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Corrigendum

Corrigendum to "Tilting modules over small Dedekind domains" [J. Pure Appl. Algebra 172 (2002) 109–117]☆

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Luigi Salce has recently discovered a problem in our paper [2]: in general, the modules $R_{(P)}$ introduced in [2, p. 111] may not be obtained by classical localization at a multiplicative subset of R.

The problem does not affect the main results of [2]. Moreover, one can use prime avoidance in the case when P is cofinite in mSpec$R$. In the general case, however, several auxiliary results have to be reformulated and reproved. This concerns Lemma 3, Proposition 4, Lemma 7, and part of the proof of Theorem 12.

The idea of our correction below is to replace localization by a direct study of divisibility conditions for the modules in case.

First, we explain why localization is not an appropriate tool here.

Example ([1]). By a classical result of Claborn, any abelian group is the class group of a Dedekind domain. In particular, there is a Dedekind domain R whose class group G contains a torsion-free element g. Since any non-zero ideal of R is uniquely the product of prime ideals, w.l.o.g., $g = [q]$ where q is a maximal ideal of R such that q is contained in the union of all the other maximal ideals of R.

Let $P = \{p \in \mathrm{mSpec}R \mid p \neq q\}$ and $S = R \setminus \bigcup_{p\in P} p$. Define $R_{(P)}$ as in [2, p. 111], that is, $R_{(P)} = \bigcap_{p\in P} R_p$. Then—as claimed correctly in Lemma 3(ii) in [2]— $R_{(P)}$ is the (unique) submodule of Q satisfying $R \subseteq R_{(P)}$, $R_{(P)}/R \cong E(R/q)$, and $Q/R_{(P)} \cong \bigoplus_{p\in P} E(R/p)$.

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However, Lemma 3(i) of [2] does not hold in this setting, as $R_{(P)}$ is not the localization of R at S: since $q \subseteq \bigcup_{p \in P} p$, S is the set of all units of R, so $R_S = R$.

Next, we indicate how to replace Lemma 3 and Proposition 4 of [2]. They are replaced by the following new Definition 3 and Proposition 4:

Definition 3. Let R be a Dedekind domain and Q be its quotient field. Let $P \subseteq \mathrm{mSpec}R$. Consider the minimal injective resolution of R

$$0 \to R \overset{\subseteq}{\to} Q \overset{\pi}{\to} \bigoplus_{p \in \mathrm{mSpec}R} E(R/p) \to 0.$$

Define

$$R_P = \pi^{-1}\left(\bigoplus_{p \in P} E(R/p)\right)$$

and

$$T_P = R_P \oplus \bigoplus_{p \in P} E(R/p).$$

By definition, R_P is the (unique) submodule of Q satisfying $R \subseteq R_P$, $R_P/R \cong \bigoplus_{p \in P} E(R/p)$, and $Q/R_P \cong \bigoplus_{q \in \mathrm{mSpec}R \setminus P} E(R/q)$.

Notice that for each $p \in \mathrm{mSpec}R$, the localization R_p of R at p coincides with $R_{P'}$ where $P' = \mathrm{mSpex}R \setminus \{p\}$. So $R_P = \bigcap_{q \in \mathrm{mSpec}R \setminus P} R_q$ is a subring of Q containing R.

Proposition 4. *Let R be a Dedekind domain and $P \subseteq mSpecR$. For each $p \in P$, let $\alpha_p > 0$ and let T' be a projective generator in* Mod-R_P. *Let*

$$T = \bigoplus_{p \in P} E(R/p)^{(\alpha_p)} \oplus T'.$$

Then T is a tilting module. The corresponding tilting torsion class $\mathcal{T}_P = T^\perp$ consists of all modules that are p-divisible for all $p \in P$.

Proof. Since $\mathrm{Add}(T) = \mathrm{Add}(T_P)$, it suffices to prove that T_P is tilting.

Clearly, T_P has projective dimension $\leqslant 1$ and there is an exact sequence $0 \to R \to R_P \to I_P \to 0$ where R_P and $I_P = \bigoplus_{p \in P} E(R/p)$ are direct summands of T_P.

So in order to prove that T_P is a tilting module, it suffices to show that $\mathrm{Ext}^1_R(T_P, T_P^{(\kappa)}) = 0$ for all cardinals κ. Since R_P is an extension of R by the injective module I_P, it suffices to prove that $\mathrm{Ext}^1_R(I_P, R_P^{(\kappa)}) = 0$. For each $p \in \mathrm{mSpec}R$, $E(R/p)$ has an infinite composition series with successive factors isomorphic to R/p. Since R/p is finitely presented, we are left to prove that $\mathrm{Ext}^1_R(R/p, R_P) = 0$ for all $p \in P$. But this follows from the fact that $\mathrm{Hom}_R(R/p, Q/R_P) = 0$ for all $p \in P$.

