Random wave propagation in a viscoelastic layered half space

Q. Gao a, J.H. Lin a, W.X. Zhong a, W.P. Howson b,*, F.W. Williams b

a Department of Engineering Mechanics, State Key Laboratory of Structural Analysis of Industrial Equipment, Dalian University of Technology, Dalian 116023, PR China
b Cardiff School of Engineering, Cardiff University, Cardiff CF24 3AA, Wales, UK

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Abstract

An extremely efficient and accurate solution method is presented for the propagation of stationary random waves in a viscoelastic, transversely isotropic and stratified half space. The efficiency and accuracy are obtained by using the pseudo excitation method (PEM) with the precise integration method (PIM). The solid is multi-layered and located above a semi-infinite space. The excitation sources form a random field which is stationary in the time domain. PEM is used to transform the random wave equation into deterministic equations. In the frequency-wavenumber domain, these equations are ordinary differential equations which can be solved precisely by using PIM. The power spectral densities (PSDs) and the variances of the ground responses can then be computed. The paper presents the full theory and gives results for instructive examples. The comparison between the analytical solutions and the numerical results confirms that the algorithm presented in this paper has exceptionally high precision. In addition, the numerical results presented show that: surface waves are very important for the wave propagation problem discussed; the ground displacement PSDs and variances are significant over bigger regions in the spatial domain when surface waves exist; and as the depth of the source increases the ground displacement PSDs decrease and the regions over which they have significant effect become progressively more restricted to low frequencies while becoming more widely distributed in the spatial domain.

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1. Introduction

Wave propagation in stratified or layered media is important in several physical or industrial fields, e.g. propagation of seismic waves in layered earth and ultrasonic wave propagation in fiber reinforced composite materials. Useful applications include exploration for deep-underground water, petroleum or minerals when using surface-explosion or remote-sensing techniques.
The probabilistic properties of wave propagation in layered materials mainly come from random characteristics of the waves or material parameters, or from irregular interfaces or boundaries (Manolis, 2002). The many publications on random interfaces and random material parameters include: use of the indirect boundary element method to simulate wave propagation in two-dimensional irregularly layered elastic media with internal line sources (Vai et al., 1999); investigation of two-dimensional wave scattering by valleys of semi-elliptical cross-section due to incident SH, SV and P waves (Fishman and Ahmad, 1995); application of two-dimensional analysis to examine the effect of a sloping bedrock half-space on the amplification of an anti-plane shear wave (Heymsfield, 2000); conversion of the propagation of planar compressional waves in a medium with vertically stochastic distribution of material properties into the propagation of one-dimensional random waves in order to present a fundamental analysis of wave propagation in a randomly heterogeneous geologic medium (Parra et al., 1999); use of an analytical solution of the stochastic wave equation to model 2D heterogeneous geological environments based on random Fourier–Stieltjes increments (Parra and Zook, 2001); investigation of the effects of the random variations of soil properties on the site amplification of seismic waves (Wang and Hong, 2002) and of harmonic wave propagation in viscoelastic, heterogeneous, random media (Manolis and Shaw, 1996a,b); and simulation of stochastic waves in a non-homogeneous, layered media with an irregular interface (Kiyono et al., 1995).

In contrast, the effect of the waves being random ones has received very little attention. In general, seismic ground motion varies both with time and space, i.e. it is random in both the time and space domains. The present paper covers the case for which the random excitations are stationary in the time domain by presenting a method for computing the propagation of such waves in transversely isotropic, viscoelastic and layered media. The method uses both the pseudo excitation method (PEM) (Lin, 1992; Lin et al., 1993, 1994a,b, 1995a,b,c,d) and the precise integration method (PIM) (Zhong, 1994, 2004; Zhong et al., 1997; Gao et al., 2004).

The stationary random excitation is a function of time and so the PSDs are specified in the frequency domain. The random ground responses are computed in three stages: PEM is used to transform the random excitations into deterministic pseudo harmonic excitations; Fourier transformation is used to transform the wave motion equation into a set of ODEs in the frequency-wavenumber domain; finally the ODEs are solved precisely by using PIM. The method computes the PSDs and the variances of ground responses due to the stationary random excitations. Finally, note that PEM is an efficient and accurate algorithm for linear random problems and also that PIM is an accurate method for solving ODEs, which does not require computation of the eigenvalues of the state equation and which can be used in anisotropic solids. Hence the method presented in this paper is extremely efficient and accurate.

2. The governing equation for stationary random waves

In this paper, the materials are assumed to be transversely isotropic and so the parameters of the soil are independent of direction in the horizontal plane. Therefore, see Fig. 1, the horizontal propagating direction of
the plane waves is taken as the x axis and the z axis points downwards, with z = 0 at the free surface. All variables are independent of the coordinate y. As shown on Fig. 1: the rth layer (r = 1, 2, ..., l) is bounded by the horizontal planes z = z_r−1 and z = z_r; the (l + 1)th layer is the semi-infinite space; and the sources are located at the bottom of layer s (0 < s ≤ l).

The first part of the theory below is in the space–time domain but the final part is in the frequency-wave-number domain. Therefore the space–time domain is indicated by putting a tilde above symbols. Hence if \( \tilde{u}(x, z, t) \), \( \tilde{v}(x, z, t) \) and \( \tilde{w}(x, z, t) \) are the displacements along the inertia coordinates, then the equations of wave motion are

\[
\frac{\partial \tilde{\sigma}_x}{\partial x} + \frac{\partial \tilde{\tau}_{xz}}{\partial z} = \rho \frac{\partial^2 \tilde{u}}{\partial t^2}, \quad \frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\sigma}_z}{\partial z} = \rho \frac{\partial^2 \tilde{v}}{\partial t^2}, \quad \frac{\partial \tilde{\tau}_{zx}}{\partial x} + \frac{\partial \tilde{\sigma}_x}{\partial z} = \rho \frac{\partial^2 \tilde{w}}{\partial t^2}
\]

(1)

in which \( \rho \) is the density and may differ from layer to layer, while the \( \tilde{\sigma} \) and \( \tilde{\tau} \) terms denote direct and shear stresses according to the usual conventions.

The strain–displacement relationships, using \( \bar{e} \) and \( \bar{\gamma} \) to denote direct and shear strains in the conventional way, are

\[
\bar{e}_x = \frac{\partial \tilde{u}}{\partial x}, \quad \bar{e}_y = 0, \quad \bar{e}_z = \frac{\partial \tilde{w}}{\partial z}, \quad \bar{\gamma}_{xy} = \frac{\partial \tilde{v}}{\partial x} \\
\bar{\gamma}_{xz} = \frac{\partial \tilde{w}}{\partial x} + \frac{\partial \tilde{u}}{\partial z}, \quad \bar{\gamma}_{yz} = \frac{\partial \tilde{v}}{\partial z} \tag{2}
\]

Now let the viscoelastic isotropic stress–strain relationships be

\[
P \begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_z \\ \bar{\tau}_{xy} \\ \bar{\tau}_{xz} \\ \bar{\tau}_{yz} \end{bmatrix} = R \times \begin{bmatrix} \bar{e}_x \\ \bar{e}_y \\ \bar{e}_z \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{xz} \\ \bar{\gamma}_{yz} \end{bmatrix}, \quad R = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}
\]

\[
P = \sum_{k=0}^{m} P_k \frac{q^k}{\Gamma^k}, \quad Q = \sum_{k=0}^{n} q_k \frac{m^k}{\Gamma^k} \tag{3}
\]

where \( \lambda \) and \( G \) are the Lamé constants, which may differ from layer to layer, and \( P \) and \( Q \) are differential operators. Finally, \( p_k \) and \( q_k \) are the viscoelastic material constants, so that: \( p_0 = 1, p_1 \neq 0 \) and \( q_1 \neq 0 \) for a Maxwell fluid; \( p_0 = 1, q_0 \neq 0 \) and \( q_1 \neq 0 \) for a Kelvin solid; and \( p_0 = 1, q_0 \neq 0, p_1 \neq 0 \) and \( q_1 \neq 0 \) with \( q_1 > p_1 \cdot q_0 \) corresponds to a three-parameter solid (Shames and Cozzarelli, 1992).

