0}^0 \times C_{k_1}^1|\ k_1 < q_{m-1} - 1)$ is a finite sequence of connected sets such that any two consecutive members of it have nonempty intersection. Therefore, the set $D_{k_0} = \bigcup_{k_1 < q_1 - 1} (C_{k_0}^0 \times C_{k_1}^1)$ is a connected set. We have $\{x_i^0;\ i < p_0\} \times \{x_i^1;\ i < p_1\} = \bigcup_{k_0 < q_0 - 1} D_{k_0}$. This proves the statement because any two consecutive members of the finite sequence $(D_{k_0}|\ k_0 < q_0 - 1)$ have nonempty intersection. $\square$

Theorem 2.15. *Let* (X_0, R_0), (X_1, R_1) *be reflexive n-ary relational systems and let* $Y_j \subseteq X_j$ *be a subset for each* $j = 0, 1$. *Then* Y_j *is connected in* (X_j, u_{R_j}) *for each* $j = 0, 1$ *if and only if* $Y_0 \times Y_1$ *is connected in* $(X_0, u_{R_0}) \times (X_1, u_{R_1})$.

Proof. Let Y_j be connected in (X_j, u_{R_j}) for each $j=0,1$ and let $(x_0, x_1), (y_0, y_1) \in Y_0 \times Y_1$ be arbitrary points. Then, for each $j = 0, 1$, there is a path $(z_i^j \mid i < p_j)$ in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ joining the points x_j and y_j which is contained in Y_j. As $\{z_i^0;\ i < p_0\} \times \{z_i^1;\ i < p_1\}$ contains the points $(x_0, x_1), (y_0, y_1)$ and is connected in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ by Lemma 2.14, there is a path in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ joining the points (x_0, x_1) and (y_0, y_1) which is contained in $\{z_i^0;\ i < p_0\} \times \{z_i^1;\ i < p_1\}$. Thus, $Y_0 \times Y_1$ is connected in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ because $\{z_i^0 \mid i < p_0\} \times \{z_i^1 \mid i < p_1\} \subseteq Y_0 \times Y_1$.

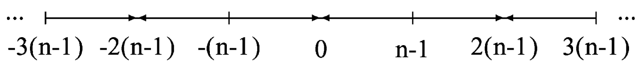
Conversely, let $Y_0 \times Y_1$ be connected in $(X_0, u_{R_0}) \times (X_1, u_{R_1})$ and let v denote the closure operation on $X_0 \times X_1$ generated by the projections $pr_j : X_0 \times X_1 \to X_j$, $j = 0, 1$. By Remark 2.8, $u \leqslant v$. Thus, as the projections $pr_j : (X_0 \times X_1, v) \to (X_j, u_{R_j})$, $j = 0, 1$, are continuous, also the projections $pr_j : (X_0, u_{R_0}) \times (X_1, u_{R_1}) \to (X_j, u_{R_j})$, $j = 0, 1$, are continuous. Consequently, $Y_j = pr_j(Y_0 \times Y_1)$ is connected in (X_j, u_{R_j}) for each $j = 0, 1$. □

3. *n*-ary digital plane

We define an n-ary relation R_n on the set $\mathbb{Z}$ of integers as follows:

$$R_n = \{(x_i \mid i < n) \in \mathbb{Z}^n;\ (x_i \mid i < n) \text{ is constant or there exists an odd number } l \in \mathbb{Z} \text{ fulfilling either } x_i = l(n-1) + i \text{ for all } i < n \text{ or } x_i = l(n-1) - i \text{ for all } i < n\}.$$

The relation R_n is demonstrated in the following figure where the nonconstant n-tuples of R_n are represented as arrows (oriented from the first to the last members of the sequences):



It is evident that R_n is terse so that u_{R_n} is an S_n^*-closure operation on $\mathbb{Z}$. Instead of u_{R_n} we will write briefly u_n. The closure operation u_2 coincides with the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k-1, 2k, 2k+1\};\ k \in \mathbb{Z}\}$—cf. [6]. The relation R_2^{-1} is nothing more than the so-called *specialization order* of u_2. Clearly, u_n is additive if and only if $n = 2$.

By Šlapal [15], we have:

Proposition 3.1. $(\mathbb{Z}, u_n)$ *is a connected* S_n^**-space in which the points* $l(n-1)$, $l \in \mathbb{Z}$ *odd, are open, while all the other points are closed* (*so that* u_n *is a* $T_{1/2}$*-closure operation*).

Let $\tilde{R}_n$ be the n-ary relation on $\mathbb{Z}$ given as follows:

$$\tilde{R}_n = \{(x_i|\ i < n) \in \mathbb{Z}^n;\ (x_i|\ i < n) \text{ is a constant } n\text{-tuple or } x_{n-1} \in \mathbb{Z} \text{ is an even number with either } x_i = x_{n-1} - 1 \text{ for all } i < n-1 \text{ or } x_i = x_{n-1} + 1 \text{ for all } i < n-1\}.$$

Further, let $f_n : \mathbb{Z} \to \mathbb{Z}$ be the surjection defined by
$f_n(l(n-1)) = l$ for each even number $l \in \mathbb{Z}$,
$f_n(l(n-1)+i) = l$ for each odd number $l \in \mathbb{Z}$ and each $i \in \mathbb{Z}$ with $|i| < n-1$.

Theorem 3.2. *$f_n : (\mathbb{Z}, R_n) \to (\mathbb{Z}, \tilde{R}_n)$ is a quotient map.*

Proof. Let $(y_i|\ i < n) \in \tilde{R}_n$. If $(y_i|\ i < n)$ is constant, say $y_i = y$ for all $i < n$, we can choose an arbitrary element $x \in f_n^{-1}(y)$. Then, putting $x_i = x$ for all $i < n$, we get $(x_i|\ i < n) \in R_n$. Let $(y_i|\ i < n)$ be not constant. Then $y_{n-1} \in \mathbb{Z}$ is even and $y_i = y_{n-1} - 1$ for all $i < n-1$ or $y_i = y_{n-1} + 1$ for all $i < n-1$. Suppose that $y_i = y_{n-1} - 1$ for all $i < n-1$ and put $x_i = (y_{n-1} - 1)(n-1) + i$ for all $i < n$. Then $(x_i|\ i < n) \in R_n$ and we have $f_n^{-1}(y_{n-1}) = \{y_{n-1}(n-1)\} = \{x_{n-1}\}$, and $x_i \in \{(y_{n-1} - 1)(n-1) + i;\ i < n\} \subseteq f_n^{-1}(y_{n-1} - 1) = f_n^{-1}(y_i)$ for any $i < n-1$. Further, suppose that $y_i = y_{n-1} + 1$ for all $i < n-1$ and put $x_i = (y_{n-1} + 1)(n-1) - i$. Again, $(x_i|\ i < n) \in R_n$ and we have $f_n^{-1}(y_{n-1}) = \{y_{n-1}(n-1)\} = \{x_{n-1}\}$, and $x_i \in \{(y_{n-1} + 1)(n-1) - i;\ i < n\} \subseteq f_n^{-1}(y_{n-1} + 1) = f_n^{-1}(y_i)$ for any $i < n-1$.

