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Ramsey numbers $r(K_3, G)$ for connected graphs G of order seven

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Abstract

The triangle-graph Ramsey numbers are determined for all 814 of the 853 connected graphs of order seven. For the remaining 39 graphs lower and upper bounds are improved.

1. Introduction

We consider finite, undirected graphs $G=(V(G),E(G))$ without loops or multiple edges. By a two-colouring (R,B) we mean a colouring of the edges of G with two different colours, say red and blue. $\langle R\rangle$ ($\langle B\rangle$) denotes the subgraph of G, which is induced by the red (blue) edges. P_p is the path, C_p the cycle and K_p the complete graph on p vertices. Also we say $K_p \to (G_1, G_2)$ if for *each* two-colouring of K_p either $\langle R\rangle$ contains G_1 (a subgraph isomorphic to G_1) or $\langle B\rangle$ contains G_2. Now we define the Ramsey number $r(G_1, G_2)$ of two graphs G_1 and G_2 as the minimum $p \in N$ where $K_p \to (G_1, G_2)$.

So far almost all Ramsey numbers for pairs of graphs G_1 and G_2 of order at most five are known: Chvatal and Harary found all Ramsey numbers $r(G_1, G_2)$ where the order of both graphs is at most four [2, 3]. In [4], Clancy listed all Ramsey numbers where $|G_1| \leqslant 4$ and $|G_2| = 5$ except for five pairs of graph. These remaining numbers were determined by Bolze and Harborth [1], Exoo et al. [6] and by Hendry [11]. Very recently $r(K_4, K_5) = 25$ has been proved by McKay and Radziszowski [12]. In [10], Hendry almost completed the list with those of $|G_1| = 5$ except for six pairs. For a complete list of all known Ramsey numbers see also [13]. In 1980 Faudree, Rousseau and Schelp published ‘All triangle-graph Ramsey numbers for connected graphs of order six’ [5]. In this paper we study the Ramsey numbers $r(K_3, G)$ for connected graphs G of order seven. According to Harary and Palmer [9] there are 853 different

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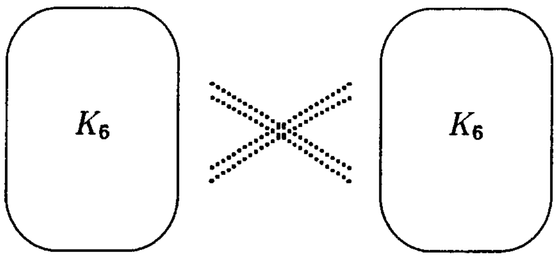
connected graphs G on seven vertices. Our Theorem 1 settles already 760 cases, but nevertheless the remaining cases require a more detailed analysis.

Like in [5] we will use further notations: For a vertex $v \in V(G)$ we denote its neighbourhood in $\langle R \rangle$ or $\langle B \rangle$ by $N_R(v)$ or $N_B(v)$, respectively. $X_R(v)$ means $N_R(v) \cap X$, where X is a subset of $V(G)$ and v is a vertex in $V(G)$ ($X_B(v) := N_B(v) \cap X$ likewise).

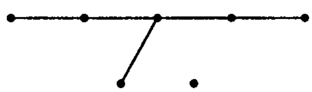
2. Results

To find a lower bound for these Ramsey numbers we have to construct a two-colouring of a complete graph which avoids both a red triangle and any connected graph of order seven. Considering the following two-colouring of K_{12} (there are two completely blue coloured K_6, joined only by red edges) we conclude that $r(K_3, G) \geqslant 13$ for all connected graphs of order seven. In this figure we indicate red edges by dotted and blue ones by normal lines.

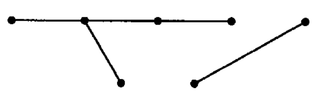


Since Graver and Yackel proved $r(K_3, K_7) = 23$ [7] the Ramsey numbers $r(K_3, G)$ for G connected and $|G| = 7$ are bounded by 13 and 23. Now we are looking for those connected graphs the triangle-graph Ramsey number of which is equal to 13. Here we are able to prove:

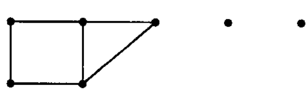
Theorem 1. (a) $r(K_3, G) \leqslant 13$ *for all connected graphs* $G \subseteq H_1$ *where*

$\overline{H_1} =$ 

(b) $r(K_3, G) \leqslant 13$ *for all connected graphs* $G \subseteq H_2$ *where*

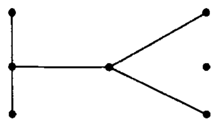
$\overline{H_2} =$ 

(c) $r(K_3, G) \leqslant 13$ *for all connected graphs* $G \subseteq H_3$ *where*

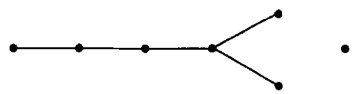
$\overline{H_3} =$ 

This theorem already reduces the number of graphs with Ramsey number possibly greater than 13 to 83. We computed a complete list of those remaining graphs. The following theorems give the exact Ramsey numbers for 64 of these graphs. For the others they improve lower and upper bounds.