Assume that the sources are discontinuities of the displacements and stresses at the bottom of the nth layer (0 < s ≤ l), i.e. at z = z_s (Kennett, 1983; Aki and Richards, 1980), then

\[
\bar{u}(x, z^+, t) = \tilde{u}(x, z^+, t) + \bar{u}(x, t), \quad \bar{v}(x, z^+, t) = \tilde{v}(x, z^+, t) + \bar{v}(x, t), \quad \bar{w}(x, z^+, t) = \tilde{w}(x, z^+, t) + \bar{w}(x, t) \\
\bar{\sigma}_x(x, z^+, t) = \tilde{\sigma}_x(x, z^+, t) + \bar{\sigma}_x(x, t), \quad \bar{\sigma}_y(x, z^+, t) = \tilde{\sigma}_y(x, z^+, t) + \bar{\sigma}_y(x, t), \quad \bar{\sigma}_z(x, z^+, t) = \tilde{\sigma}_z(x, z^+, t) + \bar{\sigma}_z(x, t) \tag{4}
\]

where \( \bar{u}, \bar{v}, \bar{w}, \bar{\sigma}_x, \bar{\tau}_{xz} \) and \( \bar{\tau}_{yz} \) are the discontinuity functions of the displacements and stresses, and \( z^+ \) and \( z^- \) represent the lower and upper faces of the interface \( z_s \). If \( \bar{u}, \bar{v}, \bar{w}, \bar{\sigma}_x, \bar{\tau}_{xz}, \) and \( \bar{\tau}_{yz} \) are random fields which are stationary in the time domain and have their attenuation functions given in the spatial domain, they can each be written in the form

\[
f(x, t) = A_f(x) f_s(t) \tag{5}
\]

where \( f \) represents \( \bar{u}, \bar{v}, \bar{w}, \bar{\sigma}_x, \bar{\tau}_{xz} \) or \( \bar{\tau}_{yz} \); \( A_f(x) \) is the attenuation function of the excitations in the spatial domain and generally is zero when \( x \) lies outside a bounded region; \( f_s(t) \) is a Fourier–Stieltjes integration of the form

\[
f_s(t) = \int_{-\infty}^{+\infty} \exp(-i\omega t) \, dx_f \tag{6}
\]

\( f_s(t) \) are stationary random components with known PSDs \( S_f(\omega) \) and \( \zeta_f(\omega) \) satisfies Eq. (7)

\[
E[dx_f^*(\omega_1) dx_f(\omega_2)] = S_f(\omega_1)\delta(\omega_1 - \omega_2) \, d\omega_1 d\omega_2 \tag{7}
\]
in which $E[\#]$ denotes the expectation value of $\#$, * denotes complex conjugate and $\omega_1$ and $\omega_2$ denote two angular frequencies.

The free boundary conditions at ground level (i.e. at $z = z_0 = 0$) are

$$\bar{\sigma}_z(x,0,t) = \bar{\tau}_{z}(x,0,t) = 0 \quad (8)$$

and the continuity conditions at the interfaces are

$$\bar{u}, \bar{v}, \bar{w}, \bar{\sigma}_z, \bar{\tau}_z \quad \text{and} \quad \bar{\tau}_c \quad \text{are continuous at} \quad z_i \quad (i = 1, 2, \ldots, s - 1; s + 1, \ldots, l) \quad (9)$$

The source of the earthquake is represented by Eq. (4) and it is also necessary to know the radiative conditions in semi-infinite space, which are presented in the next section.

3. The pseudo excitation method (PEM)

PEM is an efficient and accurate algorithm for linear random vibration problems and has been successfully used for many structures, e.g. buildings, bridges, dams and vehicles. The key idea of PEM is that it introduces a pseudo excitation to transform the stationary or non-stationary random vibration analysis into a series of simpler harmonic or transient dynamic analyses (Lin, 1992; Lin et al., 1993, 1994a,b, 1995a,b,c,d). Crucial advantages of PEM are that it is both efficient and accurate. It is used in the present paper to deal with stationary random wave propagation problems in layered solids. For such wave problems (as in the corresponding random vibration problems) the first step is to construct pseudo excitations, which for the source at $z = z_s$ are

$$\bar{f}(x,t) = \hat{A}_f(x)\sqrt{S_{ff}(\omega)} \exp(-i\omega t) \quad (10)$$

where $\bar{f}$ represents $\bar{u}, \bar{v}, \bar{w}, \bar{\sigma}_z, \bar{\tau}_z$ and $\bar{\tau}_c$. Then the pseudo responses caused are computed, after which it is easy to compute the statistical properties of the random response by using PEM. The remainder of the theory part of this paper gives the necessary details on how this is done, with the pseudo excitations given in the frequency-wavenumber domain.

First, the displacements, strains and stresses are expressed as

$$\{\bar{u}(x,z,t), \bar{v}(x,z,t), \bar{w}(x,z,t)\} = \{\hat{u}(x,z,t), \hat{v}(x,z,t), \hat{w}(x,z,t)\} \exp(-i\omega t),$$

$$\{\hat{\bar{u}}_x(x,z,t), \hat{\bar{v}}_x(x,z,t), \hat{\bar{w}}_x(x,z,t), \hat{\bar{\gamma}}_{yz}(x,z,t), \hat{\bar{\gamma}}_{xy}(x,z,t), \hat{\bar{\gamma}}_{xz}(x,z,t)\} = \{\hat{\bar{\sigma}}_x(x,z,t), \hat{\bar{\sigma}}_y(x,z,t), \hat{\bar{\sigma}}_z(x,z,t), \hat{\bar{\tau}}_{xy}(x,z,t), \hat{\bar{\tau}}_{xz}(x,z,t), \hat{\bar{\tau}}_{yz}(x,z,t)\} \exp(-i\omega t)$$

$$\{\hat{\bar{\sigma}}_z(x,z,t), \hat{\bar{\tau}}_y(x,z,t), \hat{\bar{\tau}}_y(x,z,t), \hat{\bar{\tau}}_z(x,z,t), \hat{\bar{\tau}}_z(x,z,t), \hat{\bar{\tau}}_z(x,z,t)\} \exp(-i\omega t) \quad (11)$$

where $\omega$ is the angular frequency and the left-hand sides of these three equations are, respectively, the displacements, strains and stresses in the frequency domain. Then the Fourier transform and the inverse Fourier transform are defined as, respectively,

$$g(\kappa, z, \omega) = \int_{-\infty}^{+\infty} \hat{g}(x,z,\omega) \exp(-i\kappa x) \, dx, \quad \hat{g}(x,z,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\kappa, z, \omega) \exp(i\kappa x) \, d\kappa \quad (12)$$

where $\hat{g}$ represents any displacement, strain or stress in the frequency domain, e.g. $\bar{u}, \hat{\bar{\tau}}$, etc. The Fourier transform replaces the spatial coordinate $x$ by the wavenumber $\kappa$, so that the coordinate system $(x,z,\omega)$ becomes $(\kappa, z, \omega)$.