Conversely, let $(y_i|\ i < n) \in \mathbb{Z}^n$ and for each $i < n$ let there exist $x_i \in f_n^{-1}(y_i)$ such that $(x_i|\ i < n) \in R_n$. If $(y_i|\ i < n)$ is constant, then clearly $(y_i|\ i < n) \in \tilde{R}_n$. Let $(y_i|\ i < n)$ be not constant. Then there exists an even number l with $x_{n-1} = l(n-1)$. Thus, $y_{n-1} = l$. Since $x_i = (l-1)(n-1) + i$ for all $i < n-1$ or $x_i = (l+1)(n-1) - i$ for all $i < n-1$, we have $y_i = l - 1$ for all $i < n-1$ or $y_i = l + 1$ for all $i < n-1$. Therefore, $(y_i|\ i < n) \in \tilde{R}_n$. □

Theorem 3.3. $u_{\tilde{R}_n} = u_2$.

Proof. Let $A \subseteq \mathbb{Z}$ and $x \in u_{\tilde{R}_n} A$. If $x \in A$, then $x \in u_2 A$. Suppose that $x \notin A$. Then there exist $(x_i|\ i < n) \in \tilde{R}_n$ and i_0, $0 < i_0 < n$, such that $x = x_{i_0}$ and $x_i \in A$ for all $i < i_0$. Since x_{n-1} is even and $x_{n-2} = x_{n-1} - 1$ or $x_{n-2} = x_{n-1} + 1$, we have $x_{n-2} R_2 x_{n-1}$. As clearly $i_0 = n-1$, we get $x \in u_2 A$. We have shown that $u_{\tilde{R}_n} \leqslant u_2$.

Conversely, let $x \in u_2 A$. If $x \in A$, then $x \in u_{\tilde{R}_n} A$. Suppose that $x \notin A$. Then there exists $y \in A$ such that $x \in u_2\{y\}$. Consequently, $y R_2 x$ and thus x is even and $y = x - 1$ or $y = x + 1$. Now, putting $x_i = y$ for all $i < n-1$ and $x_{n-1} = x$, we get $(x_i|\ i < n) \in \tilde{R}_n$ and $x_i \in A$ for each $i < n-1$. Therefore, $x \in u_{\tilde{R}_n} A$. We have shown that $u_2 \leqslant u_{\tilde{R}_n}$ which completes the proof. □

Corollary 3.4. *$f_n : (\mathbb{Z}, u_n) \to (\mathbb{Z}, u_2)$ is a quotient map.*

Proof. The statement follows from Theorems 3.2, 3.3 and Proposition 2.2. □

We put $(\mathbb{Z}^2, v_n) = (\mathbb{Z}, u_n) \times (\mathbb{Z}, u_n)$, so that $v_n = u_{R_n \times R_n}$. The closure space $(\mathbb{Z}^2, v_n)$ will be called the *n-ary digital plane*. Clearly, the product $(\mathbb{Z}, u_2) \times (\mathbb{Z}, u_2)$ coincides with the usual topological product and so the binary digital plane coincides with the Khalimsky plane (cf. [6]). As a consequence of 3.1 we get:

Proposition 3.5. *$(\mathbb{Z}^2, v_n)$ is a connected S_n^*-space in which a point $z = (z_1, z_2) \in \mathbb{Z}^2$ is open if and only if $z_1 = k(n-1)$ and $z_2 = l(n-1)$ where $k, l \in \mathbb{Z}$ are odd, and it is closed if and only if $z_1 \neq k(n-1)$ for all odd numbers $k \in \mathbb{Z}$ and $z_2 \neq l(n-1)$ for all odd numbers $l \in \mathbb{Z}$.*

We denote by $F_n : (\mathbb{Z}^2, v_n) \to (\mathbb{Z}^2, v_2)$ the map $F_n = f_n \times f_n$. Clearly, if $k, l \in \mathbb{Z}$, then F_n^{-1} equals the singleton $\{(k(n-1), l(n-1))\}$ whenever k, l are even, the "square" $\{(x, y) \in \mathbb{Z}^2;\ |x-k| < n-1,\ |y-l| < n-1\}$ of $(2n-3)^2$ points whenever k, l are odd, the "abscissa" $\{(x, y) \in \mathbb{Z}^2;\ |x-k| < n-1,\ y = l\}$ of $2n-3$ points whenever k is odd and l is even, and the "abscissa" $\{(x, y) \in \mathbb{Z}^2;\ x = k,\ |y-l| < n-1\}$ of $2n-3$ points whenever k is even and l is odd.

Theorem 3.6. *$F_n : (\mathbb{Z}^2, v_n) \to (\mathbb{Z}^2, v_2)$ is a quotient map with connected fibres.*

Proof. F_n is a quotient map because of Theorems 3.2, 3.3, Proposition 2.2, Definition 2.7 and the fact that relational systems are quotient productive. Let $(x, y) \in \mathbb{Z}^2$ be a point. Then $F_n^{-1}(x, y) = f_n^{-1}(x) \times f_n^{-1}(y)$ and both the sets $f_n^{-1}(x)$ and $f_n^{-1}(y)$ are connected in $(\mathbb{Z}, v_n)$ (as each of them is a singleton or the union of two connected elements in $(\mathbb{Z}, v_n)$ having a common member). Thus, $F_n^{-1}(x, y)$ is connected in $(\mathbb{Z}^2, v_n)$ by Theorem 2.15. □

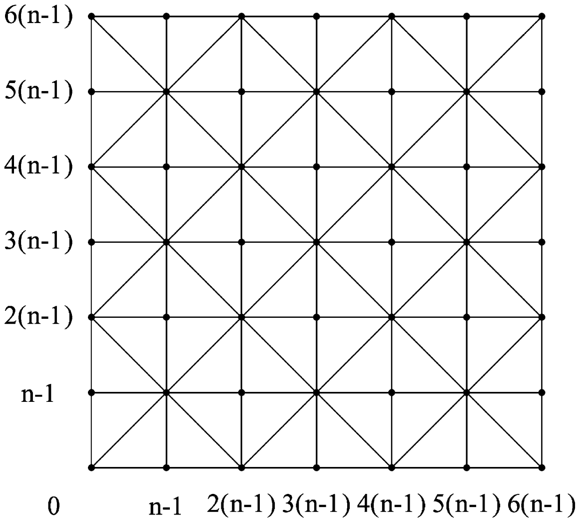
Let $J \subseteq \mathbb{Z}^2$ be a subset and $z \in J$ a point. Then we put

$$J_z = \{(z_i \mid i < n) \in R_n \times R_n;\ (z_i \mid i < n) \text{ is a non-constant } n\text{-tuple with } z \in \{z_i;\ i < n\} \subseteq J\}.$$

Definition 3.7. A *simple closed curve* in $(\mathbb{Z}^2, v_n)$ is a finite connected set $J \subseteq \mathbb{Z}^2$ that satisfies the following two conditions:

(1) For any point $z \in J$ there exists an n-tuple $(z_i \mid i < n) \in J_z$ with $z_0 = (k(n-1), l(n-1))$, $k, l \in \mathbb{Z}$.
(2) If $z \in J$ is a point with $z = (k(n-1), l(n-1))$, $k, l \in \mathbb{Z}$, then $\operatorname{card} J_z = 2$.