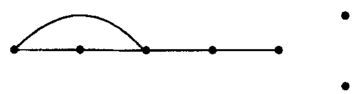
Theorem 2. (a) $r(K_3, G) \leqslant 14$ *for all connected graphs* $G \subseteq H_4$ *where*

$\overline{H_4} =$ 

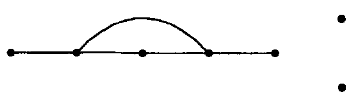
(b) $r(K_3, G) \leqslant 14$ *for all connected graphs* $G \subseteq H_5$ *where*

$\overline{H_5} =$ 

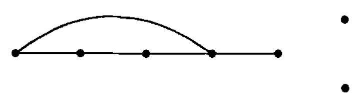
(c) $r(K_3, G) \leqslant 14$ *for all connected graphs* $G \subseteq H_6$ *where*

$\overline{H_6} =$ 

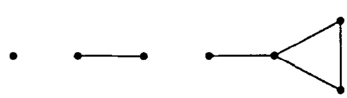
(d) $r(K_3, G) \leqslant 14$ *for all connected graphs* $G \subseteq H_7$ *where*

$\overline{H_7} =$ 

(e) $r(K_3, G) \leqslant 14$ *for all connected graphs* $G \subseteq H_8$ *where*

$\overline{H_8} =$ 

(f) $r(K_3, G) \leqslant 14$ *for all connected graphs* $G \subseteq H_9$ *where*

$\overline{H_9} =$ 

This theorem decreases the upper bound for 49 of these graphs to 14. The following theorem increases the lower bound for 36 of those graphs to 14 as well.

The fact that $r(K_3, K_5) = 14$ [4] proves Theorem 3(a). Analogously the equation $r(K_3, K_6 - K_3) = 14$ [5] gives Theorem 3(b), Theorem 3(c) is proved in the next section.

Theorem 3. (a) $r(K_3, G) \geqslant 14$ *for all connected graphs G containing a complete subgraph on* 5 *vertices.*

(b) $r(K_3, G) \geqslant 14$ *for all connected graphs G containing a* $K_6 - K_3$.

(c) $r(K_3, G) \geqslant 14$ *for all connected graphs G containing a* $K_7 - C_i$, *where* $i =$ 5, 6 or 7.

Theorem 4. (a) $r(K_3, G) \leqslant 17$ *for all connected graphs G the complement of which contains a* $P_3 \cup P_2$.

(b) $r(K_3, G) \leqslant 17$ *for all connected graphs G the complement of which contains a* P_4.

From Theorem 4 it follows that 17 is an upper bound for at least 35 of the 83 graphs.

Theorem 5. $r(K_3, G) \geqslant 17$ *for all conncected graphs containing a* $K_6 - P_2$.

Theorem 5 is a simple conclusion from the fact that $r(K_3, K_6 - P_2) = 17$ [4].

Theorem 6. $r(K_3, G) \leqslant 18$ *for each connected graph on seven vertices G the complement of which contains a path on three vertices.*

The previous theorem gives a new upper bound for 5 of the 83 graphs.

The next theorem follows from the fact that $r(K_3, K_6)$ is equal to 18 [5].

Theorem 7. $r(K_3, G) \geqslant 18$ *for all connected graphs G containing a* K_6.

Thus $r(K_3, G) = 18$ for these 5 graphs.

In 1982 Grenda and Harborth [8] proved the following theorem:

Theorem 8. $r(K_3, K_7 - P_2) = 21$.

Altogether we find that the triangle-graph Ramsey number is for at least 760 connected graphs of order seven equal to 13, for at least 36 equal to 14, for at least 11 equal to 17, for at least 5 equal to 18, for at least one equal to 21 and for exactly one equal to 23. The Ramsey numbers for the remaining 39 graphs are mostly bounded by 14 and 17 or by 13 and 14.

3. Proofs

First of all we prove three lemmas, which are often used in the sequel.