Substituting Eq. (11) into Eqs. (1)–(3) and then performing the Fourier transformation decouples the governing equation for isotropic solids into two sets of ODEs

$$\mu K_{22,m} q''_m + \mu (K_{21,m} - K_{12,m}) q'_m - (\mu K_{11,m} - \rho \omega^2 I_m) q_m = 0, \quad m = 1, 2 \quad (13)$$
in which \((#)’ = \partial(#) / \partial z\) represents differentiation with respect to \(z\) and

\[
\begin{align*}
q_1 &= \begin{bmatrix} u \\ w \end{bmatrix}, & K_{22,1} &= \begin{bmatrix} G & 0 \\ 0 & \lambda + 2G \end{bmatrix}, & K_{21,1} &= K_{12,1}^H = i\kappa \begin{bmatrix} 0 & G \\ \bar\lambda & 0 \end{bmatrix}, & K_{11,1} &= \kappa^2 \begin{bmatrix} \bar\lambda + 2G & 0 \\ 0 & G \end{bmatrix} \\
q_2 &= v, & K_{22,2} &= G, & K_{21,2} &= K_{12,2}^H = 0, & K_{11,2} &= \kappa^2 G
\end{align*}
\]

(14)

\[
\mu = \frac{\sum_{k=0}^{n}(-i\omega)^k q_k}{\sum_{k=0}^{m}(-i\omega)^k p_k}
\]

(15)

where \(i = \sqrt{-1}\); the superscript ‘\(H\)’ denotes conjugate transpose; \(I_2\) is the \((2 \times 2)\) unit matrix; \(I_1 = 1\); \(q_1\) and \(q_2\) are the displacement vectors in the frequency-wavenumber domain; and \(\mu\) is the damping that is related to the viscoelastic parameters and frequency. The ODEs with \(m = 1\) correspond to the P-SV wave (i.e. the compressional-vertically polarized shear wave) and the ODE with \(m = 2\) corresponds to the SH wave (i.e. the horizontally polarized shear wave).

This deterministic problem can be solved in many ways, e.g. by using the transfer matrix method and the Newmark integration scheme. However, PIM incorporates the \(m\) viscoelastic parameters and frequency. The ODEs with \(m = 1\) correspond to the P-SV wave (i.e. the compressional-vertically polarized shear wave) and the ODE with \(m = 2\) corresponds to the SH wave (i.e. the horizontally polarized shear wave).

Defining dual vectors

\[
\begin{align*}
p_1 &= \{\tau_x, \sigma_z\}^T, & p_2 &= \tau_{\eta x}
\end{align*}
\]

(16)

which satisfy

\[
p_m = \mu(K_{22,m}q_m + K_{21,m}p_m) \quad (m = 1, 2)
\]

(17)

enables Eq. (13) to be written in the state space as

\[
\begin{align*}
v'_m &= H_m v_m, & H_m &= \begin{bmatrix} A_m & D_m \\ B_m & C_m \end{bmatrix}, & v_m &= \{q_m, p_m\}
\end{align*}
\]

(18)

where

\[
\begin{align*}
A_m &= -K_{22,m}^{-1}K_{21,m}, & B_m &= \mu(K_{11,m} - K_{12,m}K_{22,m}^{-1}K_{21,m}) - \rho\omega^2 I_m, \\
C_m &= -A_m^H, & D_m &= \frac{1}{\mu}K_{22,m}^{-1}
\end{align*}
\]

(19)

Transforming to the boundary and interface conditions of Eqs. (8) and (9) and to the source of Eq. (11) gives, for the frequency-wavenumber domain, the following detailed working.

The free surface condition at ground level (i.e. at \(z = z_0 = 0\)) is

\[
p_m(\kappa, 0, \omega) = 0
\]

(20)

The interface continuity condition is

\[
q_m(\kappa, z, \omega) \quad \text{and} \quad p_m(\kappa, z, \omega) \quad \text{are continuous at} \quad z_i (i = 1, 2, \ldots, s - 1; s + 1, \ldots, l)
\]

(21)

The source is

\[
q_m(\kappa, z^+ \omega) = q_m(\kappa, z^- \omega) + s_m, \quad p_m(\kappa, z^+ \omega) = p_m(\kappa, z^- \omega) + t_m
\]

(22)

in which

\[
\begin{align*}
s_1(\kappa, \omega) &= \{A_{11}(\kappa)\sqrt{S_{uu}(\omega)}, A_{12}(\kappa)\sqrt{S_{vu}(\omega)}\}^T, & s_2(\kappa, \omega) &= A_{12}(\kappa)\sqrt{S_{vu}(\omega)} \\
t_1(\kappa, \omega) &= \{A_{11}(\kappa)\sqrt{S_{\eta u \eta u}(\omega)}, A_{12}(\kappa)\sqrt{S_{\eta u \eta v}(\omega)}\}^T, & t_2(\kappa, \omega) &= A_{12}(\kappa)\sqrt{S_{\eta u \eta v}(\omega)}
\end{align*}
\]