For $n > 2$, simple closed curves in the n-ary digital plane are precisely circles in the graph (a portion of which is) shown in the following figure; the vertices of the graph are all points of $\mathbb{Z}^2$ but only the points $(k(n-1), l(n-1))$, $k, l \in \mathbb{Z}$, are marked in the figure and thus, on any edge connecting a pair of points, there are another $n-2$ points:



Note that for $n=2$, condition (1) of 3.7 is always satisfied and the simple closed curves with at least four points coincide with the COTS-Jordan curves from [4]. So, by Khalimsky et al. [4] we have

Theorem 3.8. *Any simple closed curve J in $(\mathbb{Z}^2, v_2)$ having at least four points separates $(\mathbb{Z}^2, v_2)$ into precisely two components (i.e., the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, v_2)$ consists of precisely two components).*

Corollary 1.5 and Theorems 3.6 and 3.8 result in

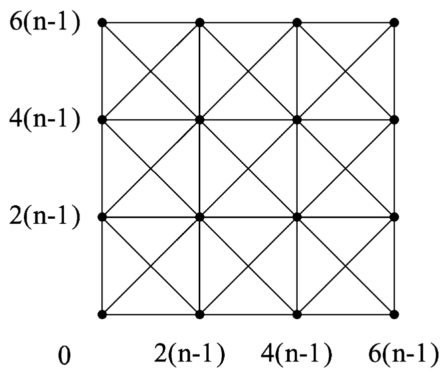
Theorem 3.9. *Let $J \subseteq \mathbb{Z}^2$ be a set such that $F_n(J)$ is a simple closed curve in $(\mathbb{Z}^2, v_2)$ having at least four points. If $F_n^{-1}(F_n(J)) = J$, then J separates $(\mathbb{Z}^2, v_n)$ into precisely two components.*

The sets J from Theorem 3.9 need not be simple closed curves in $(\mathbb{Z}^2, v_n)$ in general and so this theorem is not a satisfactory analogy of the classical Jordan curve theorem. In what follows we will give a criterion under which a given simple closed curve in the space $(\mathbb{Z}^2, v_n)$ separates this space into precisely two components.

Definition 3.10. A simple closed curve J in $(\mathbb{Z}^2, v_n)$ is said to be a *Jordan curve* provided that for any point $z \in J$ with $z = (k(n-1), l(n-1))$, $k, l \in \mathbb{Z}$ odd, from $\{((x_i, y_i)|\ i < n), ((x_i', y_i')|\ i < n)\} = J_z$ it follows that $|x_{n-1} - x_{n-1}'| = |y_{n-1} - y_{n-1}'| = 2(n-1)$.

Clearly, if $(z_i|\ i < n) \in R_n \times R_n$ and $z = (k(n-1), l(n-1)) \in \mathbb{Z}^2$ where $k, l \in \mathbb{Z}$ are odd, then from $z \in \{z_i;\ i < n\}$ it follows that $z = z_0$. Thus, the condition in Definition 3.10 means that J can turn only at the points $(k(n-1), l(n-1))$ for which $k, l \in \mathbb{Z}$ are even. It immediately follows that $F_n(J)$ has at least six points for any Jordan curve

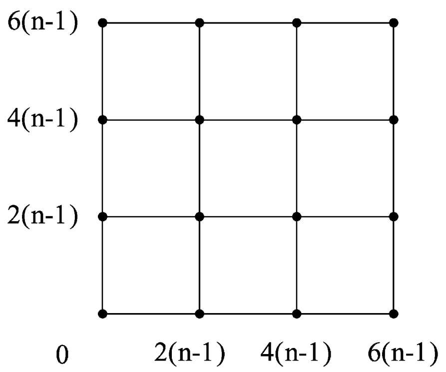
J in $(\mathbb{Z}^2, v_n)$. For $n > 2$, Jordan curves are nothing else than circles in the graph (a portion of which is) shown in the following figure; the vertices of the graph are all points of $\mathbb{Z}^2$ but only the points $(k(n-1), l(n-1))$, $k, l \in \mathbb{Z}$ even, are marked in the figure and thus, on any edge connecting a pair of points, there are another $2n-3$ points:



Theorem 3.11. *Let J be a simple closed curve in $(\mathbb{Z}^2, v_n)$ and let for any $k, l \in \mathbb{Z}$ from $(k(n-1), l(n-1)) \in J$ it follows that k or l is even. Then J is a Jordan curve in $(\mathbb{Z}^2, v_n)$ which separates (Z^2, v_n) into precisely two components.*

Proof. Clearly, J is a Jordan curve in $(\mathbb{Z}^2, v_n)$, $F_n(J)$ is a Jordan curve in $(\mathbb{Z}^2, v_2)$ having at least eight points and $F_n^{-1}(F_n(J)) = J$. Thus, the statement follows from Corollary 1.5 and Theorems 3.6 and 3.8. □

The Jordan curves from Theorem 3.11 are said to be *rectangular* because they are just the circles in the graph (a portion of which is) shown in the following figure; again, the vertices of the graph are all points of $\mathbb{Z}^2$ but only the points $(k(n-1), l(n-1))$, $k, l \in \mathbb{Z}$ even, are marked in the figure and thus, on any edge connecting a pair of points, there are another $2n-3$ points:



Definition 3.12. Let $n > 2$ and let $k, l \in \mathbb{Z}$ be odd numbers. Then we set

$$A_1(k,l) = F_n^{-1}(\{(k-1,l-1),(k,l-1),(k+1,l-1)\}),$$

$$A_2(k,l) = F_n^{-1}(\{(k-1,l+1),(k,l+1),(k+1,l+1)\}),$$

$$B_1(k,l) = F_n^{-1}(\{(k-1,l-1),(k-1,l),(k-1,l+1)\}),$$

$$B_2(k,l) = F_n^{-1}(\{(k+1,l-1),(k+1,l),(k+1,l+1)\}),$$

$$\Delta_1(k,l) = \{(x,y) \in \mathbb{Z}^2;\ |x-k(n-1)| < n-1, y = l(n-1) - x + k(n-1)\},$$

$$\Delta_2(k,l) = \{(x,y) \in \mathbb{Z}^2;\ |x-k(n-1)| < n-1, y = l(n-1) + x - k(n-1)\},$$

$$T_1(k,l) = \{(x,y) \in \mathbb{Z}^2; |x-k(n-1)| < n-1, (l-1)(n-1) < y < l(n-1) - x + k(n-1)\},$$

$$T_2(k,l) = \{(x,y) \in \mathbb{Z}^2; |x-k(n-1)| < n-1, l(n-1) + x - k(n-1) < y < (l+1)(n-1)\},$$

$$T_3(k,l) = \{(x,y) \in \mathbb{Z}^2; |x-k(n-1)| < n-1, (l-1)(n-1) < y < l(n-1) + x - k(n-1)\},$$

$$T_4(k,l) = \{(x,y) \in \mathbb{Z}^2;\ |x-k(n-1)| < n-1, l(n-1) - x + k(n-1) < y < (l+1)(n-1)\}.$$