Lemma 1. *Suppose H is a complete graph on at least* 13 *vertices with red and blue coloured edges. Assume that there is no red triangle, but a blue* K_6. *Then we conclude* *that there is also a blue F_1 where*

$\overline{F_1} =$ 

Proof. Let $X := \{x_1, \dots, x_6\}$ span the blue K_6 and let $Y := V(H) \backslash X$, $|Y| \geqslant 7$ and $Y(X,R) := \{y \in Y \mid X_R(y) > 1/2|X|\}$. Obviously $X_R(y_1) \cap X_R(y_2) \neq \emptyset$ and thus $(y_1 y_2) \in \langle B \rangle$ for all $y_1, y_2 \in Y(X,R)$. Hence the set $Y(X,R)$ spans a complete graph in $\langle B \rangle$. Now we consider two different cases:

- $|Y(X,R)| \geqslant 7$ implies that there is a K_7 in $\langle B \rangle$.
- If $|Y(X,R)| < 7$ then there is a vertex $y \in Y$ with $|X_R(y)| \leqslant 1/2|X| = 3$ which gives a graph F_1 in $\langle B \rangle$ spanned by $X \cup \{y\}$. $\square$

Lemma 2. *This time suppose H is a complete graph on at least 17 vertices. Again assume that there is no red triangle, but a blue K_6. Then we conclude that there is also a blue $K_7 - P_3$.*

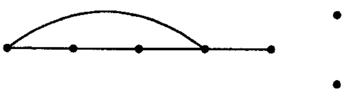
Proof. Again let $X := \{x_1, \dots, x_6\}$ span the blue K_6, $Y := V(H) \backslash X$, $|Y| \geqslant 11$ and assume that there is no blue $K_7 - P_3$. If there is a vertex $y \in Y$ with $|X_R(y)| \leqslant 2$ then we directly get a contradiction. Thus we may assume $|X_R(y)| \geqslant 3$ for all $y \in Y$. This means that at least $3 \times 11 = 33$ red edges join X and Y. Avoiding a blue K_7 we can choose $|Y_R(x_1)| = 6$ and let $Y_R(x_1) := X' = \{x_7, \dots, x_{12}\}$. Obviously, X' spans a second K_6. To avoid a blue $K_7 - P_3$ each of the remaining vertices $\{x_{13}, x_{14}, \dots\}$ has at least three red neighbours in each K_6. Say $X'' = \{x_4, x_5, x_6, x_7, x_8, x_9\}$ are the red neighbours of x_{13} which also spans a K_6. Again we either get a blue $K_7 - P_3$ or there are at least $3 \times 4 = 12$ red edges between X'' and $\{x_{14}, x_{15}, x_{16}, x_{17}\}$. The last conclusion implies that there is a vertex $x \in X''$ with two red neighbours in $\{x_{14}, x_{15}, x_{16}, x_{17}\}$, let the edges $(x_7 x_{14})$ and $(x_7 x_{15})$ be red. In addition x_7 has three red neighbours in X, and since $X'' = \{x_4, x_5, x_6, x_7, x_8, x_9\}$ spans a blue K_6 these are the vertices x_1, x_2 and x_3. Hence the set $X''' := \{x_1, x_2, x_3, x_{13}, x_{14}, x_{15}\}$ spans one more complete graph on six vertices. Following the same arguments x_4 has three red neighbours in X' and three in X''', in fact those are $x_{10}, x_{11}, x_{12}, x_{13}, x_{14}$ and x_{15}. Altogether we find five complete subgraphs of order six containing the vertices $\{x_1, \dots, x_{15}\}$. Furthermore, the vertex x_{16} has at least three red neighbours in each K_6, which implies that $|N_R(x_{16})| \geqslant 8$ and thus we either find a blue $K_7 - P_3$ or a blue K_8. Both give a contradiction to our assumption. $\square$

Lemma 3. *Again suppose H with $|V(H)| \geqslant 13$ is a complete graph, coloured blue and red. If there is no red triangle but a blue $K_6 - P_2$ then there exists a blue F_2 where*

$\overline{F_2} =$ 

Proof. If there is a blue clique of six vertices we conclude that there is also a blue F_2 by Lemma 3. Hence let $X := \{x_1, \dots, x_6\}$ span a blue $K_6 - P_2$ where the edge $(x_1 x_2)$ is red and let $Y := V(H) \backslash X, |Y| \geqslant 7$. If there is a vertex $y \in Y$ with $|X_R(y) \cap \{x_3, x_4, x_5, x_6\}| =: |X_R'(y)| \leqslant 2$ then $X \cup \{y\}$ spans a blue graph containing F_2. Hence we may assume $|X_R'(y)| \geqslant 3$ for all $y \in Y$, which in particular means $|Y(X', R)| := |\{y \in Y \mid X_R'(y) > 1/2|X'|\}| \geqslant 7$. As in the proof of Lemma 1 it follows that there is a blue K_7, spanned by $Y(X', R)$. □

Lemma 4. *Suppose H is a complete graph on at least 13 vertices with red and blue coloured edges. Assume that there is no red triangle, but a blue $K_6 - P_3$. Then we conclude that there is also a blue F_3 where*