(23)
The wave motion equations should also satisfy the radiative conditions in semi-infinite space. Suppose that the wave motion equation in the semi-infinite space is
\[
\ddot{v}_m = \mathbf{H}_m \dot{v}_m
\]  
(24)
where the overbars are used to distinguish the half space from the layers above it. For the SH wave $\mathbf{H}_1$ is a $2 \times 2$ matrix and for the P-SV wave $\mathbf{H}_2$ is a $4 \times 4$ matrix. Then if $\mathbf{T}_m$ and $\mathbf{A}_m$ (of which the Appendix A gives details) are the eigenvector and eigenvalue matrices of the matrix $\mathbf{H}_m$, let
\[
\mathbf{b}_m = \mathbf{T}_m^{-1} \ddot{v}_m
\]  
(25)
so that the vectors $\mathbf{b}_m$ satisfy
\[
\mathbf{b}_m = \mathbf{A}_m \mathbf{b}_m
\]  
(26)
and the solution for $\mathbf{b}_m$ is
\[
\mathbf{b}_m(z) = \exp[\mathbf{A}_m(z - z_i)] \mathbf{b}_{i,m} \quad z \geq z_i
\]  
(27)
With $\mathbf{A}_m$ and $\mathbf{T}_m$ ordered as in Appendix A, for an SH wave $\mathbf{b}_1(z_i)$ is a two element vector and its first and second components represent, respectively, waves travelling upwards and downwards. Similarly, for P-SV waves, $\mathbf{b}_2(z_i)$ is a four element vector and its first two components correspond to upward waves while its last two components correspond to downward waves. Furthermore, according to Eq. (25), the state vector at $z = z_i^+$ can be written as
\[
\mathbf{v}_{i,m}^+ = \begin{bmatrix} T_{UU,m} & T_{UD,m} \\ T_{DU,m} & T_{DD,m} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{U,m} \\ \mathbf{b}_{D,m} \end{bmatrix}
\]  
(28)
in which $T_{UU,m}$, $T_{UD,m}$, $T_{DU,m}$ and $T_{DD,m}$ are obtained by partitioning $\mathbf{T}_m$. The radiative conditions require that no upward travelling waves exist. Hence $\mathbf{b}_{U,1}(z_i) = \mathbf{0}$ for SH waves and $\mathbf{b}_{U,2}(z_i) = \mathbf{0}$ for P-SV waves, so that, by using Eq. (18),
\[
\mathbf{q}_{i,m}^+ = T_{UD,m} \mathbf{b}_{D,m}, \quad \mathbf{p}_{i,m}^+ = T_{DD,m} \mathbf{b}_{D,m}
\]  
(29)
So far, the derivation has obtained the governing frequency-wavenumber domain equation for random wave propagation in layered solids under pseudo excitations. Next, let $\mathbf{q}(\kappa, z, \omega)$ be the solution of Eq. (18) for the boundary conditions of Eqs. (20), (21) and (29) and for the source condition of Eq. (23). Then the pseudo responses can be obtained by using the inverse Fourier transformation
\[
\tilde{\mathbf{q}}(x, z, t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{q}(\kappa, z, \omega) \exp(i\kappa x) \, d\kappa \exp(-i\omega t)
\]  
(30)
In general, this inverse Fourier transformation cannot be performed analytically. Therefore in this paper the discrete Fourier transformation is used to approximate Eq. (30). Hence if the integration range is from $\kappa_0$ to $\kappa_f$ and the integration step is $\Delta \kappa$, Eq. (30) can be written as
\[
\tilde{\mathbf{q}}(x, z, t, \omega) \approx \frac{\Delta \kappa \exp(-i\omega t)}{2\pi} \sum_{n=\kappa_0/\Delta \kappa}^{\kappa_f/\Delta \kappa} \mathbf{q}(n\Delta \kappa, z, \omega) \exp(in\Delta \kappa x)
\]  
(31)
Then the PSDs and variances of the random ground displacement responses can be computed from Eqs. (32) which were obtained by PEM.
\[
\begin{align*}
S_{uu}(x, 0, \omega) &= \tilde{u}(x, 0, \omega) \times \text{conj}[\tilde{u}(x, 0, \omega)] \\
S_{uv}(x, 0, \omega) &= \tilde{v}(x, 0, \omega) \times \text{conj}[\tilde{v}(x, 0, \omega)] \\
S_{ww}(x, 0, \omega) &= \tilde{w}(x, 0, \omega) \times \text{conj}[\tilde{w}(x, 0, \omega)] \\
\sigma_u^2 &= \int_0^\infty S_{uu}(x, 0, \omega) \, d\omega, \quad \sigma_v^2 = \int_0^\infty S_{uv}(x, 0, \omega) \, d\omega, \quad \sigma_w^2 = \int_0^\infty S_{ww}(x, 0, \omega) \, d\omega
\end{align*}
\]  
(32)
where \( \text{conj} \) denotes complex conjugate; \( S_{uu}, S_{vv} \) and \( S_{ww} \) are the PSDs of the ground displacement; and \( \sigma_u^2, \sigma_v^2 \) and \( \sigma_w^2 \) are the variances.

For isotropic solids, Kennett (1983) proposed an algorithm for solving the ODEs which is based on the technique of wave splitting and reflection and transmission matrices. However, the application of this method to anisotropic solids is not easy, because it needs the eigenvalues of the ODEs and because properties of the eigenvalues of the state equation are very complicated they cannot be obtained analytically. Therefore PIM is used to solve the ODEs in the present paper, because it does not require the eigenvalues and so it can be used for both isotropic and anisotropic solids. The wave motion equations for P-SV and SH waves are of similar mathematical form and so can be solved identically when using PIM. Therefore the subscript \( m \) is omitted in the next section.

4. The precise integration method for random wave motion

By using PEM, the governing equation of random wave propagation in a layered solid has been transformed into deterministic ODEs in the frequency-wavenumber domain. In this section, PIM is used to solve these ODEs. PIM is an accurate method for solving ODEs with two-point boundary value conditions or initial value conditions and the precision is only dependent on the precision of the computer used (Zhong, 2004).

4.1. The interval formulation and matrix differential equations

Select an interval \([z_a, z_b]\) within an arbitrary layer, the displacements and stresses are continuous in this interval. Let \( q_a \) be the displacement vector at \( z_a \), \( p_b \) be the force vector at \( z_b \). Clearly, if \( q_a \) and \( p_b \) have been specified, the solution of \( q \) and \( p \) in the interval \([z_a, z_b]\) is well defined.

For linear systems, the following relations stand:

\[
\begin{bmatrix}
q_a \\
p_b
\end{bmatrix} =
\begin{bmatrix}
F(z_b, z_a) & -G(z_b, z_a) \\
Q(z_b, z_a) & E(z_b, z_a)
\end{bmatrix}
\begin{bmatrix}
q_b \\
p_a
\end{bmatrix}
\tag{33}
\]

in which \( F, Q, G \) and \( E \) are complex matrices to be determined. Since the material properties in the interval are constant, the matrices \( F, Q, G \) and \( E \) only depend on the thickness, i.e. \( \tau = z_b - z_a \). Assume now that \( q_b \) and \( p_b \) are given initial values and let the position of \( a \) vary. Differentiating Eq. \((33)\) with respect to \( z_a \) and noting that \( \frac{d}{dz_a} = -\frac{d}{d\tau} \), the following equations are derived

\[
\begin{align*}
q_a' &= -F'(\tau)q_b + G'(\tau)p_a - G(\tau)p_b' \\
0 &= -Q'(\tau)q_b - E'(\tau)p_a + E(\tau)p_b'
\end{align*}
\tag{34}
\]

From Eq. \((18)\), the dual equations can be written as

\[
\begin{align*}
q_a' &= A_q q_a + D_p a, \\
p_a' &= B_q q_a + C_p a
\end{align*}
\tag{35}
\]

Using Eqs. \((33)-(35)\), one obtains

\[
\begin{align*}
[(A + GB)F + F']q_b + [D - (A + GB)G - G' + GC]p_a &= 0 \\
(-Q' + EBF)q_b + (EC - E' - EBG)p_a &= 0
\end{align*}
\tag{36}
\]

Noting that the vectors \( q_b \) and \( p_a \) are mutually independent leads to

\[
\begin{align*}
F' &= -(A + GB)F, \\
G' &= D - (A + GB)G + GC \\
Q' &= EBF, \\
E' &= E(C - BG)
\end{align*}
\tag{37}
\]

Now let \( z_a \) approach \( z_b \). The boundary conditions can then be derived as

\[
G(\tau) = Q(\tau) = 0, F(\tau) = E(\tau) = 1 \text{ when } \tau \to 0
\tag{38}
\]
4.2. The combination of adjacent intervals

Consider two adjacent intervals \([z_{b}^{a}, z_{b}^{c}]\) and \([z_{c}^{a}, z_{c}^{c}]\). Assume the displacements and stresses are continuous in interval \([z_{b}^{a}, z_{b}^{c}]\) and \([z_{c}^{a}, z_{c}^{c}]\), but discontinuous at \(z_{b}\) and \(z_{c}\). Applying Eq. (33) gives

\[
q_{a}^{+} = F_{1}q_{b}^{a} - G_{1}p_{a}^{a}, \quad p_{b}^{-} = Q_{1}q_{b}^{a} + E_{1}p_{a}^{+}
\]

\[
q_{b}^{+} = F_{2}q_{c}^{b} - G_{2}p_{b}^{b}, \quad p_{c}^{-} = Q_{2}q_{c}^{b} + E_{2}p_{b}^{+}
\]

(39)

(40)