Graphically, the sets $A_1(k,l)$, $A_2(k,l)$, $B_1(k,l)$ and $B_2(k,l)$ are the lower, upper, left and right sides of the square $F_n^{-1}(k,l)$, respectively, which do not meet this square. The sets $\Delta_1(k,l)$ and $\Delta_2(k,l)$ are the increasing and decreasing diagonals of the square F_n^{-1} (which are contained in the square). Finally, $T_1(k,l)$ and $T_4(k,l)$ are the left lower and right upper triangles, respectively, obtained by separating the square F_n^{-1} by the decreasing diagonal $\Delta_1(k,l)$ (which are included in the square but do not meet the diagonal). Similarly, $T_2(k,l)$ and $T_3(k,l)$ are the left upper and right lower triangles, respectively, obtained by separating the square F_n^{-1} by the increasing diagonal $\Delta_2(k,l)$ (which are included in the square but do not meet the diagonal).

Lemma 3.13. *Let $n > 2$ and let k, l be odd numbers. Then each of the* 10 *sets introduced in Definition* 3.12 *is connected in* $(\mathbb{Z}^2, v_n)$.

Proof. The sets $A_1(k,l)$, $A_2(k,l)$, $B_1(k,l)$ and $B_2(k,l)$ are inverse images under F_n of connected sets in $(\mathbb{Z}^2, v_2)$. Thus, they are connected by Corollary 1.5 and Theorem 3.6. It is evident that also the sets $\Delta_1(k,l)$ and $\Delta_2(k,l)$ are connected because each of them is the union of a pair of connected elements having a common member. Now, consider the set $T_1(k,l)$. Then, for each $x \in \mathbb{Z}$ with $(k-1)(n-1) < x < k(n-1)$, the set $C_x = \{(x,y) \in \mathbb{Z}^2;\ (l-1)(n-1) < y < l(n-1) - x + k(n-1)\}$ is connected (as it is formed by a connected element in the case $x = k(n-1) - 1$ or is a union of a pair of connected elements having a common member otherwise). Similarly, for each $y \in \mathbb{Z}$ with $(l-1)(n-1) < y < l(n-1)$, the set $D_y = \{(x,y) \in \mathbb{Z}^2;\ (k-1)(n-1) < x < l(n-1) - y + k(n-1)\}$ is connected (as it is formed by a connected element in the case $y = l(n-1)-1$ or is a union of a pair of connected elements having a common member otherwise). Clearly, $T_1(k,l) = \bigcup\{C_x;\ (k-1)(n-1) < x < k(n-1)\} \cup \bigcup\{D_y;\ (l-1)(n-1) < y < l(n-1)\}$. For each natural number $i < n-2$ put $C'_i = C_{(k-1)(n-1)+1+i}$, $D'_i = D_{(l-1)(n-1)+1+i}$ and $E_i = C'_i \cup D'_i$. Then $(C'_i \mid i < n-2) = (C_x \mid (k-1)(n-1) < x < k(n-1))$ and $(D'_i \mid i < n-2) = (D_y \mid (l-1)(n-1) < y < l(n-1))$. Thus, $T_1(k,l) = \bigcup_{i<n-2} E_i$ where E_i is a connected set for each $i < n-2$ (because $C'_i \cap D'_i = \{(x_{(k-1)(n-1)+1+i}, y_{(l-1)(n-1)+1+i})\}$). Since $D'_i \cap C'_0 = \{(x_{(k-1)(n-1)+1}, y_{(l-1)(n-1)+1+i})\}$ for each $i < n-2$, we have $E_i \cap E_0 \neq \emptyset$ for each $i < n-2$. Consequently, $T_1(k,l)$ is connected in $(\mathbb{Z}^2, v_n)$. For $T_2(k,l)$, $T_3(k,l)$ and $T_4(k,l)$ the proofs are analogous. □

We will need the following, quite obvious statement.

Lemma 3.14. *Let J be a simple closed curve in $(\mathbb{Z}^2, v_2)$ having at least four points and let $k, l \in \mathbb{Z}$ be odd numbers with $(k,l) \in J$. If $(k-1, l-1) \in J$ and $(k+1, l+1) \in J$, then the points $(k-1, l)$ and $(k, l+1)$ belong to one component of $\mathbb{Z}^2 - J$ while the points $(k, l-1)$ and $(k+1, l)$ belong to the other. Similarly, if $(k-1, l+1) \in J$ and $(k+1, l-1) \in J$, then the points $(k, l-1)$ and $(k-1, l)$ belong to one component of $\mathbb{Z}^2 - J$ while the points $(k, l+1)$ and $(k+1, l)$ belong to the other.*

Theorem 3.15. *Let J be a Jordan curve in $(\mathbb{Z}^2, v_n)$ satisfying the condition that, whenever there are odd numbers $k, l \in \mathbb{Z}$ such that $(k(n-1), l(n-1)) \in J$, J contains just two of the four points $((k \pm 1)(n-1), (l \pm 1)(n-1))$. Then J separates $(\mathbb{Z}^2, v_n)$ into precisely two components.*

Proof. Clearly, the condition from Theorem 3.15 is necessary and sufficient for $F_n(J)$ to be a simple closed curve in $(\mathbb{Z}^2, v_2)$ (having at least six points). For $n = 2$, the statement immediately follows from Theorem 3.8. Suppose that $n > 2$. If $(k(n-1), l(n-1)) \notin J$ whenever $k, l \in \mathbb{Z}$ are odd, then J is rectangular and the statement follows from Theorem 3.11. So, let there be odd numbers $k, l \in \mathbb{Z}$ with $(k(n-1), l(n-1)) \in J$ and put $J' = F_n^{-1}(F_n(J))$. By Corollary 1.5 and Theorems 3.6 and 3.8, J' separates $(\mathbb{Z}^2, v_n)$ into precisely two components and $J \subseteq J'$, $J \neq J'$. Clearly, J contains just one of the sets $\Delta_1(k,l)$, $\Delta_2(k,l)$. We will assume, without loss of generality, that $\Delta_1(k,l) \subseteq J$. Then $F_n^{-1}(k-1, l+1) \cup F_n^{-1}(k+1, l-1) \subseteq J$, thus $(k-1, l+1) \in F_n(J)$ and $(k+1, l-1) \in F_n(J)$. Clearly, for any $(x,y) \in \Delta_1(k,l)$ we have $F_n^{-1}(F_n(x,y)) = T_1(k,l) \cup \Delta_1(k,l) \cup T_4(k,l)$. The condition from Theorem 3.15 implies that $F_n^{-1}(k-1, l-1) \cap J = \emptyset$ and $F_n^{-1}(k+1, l+1) \cap J = \emptyset$. Consequently, $F_n^{-1}(k, l-1) \cap J = F_n^{-1}(k, l+1) \cap J = F_n^{-1}(k-1, l) \cap J = F_n^{-1}(k+1, l) \cap J = \emptyset$. Put $z_1 = (k(n-1)-1, (l-1)(n-1))$ and $z_2 = (k(n-1)+1, (l+1)(n-1))$. Then $z_1 \in F_n^{-1}(k, l-1)$ and $z_2 \in F_n^{-1}(k, l+1)$, hence $z_1 \notin J'$ and $z_2 \notin J'$. Let E_1, E_2 be the components of $\mathbb{Z}^2 - J$ with $z_1 \in E_1$ and $z_2 \in E_2$. By Corollary 1.5, Theorem 3.6 and Lemma 3.14, E_1 and E_2 are different. Clearly, we have $z_1 \in v_n T_1(k,l)$ and $z_2 \in v_n T_4(k,l)$. Thus, as $T_1(k,l)$ and $T_4(k,l)$ are connected by Lemma 3.13, the sets $E_1 \cup T_1(k,l)$ and $E_2 \cup T_4(k,l)$ are connected in the subspace