The previous reasoning also shows that $\mathcal{T}_P = T_P^\perp = I_P^\perp = (\bigoplus_{p \in P} R/p)^\perp$, so $\mathcal{T}_p = \{M \in \text{Mod-}R \mid \mathrm{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\}$. By Lemma 1 of [2], $\mathcal{T}_P$ is just the class of all modules that are p-divisible for all $p \in P$. $\square$

Next, we state the correct version of Lemma 7. Recall that by [2, p. 113], the nucleus, $\mathrm{nuc}(M)$, of a torsion-free module M is defined as the largest subring R' of Q such that M is an R'-module. So $\mathrm{nuc}(M) = \{q \in Q \mid qM \subseteq M\}$.

Lemma 7. *Let M be a torsion-free module. Let P be the set of all $p \in mSpecR$ such that M is p-divisible. Then* $\mathrm{nuc}(M) = R_P$.

Proof. For each $p \in \mathrm{mSpec}R$, let $(C_{pi} \mid i < \omega)$ be the composition series of $E(R/p)$, and let $D_{pi} = \pi^{-1}(C_{pi})$ $(i < \omega)$.

Let $p \in P$. By induction on i, we prove that $D_{pi} \subseteq \mathrm{nuc}(M)$. Clearly, $D_{p0} = R \subseteq \mathrm{nuc}(M)$. Assume $D_{pi} \subseteq \mathrm{nuc}(M)$ and let $x \in D_{p,i+1} \setminus D_{pi}$. Then $xp \subseteq \mathrm{nuc}(M)$, so $xM = x(pM) \subseteq M$. Hence $x \in \mathrm{nuc}(M)$. This proves that $\pi^{-1}(E(R/p)) \subseteq \mathrm{nuc}(M)$, and $R_P \subseteq \mathrm{nuc}(M)$.

Assume $R_P \neq \mathrm{nuc}(M)$, so there is $q \in \mathrm{mSpec}R \setminus P$ such that $D_{q1} \subseteq \mathrm{nuc}(M)$. There is an exact sequence $0 \to R \overset{\subseteq}{\to} D_{q1} \to R/q \to 0$. So $q \subseteq qD_{q1} \subseteq R$, and $qR_q \subsetneq R_q \subsetneq D_{q1} \bigotimes_R R_q \subseteq Q_q$. Since R_q is a valuation domain, the module $N = (D_{q1} \bigotimes_R Rq)/qRq$ is uniserial, but not simple. So $qN \neq 0$. It follows that $qD_{q1} = R$. Then $M = qD_{q1}M \subseteq qM$, that is, $q \in P$, a contradiction. □

In the proof of Proposition 8 in [2], it is shown that the modules N and N' are divisible by the same primes in mSpec$R$. So $\mathrm{nuc}(N) = \mathrm{nuc}(N')$ by the new Lemma 7 above.

Finally, we indicate the changes needed in the last part of the proof of Theorem 12 in [2]. Recall that Theorem 12 says that under V = L, tilting modules are exactly the modules T from the new Proposition 4 for some $P \subseteq \mathrm{mSpec}R$, where $\alpha_p > 0$ for all $p \in P$, and T' is a projective generator in Mod-R_P.

We proceed as in that proof until the point showing that T' is a projective $\mathrm{nuc}(T')$-module (p. 116, line 8). Now, by the new Lemma 7, there is $P \subseteq P' \subseteq \mathrm{mSpec}R$ such that $R_{P'} = \mathrm{nuc}(T')$.

Assume there is $q \in P' \setminus P$. Let $R' = \bigcap_{q \neq p \in \mathrm{mSpec}R} R_p$. Then $R' \subseteq R_{P'}$. As in the proof of Theorem 12, we see that $\mathrm{Ext}^1_R(R', \bigoplus_{n \in \omega} R/q^n) = 0$, and hence $\mathrm{Ext}^1_R(Q, \bigoplus_{n \in \omega} R/q^n) = 0$, a contradiction. This gives $P' = P$.

Finally, since R_p is p-divisible for all $p \in P$, we have $R_P \in T^{\perp} = \mathrm{Gen}(T) = \mathrm{Gen}(T')$ (the latter equality holds because $\mathrm{Hom}_R(I_p, R_P) = 0$), so T' is a generator of Mod-R_P.

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