If the discontinuous value of the displacements and stresses at \(z_{b}\) and \(z_{c}\) are

\[
q_{b}^{+} = q_{b}^{c} + s_{b}, \quad p_{b}^{-} = p_{b}^{c} + t_{b}, \quad q_{c}^{+} = q_{c}^{c} + s_{c}, \quad p_{c}^{-} = p_{c}^{c} + t_{c}.
\]

(41)

then we have

\[
q_{a}^{+} = F_{1}q_{b}^{a} - G_{1}p_{a}^{a} + q_{1}, \quad p_{b}^{-} = Q_{1}q_{b}^{a} + E_{1}p_{a}^{+} + p_{1}
\]

\[
q_{b}^{+} = F_{2}q_{c}^{b} - G_{2}p_{b}^{b} + q_{2}, \quad p_{c}^{-} = Q_{2}q_{c}^{b} + E_{2}p_{b}^{+} + p_{2}
\]

\[
q_{c}^{+} = F_{2}q_{c}^{c} - G_{2}p_{c}^{c} + q_{12}, \quad p_{c}^{-} = Q_{2}q_{c}^{c} + E_{2}p_{c}^{+} + p_{12}
\]

(42)

(43)

The intervals \([z_{a}^{a}, z_{b}^{c}]\) and \([z_{b}^{b}, z_{c}^{c}]\) can be merged into a single interval \([z_{a}^{a}, z_{c}^{c}]\) with

\[
q_{a}^{+} = F_{12}q_{b}^{a} - G_{12}p_{a}^{a} + q_{12}, \quad p_{c}^{-} = Q_{12}q_{c}^{a} + E_{12}p_{a}^{+} + p_{12}
\]

(44)

Comparing Eqs. (42) and (43) with Eq. (44) gives

\[
F_{12} = F_{1}(I + G_{2}Q_{1})^{-1}F_{2}, \quad Q_{12} = Q_{2} + E_{2}Q_{1}(I + G_{2}Q_{1})^{-1}F_{2}
\]

\[
E_{12} = E_{2}(I + Q_{1}G_{2})^{-1}E_{1}, \quad G_{12} = G_{1} + F_{1}G_{2}(I + Q_{1}G_{2})^{-1}E_{1}
\]

\[
q_{12} = q_{1} + F_{1}(I + G_{2}Q_{1})^{-1}(q_{2} - G_{2}p_{1})
\]

\[
p_{12} = p_{2} + E_{2}(I + Q_{1}G_{2})^{-1}(Q_{1}q_{2} + p_{1})
\]

(45)

(46)

The above equations are derived for the combination of adjacent intervals. They are important for the solution of eigenvalue problems and ODEs.

4.3. Initialization of interval matrices

Eq. (45) shows how the interval matrices \(F, Q, G\) and \(E\) are merged. So far, however, no interval matrices have been given; i.e. only the system matrices \(A, B, C\) and \(D\) are available to us. Next, we will derive a set of \(F, Q, G\) and \(E\) from \(A, B, C\) and \(D\).

Assume the thickness of the \(i\)th layer is \(h_{i} = z_{i} - z_{i-1}\). It is firstly divided into many \((2^{N})\) sub-layers with equal thickness \(h_{i} = h_{i}/2^{N_{i}}\). Secondly, each sub-layer is further divided into \(2^{N_{0}}\) (\(N_{0} = 20\) in this paper) mini-layers with equal thickness \(\tau\) (Zhong et al., 1997; Gao et al., 2004), thus

\[
\tau = \tilde{h}_{i}/2^{N_{0}}
\]

(47)

The \(\tau\) of such intervals is extremely small and the associated matrices \(F, Q, G\) and \(E\) can be computed in terms of a Taylor series expansion

\[
Q(\tau) = \theta_{1}\tau + \theta_{2}\tau^{2} + \theta_{3}\tau^{3} + \theta_{4}\tau^{4}, \quad G(\tau) = \gamma_{1}\tau + \gamma_{2}\tau^{2} + \gamma_{3}\tau^{3} + \gamma_{4}\tau^{4}
\]

\[
\bar{F}(\tau) = \phi_{1}\tau + \phi_{2}\tau^{2} + \phi_{3}\tau^{3} + \phi_{4}\tau^{4}, \quad \bar{F}(\tau) = \bar{F}(\tau)
\]

\[
\bar{E}(\tau) = \psi_{1}\tau + \psi_{2}\tau^{2} + \psi_{3}\tau^{3} + \psi_{4}\tau^{4}, \quad \bar{E}(\tau) = \bar{E}(\tau)
\]

(48)
Substituting the above equations into Eq. (45) gives

\[ \begin{align*}
\theta_1 &= B, \quad \gamma_1 = D, \quad \phi_1 = -A, \quad \psi_1 = C \\
\theta_2 &= (\psi_1 B + B \phi_1)/2, \quad \gamma_2 = (A \gamma_1 - \gamma_1 C)/2 \\
\phi_2 &= -(A \phi_1 + \gamma_1 B)/2, \quad \psi_2 = -(B \gamma_1 - \psi_1 C)/2 \\
\theta_3 &= (\psi_2 B + B \phi_2 + \psi_1 B \phi_1)/3, \quad \gamma_3 = (A \gamma_2 - \gamma_2 C + \gamma_1 B \gamma_1)/3 \\
\phi_3 &= -(A \phi_2 + \gamma_2 B + \gamma_1 B \phi_1)/3, \quad \psi_3 = -(B \gamma_2 + \psi_1 B \gamma_1 - \psi_2 C)/3 \\
\theta_4 &= (\psi_3 B + B \phi_3 + \psi_2 B \phi_2 + \psi_1 B \phi_2)/4 \\
\gamma_4 &= (A \gamma_3 - \gamma_3 C + \gamma_2 B \gamma_1 + \gamma_1 B \gamma_2)/4 \\
\phi_4 &= -(A \phi_3 + \gamma_2 B + \gamma_1 B \phi_1 + \gamma_1 B \phi_2)/4 \\
\psi_4 &= -(B \gamma_3 + \psi_1 B \gamma_2 + \psi_2 B \gamma_1 - \psi_3 C)/4
\end{align*} \]  

(49)

(50)

(51)

(52)

Now \( F = I + \tilde{F} \) and \( E = I + \tilde{E} \), in which \( \tilde{F} \) and \( \tilde{E} \) are very small because \( \tau \) is very small. It is important to note that \( \tilde{F} \) and \( \tilde{E} \) must be computed and stored independently to avoid losing effective digits. Hence it is necessary to replace Eq. (45) in the computation by

\[ \begin{align*}
G_{12} &= G + (I + \tilde{F})G(I + QG)^{-1}(I + \tilde{E}), \quad Q_{12} = Q + (I + \tilde{E})Q(I + QG)^{-1}(I + \tilde{F}) \\
\bar{E}_{12} &= (\tilde{E} - QG/2)(I + QG)^{-1} + (I + QG)^{-1}(\tilde{E} - QG/2) + \tilde{F}(I + QG)^{-1}\tilde{F} \\
\bar{E}_{12} &= (E - QG/2)(I + QG)^{-1} + (I + QG)^{-1}(E - QG/2) + \tilde{E}(I + QG)^{-1}\tilde{E}
\end{align*} \]  

(53)

4.4. The \( 2^N \) algorithm

Once \( \tilde{F}(\tau), \tilde{E}(\tau), Q(\tau) \) and \( G(\tau) \) in the interval \( \tau \) have been obtained, Eqs. (53) can be used to obtain \( F(h), \tilde{E}(h), Q(h) \) and \( G(h) \). As all intervals have equal thickness, \( F_1 = F_2, Q_1 = Q_2, \) and so on. Each pass through such interval combinations reduces the number of intervals by a half. When all \( N_0 \) passes have been completed, \( F(h), \tilde{E}(h), Q(h) \) and \( G(h) \) for a typical sub-layer are obtained. The process is summarized in Fig. 2. Then compute \( Q(h), G(h), F(h) \) and \( E(h) \) from all sub-layers, as summarized in Fig. 3. The combination of layer matrices into global matrices is similar to the above and is summarized in Fig. 4.