$\mathbb{Z}^2 - J_1$ of $(\mathbb{Z}^2, v_n)$ where $J_1 = J' - (T_1(k,l) \cup T_4(k,l))$. Obviously, $E_1 \cup T_1(k,l)$ and $E_2 \cup T_4(k,l)$ are disjoint and $E_1 \cup T_1(k,l) \cup E_2 \cup T_4(k,l) = \mathbb{Z}^2 - J_1$. As $\mathbb{Z}^2 - J_1$ is not connected (because $F_n(\mathbb{Z}^2 - J_1) = \mathbb{Z}^2 - F_n(J_1) = \mathbb{Z}^2 - F_n(J)$ is not connected), the sets $E_1 \cup T_1(k,l)$ and $E_2 \cup T_4(k,l)$ are components of $\mathbb{Z}^2 - J_1$. Consequently, J_1 separates $(\mathbb{Z}^2, v_n)$ into precisely two components. Now, if there are odd numbers $k_1, l_1 \in \mathbb{Z}$ such that $(k_1(n-1), l_1(n-1)) \in J$, in the next step we repeat the previous considerations for k_1, l_1 and J_1 instead of k, l and J. In this way, we obtain a set J_2 which separates $(\mathbb{Z}^2, v_n)$ into precisely two components. After a finite number m of steps we will get a set J_m which separates $(\mathbb{Z}^2, v_n)$ into precisely two components and contains no point $(k(n-1), l(n-1))$ with $k, l \in \mathbb{Z}$ odd. Clearly, $J_m = J$ which proves the statement. □

The Jordan curves J in $(\mathbb{Z}^2, v_n)$ satisfying the condition from Theorem 3.15 are said to be *without acute angles*. Thus, every rectangular Jordan curve is without acute angles.

Whenever $n > 2$, each Jordan curve J in $(\mathbb{Z}^2, v_n)$ clearly satisfies $\operatorname{card} J \geqslant 6(n-1)$. The shortest Jordan curves J in $(\mathbb{Z}^2, v_n)$, i.e., those satisfying $\operatorname{card} J = 6(n-1)$, are said to be *elementary*. Of course, if J is an elementary Jordan curve in $(\mathbb{Z}^2, v_n)$, then there are odd numbers $k, l \in \mathbb{Z}$ such that J is the union of one of the sets $A_1(k,l), A_2(k,l)$, one of the sets $B_1(k,l), B_2(k,l)$, and one of the sets $\Delta_1(k,l), \Delta_1(k,l)$.

Theorem 3.16. *Let $n > 2$ and let J be an elementary Jordan curve in $(\mathbb{Z}^2, v_n)$. Then J separates $(\mathbb{Z}^2, v_n)$ into precisely two components.*

Proof. We will suppose, without loss of generality, that $J = A_1(k,l) \cup B_1(k,l) \cup \Delta_1(k,l)$. Put $J' = A_1(k,l) \cup B_1(k,l) \cup A_2(k,l) \cup B_2(k,l)$. Then $F_n(J') = \{(x,y) \in \mathbb{Z}^2;\ \max\{|x-k|, |y-l|\} = 1\}$ is a simple closed curve in $(\mathbb{Z}^2, v_2)$ with $\operatorname{card} F_n(J') = 8$. Thus, $F_n(J')$ separates $(\mathbb{Z}^2, v_2)$ into precisely two components by Theorem 3.8. As $J' = F_n^{-1}(F_n(J'))$, J' separates $(\mathbb{Z}^2, v_n)$ into precisely two components by Corollary 1.5 and Theorem 3.6. Clearly, one of these components is the set $C = F_n^{-1}(k,l) = T_1(k,l) \cup T_4(k,l) \cup \Delta_1(k,l)$, hence the other is the set $\mathbb{Z}^2 - (C \cup J')$. By Lemma 3.13, $T_1(k,l)$ is connected in $(\mathbb{Z}^2, v_n)$ and thus it is connected also in the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, v_n)$ because $T_1(k,l) \subseteq \mathbb{Z}^2 - J$. We have $\mathbb{Z}^2 - (J \cup T_1(k,l)) = (\mathbb{Z}^2 - (C \cup J')) \cup (A_2(k,l) - \{z_1\}) \cup (B_2(k,l) - \{z_2\}) \cup T_4(k,l)$ where $\{z_1\} = F_n^{-1}(k-1, l+1)$ and $\{z_2\} = F_n^{-1}(k+1, l-1)$. Clearly, $\mathbb{Z}^2 - (C \cup J') \subseteq \mathbb{Z}^2 - J$ and $(A_2(k,l) - \{z_1\}) \cup (B_2(k,l) - \{z_2\}) \subseteq v_n(\mathbb{Z}^2 - (C \cup J'))$. Thus, $(\mathbb{Z}^2 - (C \cup J')) \cup (A_2(k,l) - \{z_1\}) \cup (B_2(k,l) - \{z_2\})$ is connected in $\mathbb{Z}^2 - J$. Let $z_0 = ((k+1)(n-1), l(n-1)+1)$. Then $z_0 \in (B_2(k,l) - \{z_2\}) \cap v_n T_4(k,l)$. Consequently, as $T_4(k,l)$ is connected in $\mathbb{Z}^2 - J$ by Lemma 3.13, the set $\mathbb{Z}^2 - (J \cup T_1(k,l))$ is connected in $\mathbb{Z}^2 - J$ too. As any path in $(\mathbb{Z}^2, v_n)$ connecting a point of $T_1(k,l)$ with another one of $\mathbb{Z}^2 - (J \cup T_1(k,l))$ clearly contains a point of J, the sets $\mathbb{Z}^2 - (J \cup T_1(k,l))$ and $T_1(k,l)$ are components of the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, v_n)$. □

Lemma 3.17. *Let $n > 2$ and let $k, l \in \mathbb{Z}$ be odd numbers. Put $A = T_1(k,l) \cup F_n^{-1}(\{(k-1, l), (k-1, l+1)\}) \cup \Delta_1(k,l)$ and $W = T_4(k-2, l) \cup F_n^{-1}(k-2, l+1) \cup T_3(k-1, l-1) \cup F_n^{-1}(k-1, l+2) \cup T_1(k, l+2) \cup F_n^{-1}(k, l+1) \cup T_4(k,l)$. Let $B \supseteq A$ be a connected set in $(\mathbb{Z}^2, v_n)$ such that $B \cap F_n^{-1}(\{(k-1, l-1), (k, l-1), (k+1, l-1)\}) = \emptyset$ and*

$W \subseteq B$. If $F_n^{-1}(k+1,l) \subseteq B$ or $F_n^{-1}(k+1,l) \cap B = \emptyset$, then $B-A$ is a connected set in $(\mathbb{Z}^2, v_n)$.