When the computation process of Fig. 4 is finished, the interval matrices and vectors \( Q_{12}, G_{12}, F_{12}, E_{12}, q_{12} \) and \( p_{12} \) have been obtained. \( z_0^n \) corresponds to end \( a \) and \( z_1^n \) corresponds to end \( b \), so we have the following equation

\[ \begin{align*}
q_0^+ &= F_{12}q_i^+ + G_{12}p_i^+ + q_{12}, \quad p_i^+ = Q_{12}q_i^+ + E_{12}p_0^+ + p_{12} \\
q_0^+ &= F_{12}q_i^+ + q_{12}, \quad p_i^+ = Q_{12}q_i^+ + p_{12}, \quad p_i^+ = T_{DD}T_{UD}^{-1}q_i^+
\end{align*} \]  

(54)

(55)

Eq. (55) gives

\[ q_0^+ = F_{12}T_{UD}(T_{DD} - Q_{12}T_{UD})^{-1}p_{12} + q_{12} \]  

(56)

\[ \text{Compute } \tilde{F}(\tau), \tilde{E}(\tau), Q(\tau), G(\tau) \text{ from Eqs. (48)-(52);} \]

\[ \text{for (itera=0; itera}<N_0; \text{itera++}) { \}
\]

\[ \text{Execute combination formulas of Eq. (53);} \]

\[ Q = Q_{12}, \quad G = G_{12}, \quad F = F_{12}, \quad E = E_{12} \]

\[ } \]

\[ Q(h) = Q_{12}, \quad G(h) = G_{12}, \quad F(h) = I + \tilde{F}_{12}, \quad E(h) = I + \tilde{E}_{12} \]

Fig. 2. Determination of sub-layer matrices by interval combination.
This enables the pseudo ground response to be obtained by using Eq. (31), after which the ground displacement PSDs and variances can be obtained using Eq. (32).
5. Numerical examples

In the following examples, the source is assumed to be an interface crack of the form:

\[
\begin{align*}
\bar{u}(x, t, z^+) &= u(x, t, z^+) + \tilde{A}(x)\tilde{u}_0(t), & \bar{v}(x, t, z^+) &= v(x, t, z^+) + \tilde{A}(x)\tilde{v}_0(t) \\
\bar{w}(x, t, z^+) &= w(x, t, z^+) + \tilde{A}(x)\tilde{w}_0(t), & \bar{\sigma}_z(x, t, z^+) &= \sigma_z(x, t, z^+) + \tilde{A}(x)\tilde{\sigma}_{zz}(t) \\
\bar{\tau}_{xz}(x, t, z^+) &= \tau_{xz}(x, t, z^+) + \tilde{A}(x)\bar{\tau}_{xz}(t), & \bar{\tau}_{zx}(x, t, z^+) &= \tau_{zx}(x, t, z^+) + \tilde{A}(x)\bar{\tau}_{zx}(t)
\end{align*}
\]

in which \(\tilde{A}(x)\) is the attenuation function shared by all of the excitations (in the spatial domain), and so has the profile shown in Fig. 5, and \(\bar{u}_0(t), \bar{v}_0(t), \bar{w}_0(t)\), \(\bar{\sigma}_{zz}(t)\), \(\bar{\tau}_{xz}(t)\) and \(\bar{\tau}_{zx}(t)\) are stationary processes with known PSDs \(S_{\bar{u}\bar{u}}(\omega)\), \(S_{\bar{v}\bar{v}}(\omega)\), \(S_{\bar{w}\bar{w}}(\omega)\), \(S_{\bar{\sigma}_{zz}\bar{\sigma}_{zz}}(\omega)\), \(S_{\bar{\tau}_{xz}\bar{\tau}_{xz}}(\omega)\) and \(S_{\bar{\tau}_{zx}\bar{\tau}_{zx}}(\omega)\).

Example 1. This example concerns SH random waves for a layered half space which consists of a viscoelastic layer overlaying a semi-infinite space. The shear modulus, density and depth of the viscoelastic layer are \(G_1\), \(\rho_1\) and \(L\), while the shear modulus and density of the semi-infinite space are \(G_2\) and \(\rho_2\). The viscoelastic parameters are \(\rho_0 = 1.0, \quad \rho_1 = 0.05, \quad q_0 = 1\) and \(q_1 = 0.1\). The source is the interface crack at \(z = L\) and the PSDs of the stationary random excitation are \(S_{\bar{u}_0} = 1.0\) (m²/s) and \(S_{\bar{\tau}_{xz}\bar{\tau}_{xz}} = 0.0\) (N²/m⁴). For this simple case, the pseudo response of the ground is obtained analytically in Appendix B as

\[
\begin{align*}
\bar{v}(x, 0, \omega) &= -\frac{\beta_2 G_2}{\pi} S_{\bar{v}\bar{v}}(\omega) \int_{-\infty}^{+\infty} A_i(\kappa) \exp(-\beta_1 L) \exp[i(\kappa x - \omega t)] d\kappa \\
\beta_1 &= \sqrt{\kappa^2 - \frac{\rho_1 \omega^2}{\mu G_1}}, \quad \beta_2 = \sqrt{\kappa^2 - \frac{\rho_2 \omega^2}{\mu G_2}}
\end{align*}
\]

Here \(\beta_1\) and \(\beta_2\) are complex and the root taken is the one with a positive real part. The damping coefficient \(\mu\) is computed by Eq. (15). Eq. (58) is now used to verify the precision of the PIM.

Case 1. For this case: \(G_1 = 0.35 \times 10^{11}\) (N/m²); \(\rho_1 = 3.0 \times 10^3\) (kg/m³); \(G_2 = 0.75 \times 10^{11}\) (N/m²); \(\rho_2 = 3.0 \times 10^3\) (kg/m³); \(L = 1.9 \times 10^4\) (m); and the discrete Fourier transformation used \(\kappa_0 = -0.005\) (m⁻¹), \(\kappa_f = 0.005\) (m⁻¹) and \(\Delta \kappa = 2.0 \times 10^{-6}\) (m⁻¹) as the integral limits and step. Fig. 6(a) and (c) gives the ground response and the variance computed by the present method and Fig. 7(a) shows the relative error of the variance when compared to the analytical formula of Eq. (58). Clearly, the relative error is of the order of \(10^{-12}\), which confirms the high precision of the method presented in this paper.

Case 2. For this case: \(G_1 = 0.35 \times 10^{11}\) (N/m²); \(\rho_1 = 3.0 \times 10^3\) (kg/m³); \(G_2 = 0.15 \times 10^{11}\) (N/m²); \(\rho_2 = 3.0 \times 10^3\) (kg/m³); \(L = 1.9 \times 10^4\) (m) and \(\kappa_0 = -0.005\) (m⁻¹), \(\kappa_f = 0.005\) (m⁻¹) and \(\Delta \kappa = 2.0 \times 10^{-6}\) (m⁻¹). Fig. 6(b) and (d) and Fig. 7(b) correspond to, respectively, Fig. 6(a) and (c) and Fig. 7(a) and again yield an error of the order of \(10^{-12}\), i.e. the high precision of the present method is again verified.

In case 1 \(\frac{G_1}{\rho_1} < \frac{G_2}{\rho_2}\) and so a Love surface wave exists for this SH wave problem, whereas in case 2 \(\frac{G_1}{\rho_1} > \frac{G_2}{\rho_2}\) and so there is no Love surface wave. Fig. 6(a) and (b) shows that the properties of the ground response are substantially different for these two cases. For the case in which Love surface waves exist, see Fig. 6(a), the ground response is

Fig. 5. The attenuation function of the excitations.
displacement PSD varies in a more complicated way than when they do not exist, with the effect in the spatial domain being larger in the low frequency range. This shows that surface waves are very important when random waves propagate in layered solids.

**Example 2.** This example concerns both SH and P-SV random waves for a layered half space which consists of four layers, of which the fourth layer is a semi-infinite space. The layer parameters are given in Table 1 and

![Fig. 6. SH stationary random wave responses for cases 1 and 2 of example 1: (a, b) PSD of \( v \) and (c, d) variance of \( v \).](image-url)

![Fig. 7. The relative error of the variance obtained by PIM.](image-url)

**Table 1**

<table>
<thead>
<tr>
<th>Layer</th>
<th>( \lambda ) ((10^{10} \text{ N/m}^2))</th>
<th>( G ) ((10^{10} \text{ N/m}^2))</th>
<th>( \rho ) ((10^3 \text{ kg/m}^3))</th>
<th>Thickness ((10^4 \text{ m}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.3</td>
<td>3.5</td>
<td>2.74</td>
<td>1.9</td>
</tr>
<tr>
<td>2</td>
<td>4.4</td>
<td>4.3</td>
<td>3.00</td>
<td>1.9</td>
</tr>
<tr>
<td>3</td>
<td>8.0</td>
<td>7.2</td>
<td>3.32</td>
<td>1.2</td>
</tr>
<tr>
<td>4</td>
<td>8.2</td>
<td>7.0</td>
<td>3.34</td>
<td>Semi-infinite</td>
</tr>
</tbody>
</table>
were chosen from the Gutenberg model (Aki and Richards, 1980). The viscoelastic parameters for all layers are $p_0 = 1.0$, $p_1 = 0.05$, $q_0 = 1$ and $q_1 = 0.1$ while the discrete Fourier transformation used $\kappa_0 = -0.005$ (m$^{-1}$), $\kappa_f = 0.005$ (m$^{-1}$) and $\Delta \kappa = 2.0 \times 10^{-6}$ (m$^{-1}$) as the integral limits and step. The PSDs and variances of the ground displacement were computed for the following three cases, for each of which three alternative source positions were considered, namely the source was at the bottom of the first, second or third layer.

Case 1. This case is for an SH random wave and the source is a stationary excitation induced by an interface crack in the $y$ direction with PSDs of $S_{\dot{v}y} = 1.0$ (m$^2$ s) and $S_{\dot{r}_x \dot{r}_y} = 0.0$ (N$^2$ s/m$^4$). Fig. 8 gives the ground displacement response PSDs and variances for the $y$ direction. Thus (a), (b) and (c) are the PSDs when the source is at the bottom of the first, second and third layers, respectively, while (d) gives the variances for these three cases. Conclusions which may be drawn from Fig. 8 include the following.

Because the attenuation function is symmetric about the $x$ axis, the ground displacement PSDs and variances of Fig. 8 are also symmetric about the $x$ axis and they have their maximum values at $x = 0$. The results show that as the depth of the source increases, the PSDs and variances decrease substantially, but are significant for a much wider range in the spatial domain. Moreover the PSDs are high only in the lower frequency range, particularly as the depth of the source increases, i.e. the high frequency components of the response decay faster for deeper sources.

Case 2. This case is for a P-SV random wave and the stationary source is an interface crack in the $x$ direction with PSDs of $S_{\dot{u}x} = 1.0$ (m$^2$ s), $S_{\dot{w}x} = S_{\dot{\sigma}_x \dot{r}_z} = S_{\dot{r}_x \dot{r}_z} = 0$. Fig. 9 gives, as (a)–(f), the random ground displacement PSDs and variances in the $x$ and $z$ directions for all three alternative locations of the source. Fig. 9 also gives the corresponding variances, see (g) and (h). The main conclusions are as follows.

The PSDs and variances are again all symmetric about the $x$ axis, but while the displacement variances for the $x$ direction have maximum values at $x = 0$, those for the $z$ direction have zeros (i.e. minima) there while their symmetrically located maxima are close to $x = 0$. Similarly to case 1, as the depth of the source increases the ground displacement PSDs for the $x$ and $z$ directions: become smaller; become more highly concentrated in the low frequency range; and become increasingly significant over a wider range of the spatial domain. Note that the maxima for the $z$ direction PSDs become more widely separated as the depth of the source increases.

Fig. 8. Stationary random response details for SH wave in case 1 of example 2, giving the PSDs of $v$ when the source is at the bottom of: (a) layer 1; (b) layer 2; (c) layer 3; (d) gives the variances of $v$ for these three cases.
and that these PSDs are large for a much wider range in the spatial domain than are either the PSDs for the \( x \) direction or the PSDs of case 1.

**Case 3.** This case is for a P-SV random wave and the stationary source is an interface crack in the \( z \) direction with PSDs of \( S_{uuz} = 1.0 \) (m\(^2\) s) and \( S_{wuz} = S_{\sigma_{uz}\sigma_{uz}} = S_{\tau_{uz}\tau_{uz}} = 0.0 \). Fig. 10 is identical to Fig. 9 except for this change of the source. The main conclusions are identical to those for case 2 except that the conclusions drawn in case 2 for the \( z(x) \) direction PSDs now apply instead to the \( x(z) \) direction PSDs.

**Fig. 9.** Stationary random response details for P-SV wave in case 2 of example 2, giving the PSDs of \( u \) and \( w \) when the source is at the bottom of: (a, b) layer 1; (c, d) layer 2; (e, f) layer 3. (g, h) gives the variances of \( u \) and \( w \) for these three cases.
6. Conclusions

This paper shows that the pseudo excitation method (PEM) in combination with the precise integration method (PIM) yields an effective approach to analysing the propagation of stationary random waves in layered half space solids with an inter-layer excitation. Because PEM is efficient and accurate for linear random problems and PIM is accurate for solving ODEs, combining these two methods gives very efficient and accurate solutions. The first example presented was simple enough to have an analytical solution, comparison with which confirmed that the algorithm presented in this paper has exceptionally high precision. The numerical

Fig. 10. Stationary random response details for P-SV wave in case 3 of example 2, giving the PSDs of $u$ and $w$ when the source is at the bottom of: (a, b) layer 1; (c, d) layer 2; (e, f) layer 3; (g, h) gives the variances of $u$ and $w$ for these three cases.