Proof. Note that A and W are disjoint. The sets $F_n^{-1}(k-2,l+1)$, $F_n^{-1}(k-1,l+2)$, $F_n^{-1}(k,l+1)$ and $F_n^{-1}(k+1,l)$ are connected because F_n has connected fibres. By Lemma 3.13, also the sets $T_4(k-2,l)$, $T_3(k-1,l-1)$, $T_1(k,l+2)$ and $T_4(k,l)$ are connected. We clearly have $((k-1)(n-1)-1,(l+1)(n-1)) \in F_n^{-1}(k-2,l+1) \cap v_n T_4(k-2,l) \cap v_n T_3(k-1,l-1)$, $((k-1)(n-1),(l+1)(n-1)+1) \in F_n^{-1}(k-1,l+2) \cap v_n T_3(k-1,l-1) \cap v_n T_1(k,l+2)$, $((k-1)(n-1)+1,(l+1)(n-1)) \in F_n^{-1}(k,l+1) \cap v_n T_1(k,l+2)$ and $(k(n-1)+1,(l+1)(n-1)) \in F_n^{-1}(k,l+1) \cap v_n T_4(k,l)$. Consequently, W is a connected set. Further, we clearly have $((k-1)(n-1)-1,(l-1)(n-1)+1) \in v_n T_4(k-2,l) \subseteq v_n W$. Therefore, $W \cup \{((k-1)(n-1)-1,(l-1)(n-1)+1)\}$ is connected. As $F_n^{-1}(k+1,l+1) \in v_n F_n^{-1}(k,l+1) \in v_n W$, also $W \cup F_n^{-1}(k+1,l)$ is connected.

We will show that for any path connecting two different points of $B-A$ which is contained in B and meets A there is a path connecting these points which is contained in $B-A$. To this end, let $C=(z_i|\ i<m)$ be a path contained in B such that $z_0, z_{m-1} \in B-A$ and $\{z_i;\ i<m\} \cap A \neq \emptyset$. Then there is a finite increasing sequence $(j_k|\ k<p)$ of natural numbers with $j_0=0$ and $j_{p-1}=m-1$ such that $(z_j|\ j_k \leqslant j \leqslant j_{k+1})$ is a connected element for each $k<p-1$. Let $k_1<p-1$ be the least natural number having the property that the connected element $C_{k_1}=(z_i|\ j_{k_1} \leqslant i \leqslant j_{k_1+1})$ meets A and let $k_2<p-1$ be the greatest natural number having the property that the connected element $C_{k_2}=(z_i|\ j_{k_2} \leqslant i \leqslant j_{k_2+1})$ meets A. Then $z_{j_{k_1}} \in B-A$, $z_{j_{k_2+1}} \in B-A$ and both C_{k_1} and C_{k_2} meet $A-T_1(k,l)$.

First, suppose that $C_{k_1} \cap \Delta_1(k,l) = \emptyset = C_{k_2} \cap \Delta_1(k,l)$. Then it is evident that $C_{k_1} \in R_n \times R_n$, $C_{k_1} \cap A = \{z_{j_{k_1+1}}\}$ and $z_{j_{k_1+1}-1} \in W \cup E$ where $E=\emptyset$ if $((k-1)(n-1)-1,(l-1)(n-1)+1) \notin B$ and $E=\{((k-1)(n-1)-1,(l-1)(n-1)+1)\}$ if $((k-1)(n-1)-1,(l-1)(n-1)+1) \in B$. Similarly, denoting the inversion of C_{k_2} by $C_{k_2}^-$, we get $C_{k_2}^- \in R_n \times R_n$, $C_{k_2} \cap A = \{z_{j_{k_2}}\}$ and $z_{j_{k_2}+1} \in W \cup E$. Thus, $C'_{k_1}=(z_i|\ j_{k_1} \leqslant i < j_{k_1+1})$ and $C'_{k_2}=(z_i|\ j_{k_2} < i \leqslant j_{k_2+1})$ are connected elements contained in $B-A$. As $W \cup E$ is connected, there is a path C' connecting $z_{j_{k_1+1}-1}$ and $z_{j_{k_2}+1}$ which is contained in $W \cup E \subseteq B-A$. Consequently, the union C'' of the paths C'_{k_1}, C' and C'_{k_2} is a path connecting $z_{j_{k_1}}$ and $z_{j_{k_2+1}}$ which is contained in $B-A$. Now, the union of the paths $(z_i|\ i \leqslant j_{k_1})$, C'' and $(z_i|\ j_{k_2+1} \leqslant i < m)$ is a path connecting z_0 and z_{m-1} which is contained in $B-A$.

Next, let $C_{k_1} \cap \Delta_1(k,l) \neq \emptyset$ or $C_{k_2} \cap \Delta_1(k,l) \neq \emptyset$. We can suppose, without loss of generality, that $C_{k_1} \cap \Delta_1(k,l) \neq \emptyset$ (because otherwise we can replace C by its inversion). Then $z_{j_{k_1}} \in W \cup G \cup H$ where $G=\emptyset$ if $F_n^{-1}(k+1,l+1) \cap B = \emptyset$ and $G=F_n^{-1}(k+1,l+1)$ if $F_n^{-1}(k+1,l+1) \subseteq B$, and $H=\emptyset$ if $F_n^{-1}(k+1,l) \cap B = \emptyset$ and $H=F_n^{-1}(k+1,l)$ if $F_n^{-1}(k+1,l) \subseteq B$. First, suppose that $C_{k_2} \cap \Delta_1(k,l) = \emptyset$. Then $C'_{k_2}=(z_i|\ j_{k_2} < i \leqslant j_{k_2+1})$ is a connected element with $z_{j_{k_2}+1} \in W \cup E$ which is contained in $B-A$ and whose inversion belongs to $R_n \times R_n$. Thus, there is a path C' connecting $z_{j_{k_1}}$ and $z_{j_{k_2}+1}$ which is contained in $W \cup E \cup G \cup H \subseteq B-A$. Hence, the union C'' of the paths C' and C'_{k_2} is a path connecting $z_{j_{k_1}}$ and $z_{j_{k_2+1}}$ which is contained in $B-A$. Now, the union of the paths $(z_i|\ i \leqslant j_{k_1})$, C'' and $(z_i|\ j_{k_2+1} \leqslant i \leqslant m)$ is a path connecting z_0 and z_{m-1} which is contained in $B-A$. Finally, suppose that $C_{k_2} \cap \Delta_1(k,l) \neq \emptyset$. Then $z_{j_{k_2+1}} \in W \cup G \cup H$, thus there is a path C' connecting $z_{j_{k_1}}$ and $z_{j_{k_2+1}}$ which is contained in $W \cup G \cup H \subseteq B-A$.