6. Conclusions

This paper shows that the pseudo excitation method (PEM) in combination with the precise integration method (PIM) yields an effective approach to analysing the propagation of stationary random waves in layered half space solids with an inter-layer excitation. Because PEM is efficient and accurate for linear random problems and PIM is accurate for solving ODEs, combining these two methods gives very efficient and accurate solutions. The first example presented was simple enough to have an analytical solution, comparison with which confirmed that the algorithm presented in this paper has exceptionally high precision. The numerical
examples also show that: surface waves are very important for the wave propagation; the ground displacement PSDs and variances are significant over bigger regions in the spatial domain when surface waves exist; and that as the depth of the source increases the ground displacement PSDs decrease and the regions over which they have significant effect become progressively more restricted to low frequencies while becoming more widely distributed in the spatial domain. Note that the dominance of low frequencies is related to the damping used in the examples being frequency dependent and so being smaller at low frequencies.

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Appendix A. Eigensolution of system matrix \( H_m \)

For the SH wave

\[
H_1 = \begin{bmatrix}
0 & \frac{1}{\mu G} \\
\kappa^2 \mu G - \rho \omega^2 & 0
\end{bmatrix}
\]  

(A1)

(1) When \( \kappa^2 \mu G - \rho \omega^2 \neq 0 \)

\[
\Lambda_1 = \text{diag}(\alpha, -\alpha), \quad T_1 = \begin{bmatrix} 1 & 1 \
\alpha G & -\alpha G \end{bmatrix}, \quad \alpha = \sqrt{\kappa^2 - \frac{\rho \omega^2}{\mu G}}
\]  

(A2)

in which \( \alpha \) takes the root with the positive real part.

(2) When \( \kappa^2 \mu G - \rho \omega^2 = 0 \)

\[
\Lambda_1 = \begin{bmatrix} 0 & 0 \\
1 & 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 1 \\
G & 0 \end{bmatrix}
\]  

(A3)

For the P-SV wave

\[
H_2 = \begin{bmatrix}
0 & -i\kappa & \frac{1}{\mu G} & 0 \\
-i\kappa & \frac{\lambda}{\lambda + 2G} & 0 & \frac{1}{\mu(\lambda + 2G)} \\
\kappa^2 \mu \left( \frac{4\lambda G + 4G^2}{\lambda + 2G} \right) - \rho \omega^2 & 0 & 0 & -i\kappa \frac{\lambda}{\lambda + 2G} \\
0 & -\rho \omega^2 & -i\kappa & 0
\end{bmatrix}
\]  

(A4)

(1) When \( \kappa = 0, \omega = 0 \)

\[
\Lambda_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
G & 0 & 0 & 0 \\
0 & \lambda + 2G & 0 & 0 \end{bmatrix}
\]  

(A5)
Appendix B. Derivation of Eq. (58)

The wave motion equations for the finite viscoelastic layer are

\[
G_1 \left( q_0 + q_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 \tilde{v}_1(x,z,t)}{\partial x^2} + \frac{\partial^2 \tilde{v}_1(x,z,t)}{\partial z^2} = \rho_1 \left( p_0 + p_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 \tilde{v}_1(x,z,t)}{\partial t^2}
\]

and the wave motion equations for the semi-infinite space are

\[
G_2 \left( q_0 + q_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 \tilde{v}_2(x,z,t)}{\partial x^2} + \frac{\partial^2 \tilde{v}_2(x,z,t)}{\partial z^2} = \rho_2 \left( p_0 + p_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 \tilde{v}_2(x,z,t)}{\partial t^2}
\]

The pseudo excitations at \( z = L \) are

\[
\tilde{v}_2(x,L,t) - \tilde{v}_1(x,L,t) = \hat{A}_v(x) \sqrt{S_\text{in}(\omega)} \exp(-i\omega t), \quad \tilde{\tau}_{1z}(x,L,t) = \tilde{\tau}_{1z}(x,L,t)
\]

in which \( \hat{A}_v(x) \) is a deterministic function and \( \sqrt{S_\text{in}(\omega)} \) is the PSD of the stationary random field \( \tilde{v} \). The free boundary condition at \( z = 0 \) is

\[
\tilde{\tau}_{1z}(x,0,t) = 0
\]

By using Fourier transformation, the wave motion equations, boundary conditions and pseudo excitations can be transformed into the frequency-wavenumber domain to give

\[
v''_1(k,z,\omega) = \left( \kappa^2 - \frac{\rho_1 \omega^2}{\mu G_1} \right) v_1(k,z,\omega)
\]
\[ v_2'(\kappa, z, \omega) = \left( \kappa^2 - \frac{\rho_2 \omega^2}{\mu G_2} \right) v_2(\kappa, z, \omega) \quad (B6) \]

\[ \tau_{1z}(\kappa, 0, \omega) = 0 \quad (B7) \]

\[ v_2(\kappa, L, \omega) - v_1(\kappa, L, \omega) = A_\varepsilon(\kappa) \sqrt{S_{zz}(\omega)}, \quad \tau_{2z}(\kappa, L, \omega) - \tau_{1z}(\kappa, L, \omega) = 0 \quad (B8) \]

The solution which satisfies both Eq. (B5) and the free boundary condition (B7) is

\[ v_1(\kappa, z, \omega) = C_1[\exp(\beta_1 z) + \exp(-\beta_1 z)], \quad \beta_1 = \sqrt{\kappa^2 - \frac{\rho_1 \omega^2}{\mu G_1}} \quad (B9) \]

and the solution which satisfies Eq. (B6) is

\[ v_2(\kappa, z, \omega) = C_2 \exp(-\beta_2 z) + C_3 \exp(\beta_2 z), \quad \beta_2 = \sqrt{\kappa^2 - \frac{\rho_2 \omega^2}{\mu G_2}} \quad (B10) \]

Here \( \beta_1 \) and \( \beta_2 \) are complex and the root taken is the one with a positive real part. The radiative boundary condition requires that there are no upwards traveling waves in the semi-infinite space, so \( C_3 \) in Eq. (B10) must be zero, which gives

\[ v_2(\kappa, z, \omega) = C_2 \exp(-\beta_2 z) \quad (B11) \]

Substituting Eqs. (B9) and (B11) into Eq. (B8) gives

\[
\begin{pmatrix}
-\exp(\beta_1 L) + \exp(-\beta_1 L) & \exp(-\beta_2 L) \\
-\beta_1 G_1 \exp(\beta_1 L) - \exp(-\beta_1 L) & -\beta_2 G_2 \exp(-\beta_2 L)
\end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} A_\varepsilon(\kappa) \sqrt{S_{zz}(\omega)} \\ 0 \end{pmatrix} \quad (B12)
\]

The solution of Eq. (B12) is

\[ C_1 = \frac{-\beta_2 G_2 A_\varepsilon(\kappa) \sqrt{S_{zz}(\omega)} \exp(-\beta_1 L)}{(\beta_1 G_1 + \beta_2 G_2) + \exp(-2\beta_1 L)(\beta_2 G_2 - \beta_1 G_1)} \quad (B13) \]

Then the pseudo response at ground level in the frequency-wavenumber domain is

\[ v_1(\kappa, 0, \omega) = 2C_1 \quad (B14) \]

Then the pseudo response at ground level in the time–space domain is

\[
\tilde{v}(x, 0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} v_1(\kappa, 0, \omega) \exp(ik\kappa) \, dk
\]

\[
= \frac{-\beta_2 G_2 \sqrt{S_{zz}(\omega)}}{\pi} \int_{-\infty}^{+\infty} \frac{A_\varepsilon(\kappa) \exp(-\beta_1 L) \exp[i(k\kappa - \omega t)]}{(\beta_1 G_1 + \beta_2 G_2) + \exp(-2\beta_1 L)(\beta_2 G_2 - \beta_1 G_1)} \, dk \quad (B15)
\]

from which Eq. (58) follows.

References