Therefore, the union of the paths $(z_i|\ i \leqslant j_{k_1})$, C' and $(z_i|\ j_{k_2+1} \leqslant i < m)$ is a path connecting z_0 and z_{m-1} which is contained in $B - A$. □

Remark 3.18. Of course, statements analogous to Lemma 3.17 are valid also for $A = T_4(k,l) \cup F_n^{-1}(\{(k+1,l-1),(k+1,l)\}) \cup \Delta_1(k,l)$, $A = T_2(k,l) \cup F_n^{-1}(\{(k-1,l-1),(k-1,l)\}) \cup \Delta_2(k,l)$, and $A = T_3(k,l) \cup F_n^{-1}(\{(k+1,l),(k+1,l+1)\}) \cup \Delta_2(k,l)$.

The following digital analogy of the classical Jordan curve theorem shows that, for any $n > 1$, the n-ary digital plane provides a structure suitable for solving problems of digital image processing:

Theorem 3.19. *Let J be a Jordan curve in $(\mathbb{Z}^2, v_n)$. Then J separates $(\mathbb{Z}^2, v_n)$ into precisely two components.*

Proof. For $n = 2$ the statement follows from Theorem 3.8. Let $n > 2$. If J is without acute angles or elementary, then the statement follows from Theorem 3.15 or 3.16, respectively. Suppose that J is neither without acute angles nor elementary. Then there are odd numbers $k_1, l_1 \in \mathbb{Z}$ such that $\Delta_1(k_1,l_1) \subseteq J$ or $\Delta_2(k_1,l_1) \subseteq J$. We will suppose, without loss of generality, that $\Delta_1(k_1,l_1) \subseteq J$ (so that $J \cap \Delta_2(k_1,l_1) = \emptyset$). Clearly, J contains at least one of the sets $A_1(k_1,l_1)$, $A_2(k_1,l_1)$, $B_1(k_1,l_1)$, $B_2(k_1,l_1)$. We will suppose, without loss of generality, that $B_1(k_1,l_1) \subseteq J$. Obviously, $A_2(k_1,l_1)$ is not a subset of J and, as J is not elementary, neither $A_1(k_1,l_1)$ is a subset of J. Put $J_1 = (J - (B_1(k_1,l_1) \cup \Delta_1(k_1,l_1))) \cup A_1(k_1,l_1)$. Clearly, J_1 is a Jordan curve in $(\mathbb{Z}^2, v_n)$. Now, if J_1 is neither without acute angles nor elementary, there are odd numbers $k_2, l_2 \in \mathbb{Z}$ such that $\Delta_1(k_2,l_2) \subseteq J$ or $\Delta_2(k_2,l_2) \subseteq J$. In the next step, we apply the previous considerations to J_1 and (k_2,l_2) to obtain a new Jordan curve $J_2 = (J_1 - (B_1(k_2,l_2) \cup \Delta_1(k_2,l_2))) \cup A_1(k_2,l_2)$, and so on. After a finite number m of steps we get a Jordan curve $J_m = (J_{m-1} - (B_1(k_m,l_m) \cup \Delta_1(k_m,l_m))) \cup A_1(k_m,l_m)$ which is without acute angles or elementary.

By Theorem 3.15 or 3.16, J_m separates $(\mathbb{Z}^2, v_n)$ into precisely two components. Denote these components by D_m and E_m. As $T_1(k_m,l_m)$ is connected (by Lemma 3.13) and $T_1(k_m,l_m) \cap J_m = \emptyset$, $T_1(k_m,l_m)$ is contained in just one of the components D_m, E_m. We will suppose, without loss of generality, that $T_1(k_m,l_m) \subseteq E_m$. Clearly, we have $J_{m-1} = (J_m - A_1(k_m,l_m)) \cup B_1(k_m,l_m) \cup F_n^{-1}(k_m+1,l_m-1)$. Put $D_{m-1} = D_m \cup F_n^{-1}(k_m,l_m-1) \cup T_1(k_m,l_m)$ and $E_{m-1} = E_m - (T_1(k_m,l_m) \cup F_n^{-1}(\{(k_m-1,l_m),(k_m-1,l_m+1)\}) \cup \Delta_1(k_m,l_m))$. Then $D_{m-1} \cap E_{m-1} = \emptyset$ and $D_{m-1} \cup E_{m-1} = \mathbb{Z}^2 - J_{m-1}$. Evidently, $T_2(k_m,l_m-2) \subseteq D_m$ or $T_4(k_m,l_m-1) \subseteq D_m$. We will suppose, without loss of generality, that $T_2(k_m,l_m-2) \subseteq D_m$. Then $(k_m(n-1)-1,(l_m-1)(n-1)) \in v_n T_2(k_m,l_m-2) \subseteq v_n D_m$ and $(k_m(n-1)-1,(l_m-1)(n-1)) \in F_n^{-1}(k_m,l_m-1)$. As F_n has connected fibres, $F_n^{-1}(k_m,l_m-1)$ is connected and thus $D_m \cup F_n^{-1}(k_m,l_m-1)$ is connected too. Further, $T_1(k_m,l_m)$ is connected by Lemma 3.13 and hence $T_1(k_m,l_m) \cup F_n^{-1}(k_m,l_m-1)$ is connected (because $(k_m(n-1)-1,(l_m-1)(n-1)) \in v_n T_1(k_m,l_m)$). Consequently, D_{m-1} is connected. From Lemma 3.17 it follows that also E_{m-1} is connected.

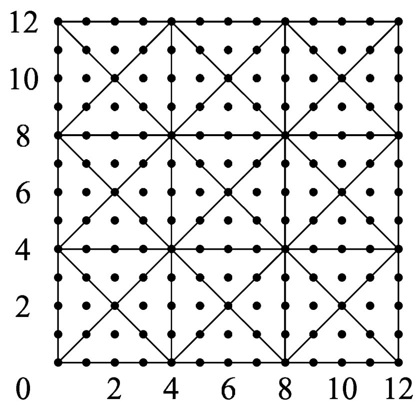
Let $(z_i|\ i < n) \in R_n \times R_n$ be an n-tuple with $z_0 \in D_{m-1}$ and $z_{i_0} \in E_{m-1}$ for some i_0, $0 < i_0 < n$. If $z_0 \in D_m$, then there is $i_1, 0 < i_1 < i_0$, such that $z_{i_1} \in J_m - A_1(k_m,l_m) \subseteq$

J_{m-1} because D_m is closed in $\mathbb{Z}^2 - J_m$ and any n-tuple of $R_n \times R_n$ whose first member does not belong to $A_1(k_m, l_m)$ can meet $A_1(k_m, l_m)$ in the last member only. Let $z_0 \notin D_m$, i.e., let $z_0 \in F_n^{-1}(k_m, l_m - 1) \cup T_1(k_m, l_m)$. Then $z_0 \in \{(x, y) \in \mathbb{Z}^2;\ (k_m - 1)(n - 1) < x < k_m - 1,\ y = l_m - 1\} \cup \{(x, y) \in \mathbb{Z}^2;\ x = k_m(n-1),\ (l_m - 1)(n-1) < y < l_m - 1\}$. Thus, there is i_1, $0 < i_1 < i_0$, such that $z_{i_1} \in \Delta_1(k_m, l_m) \subseteq J_{m-1}$. Consequently, D_{m-1} is closed in $\mathbb{Z}^2 - J_{m-1}$.

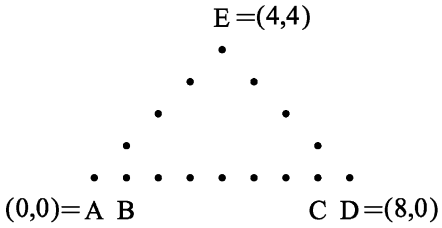
Similarly, let $(z_i |\ i < n) \in R_n \times R_n$ be an n-tuple with $z_0 \in E_{m-1}$ and $z_{i_0} \in D_{m-1}$ for some i_0, $0 < i_0 < n$. If $z_{i_0} \in D_m$, then there is i_1, $0 < i_1 < i_0$, such that $z_{i_1} \in J_m - A_1(k_m, l_m) \subseteq J_{m-1}$ because D_m is closed in $\mathbb{Z}^2 - J_m$ and any n-tuple of $R_n \times R_n$ whose first member does not belong to $A_1(k_m, l_m)$ can meet $A_1(k_m, l_m)$ in the last member only. Let $z_{i_0} \notin D_m$. Then $z_0 \in \{(x, y) \in \mathbb{Z}^2;\ x = k_m(n-1),\ l_m(n-1) < y < (l_m + 1)(n-1) - 1\} \cup \{(x, y) \in \mathbb{Z}^2;\ k_m(n-1) < x < (k_m + 1)(n-1) - 1,\ y = l_m(n-1)\}$. But then there is i_1, $0 < i_1 < i_0$, such that $z_{i_1} \in \Delta_1(k_m, l_m) \subseteq J_{m-1}$. Consequently, D_{m-1} is closed in $\mathbb{Z}^2 - J_{m-1}$.

As both D_{m-1} and E_{m-1} are closed in $\mathbb{Z}^2 - J_{m-1}$, they are components of $\mathbb{Z}^2 - J_{m-1}$, i.e., J_{m-1} separates $(\mathbb{Z}^2, v_n)$ into precisely two components. If $m > 1$, in the next step we apply the previous considerations to J_{m-1} instead of J_m and show that J_{m-2} separates $(\mathbb{Z}^2, v_n)$ into precisely two components D_{m-2} and E_{m-2}. After m steps we get a set J_0 which separates $(\mathbb{Z}^2, v_n)$ into precisely two components D_0 and E_0. As obviously $J_0 = J$, the proof is complete. □

If $n > 2$, then Jordan curves in the n-ary digital plane can turn (at the points $(k(n-1), l(n-1))$ with $k, l \in \mathbb{Z}$ even) at all angles $p\pi/4$, $p = 1, 2, 3$. But this is not true for $n = 2$ because COTS-Jordan curves in the Khalimsky plane cannot turn at the acute angle $\pi/4$ (and at the so-called *mixed points* of $\mathbb{Z}^2$, i.e., the points one coordinate of which is even and the second is odd, they can never turn). So, it can be useful to work with an n-ary digital plane where $n > 2$. Particularly, for $n = 3$ the graph (segment) placed after Definition 3.10 has the following form:



Example 3.20. Consider the following (digital picture of a) triangle:



While in the ternary digital plane the triangle ADE is a Jordan curve, in the Khalimsky plane it is not even a COTS-Jordan curve. In order that this triangle be a COTS-Jordan curve in the Khalimsky plane, we have to delete the points A, B, C and D (and then it will become even a Jordan curve). But this will lead to a deformation of the triangle.

Theorem 3.21. *Let J be a Jordan curve in $(\mathbb{Z}^2, v_2)$ and let D, E be the components of $\mathbb{Z}^2 - J$. Then both $D \cup J$ and $E \cup J$ are connected subsets of $(\mathbb{Z}^2, v_2)$.*

Proof. First, suppose that J is rectangular. Then there is a point $z = (k(n-1)+1, l(n-1))$ with $k, l \in \mathbb{Z}$ even such that $z \in J$. Put $A_1 = \{(k(n-1)+1, (l+1)(n-1)-i);\ 0 \leqslant i < n-1\}$ and $A_2 = \{(k(n-1)+1, (l-1)(n-1)+i);\ 0 \leqslant i < n-1\}$. Clearly, A_1 and A_2 are contained in different components of $\mathbb{Z}^2 - J$. We can suppose, without loss of generality, that $A_1 \subseteq D$ and $A_2 \subseteq E$. As we obviously have $z \in v_n A_1 \cup v_n A_2$, $D \cup \{z\}$ and $E \cup \{z\}$ are connected. Consequently, $D \cup J$ and $E \cup J$ are connected.

Further, suppose that J is not rectangular. Then there is a point $z = (k(n-1), l(n-1))$ with $k, l \in \mathbb{Z}$ odd such that $z \in J$. Put $z_1 = (k(n-1)-1, l(n-1))$ and $z_2 = (k(n-1)+1, l(n-1))$. Clearly, z_1 and z_2 belong to different components of $\mathbb{Z}^2 - J$. We can suppose, without loss of generality, that $z_1 \in D$ and $z_2 \in E$. Obviously, we have $\{z_1, z_2\} \subseteq v_n\{z\}$ and thus $D \cup \{z\}$ and $E \cup \{z\}$ are connected. Consequently, $D \cup J$ and $E \cup J$ are connected. □

Remark 3.22. If J is a COTS-Jordan curve in $(\mathbb{Z}^2, v_2)$ and D, E are the components of $\mathbb{Z}^2 - J$, then $D \cup \{z\}$ and $E \cup \{z\}$ are connected subsets of $(\mathbb{Z}^2, v_2)$ for each point $z \in J$. For a Jordan curve J in $(\mathbb{Z}^2, v_n)$, $n > 2$, this is not true in general (it is true, however, whenever J is rectangular). Indeed, if $n > 2$ and J is an elementary Jordan curve in $(\mathbb{Z}^2, v_n)$ such that, say, $\Delta_1(k,l) \subseteq J$ and $((k-1)(n-1), (k+1)(n-1)) \in J$ where $k, l \in \mathbb{Z}$ are odd numbers, and if D is the finite component of $\mathbb{Z}^2 - J$ and $z = ((k-1)(n-1), (k+1)(n-1))$, then $D \cup \{z\}$ is not connected.

As a concluding remark let us note that the n-ary digital planes with $n > 2$ can be useful also in the cases when we want to work with a topological structure of $\mathbb{Z}^2$ that is more "continuous" than the Alexandroff topology provided by the Khalimsky topology.

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