$\overline{F_3} =$ 

Proof. Using Lemma 1 and Lemma 3 we assume that $X := \{x_1, \dots, x_6\}$ spans an induced blue $K_6 - P_3$ where $(x_1 x_2)$ and $(x_2 x_3)$ are red edges and let $Y := V(H) \backslash X, |Y| \geqslant 7$, $X' := \{x_4, x_5, x_6\}$. Again we consider the set $|Y(X', R)| := |\{y \in Y \mid X_R'(y) > 1/2|X'|\}$. Since $(y_i y_j) \in \langle B \rangle$ for all $y_i, y_j \in Y(X', R)$ (proof of Lemma 1) $|Y(X', R)| \geqslant 6$ would give a complete graph on six vertices in $\langle B \rangle$ and thus by Lemma 1 in particular a blue graph F_3. Hence we may assume that there is a vertex $y \in Y$ with $|X_R'(y)| \leqslant 1$. Avoiding a red triangle the set $X \cup \{y\}$ spans either a blue F_3 or a blue graph containing F_3. □

Proof of Theorem 1(a). Let $V(K_{13}) = \{a, b, c, d, e, f, y_1, \dots, y_7\}$ and suppose there is a two-colouring of K_{13} which avoids both a red triangle and a blue H_1. In [4] Faudree et al. proved $r(K_3, K_6 - 2P_2) = 13$. Since our two-colouring avoids a red K_3, assume that $X := \{a, b, c, d, e, f\}$ spans a K_6, a $K_6 - P_2$ or a $K_6 - 2P_2$ in $\langle B \rangle$.

- X spans a blue K_6. By Lemma 1 there exists a blue F_1.
- X spans a blue $K_6 - P_2$. By Lemma 3 there exists a blue F_2.
- Let X span a blue $K_6 - 2P_2$ where $(ab), (cd)$ are the red edges. Avoiding a red triangle we also know $|N_R(y) \cap \{ab\}| \leqslant 1$ and $|N_R(y) \cap \{cd\}| \leqslant 1$ for all $y \in Y := V(K_{13}) \backslash X$. Furthermore, $\langle B \rangle$ contains no K_7. Thus we find at least one red edge, say $(y_1 y_2)$, in Y. Also we note $|N_R(y_1) \cap \{e, f\}| + |N_R(y_2) \cap \{e, f\}| \leqslant 2$. This implies that either $\{y_1\} \cup X$ or $\{y_2\} \cup X$ spans a blue H_1.

Altogether we know that each two-colouring of a K_{13} which avoids any red triangle contains a blue F_1, a blue F_2 or a blue H_1. Since $H_1 \subseteq F_2 \subseteq F_1$ we conclude that all these two-colourings contain at least a blue H_1 which contradicts our assumption. □

Proof of Theorem 1(b). We assume that there is a two-colouring of K_{13} which avoids a blue H_2 as well as a red K_3. Again we conclude from $r(K_3, K_6 - 2P_2) = 13$ that there is a blue K_6, an induced blue $K_6 - P_2$ or an induced blue $K_6 - 2P_2$. Following

Lemma 1 and Lemma 3 we conclude that the existence of a blue K_6 or an induced blue $K_6 - P_2$ implies a blue F_1, F_2, respectively. In particular, a blue K_6 and an induced blue $K_6 - P_2$ give a blue H_2. Thus assume that there is a blue $K_6 - 2P_2$ spanned by $X := \{a,b,c,d,e,f\}$, where (ab) and (cd) are red edges. Let $Y := V(K_{13})\backslash X$, $A := \{a,b\}$, $C := \{c,d\}$ and $E := \{e,f\}$. If there is a vertex $y \in Y$ with $|N_R(y) \cap \{a,b,c,d\}| \leqslant 1$ then there is a blue H_2 – spanned by $\{a,b,c,d,e,f,y\}$. Hence $|A_R(y)| = 1$ and $|C_R(y)| = 1$ for all $y \in Y$ which implies $|X_R(A)| = 7$ and $|X_R(C)| = 7$. This and the fact that the existence of a blue K_6 contradicts our assumption means that one vertex of a and b and one of c and d has exactly four red neighbours in Y. Say the edges $(ay_1), (ay_2), (ay_3), (ay_4)$, $(by_5), (by_6)$ and (by_7) are red. To avoid a blue $K_6 - P_2$ (Lemma 3) c as well as d has exactly two red neighbours in $\{y_1, y_2, y_3, y_4\}$. Without loss of generality let $(cy_1), (cy_2), (dy_3)$ and (dy_4) be red. Furthermore, each vertex of $\{y_5, y_6, y_7\}$ has one red neighbour in C – say $(cy_5), (cy_6), (dy_7) \in \langle R \rangle$. Now we consider different cases:

- If $\{y_3, y_4, y_5\} \subseteq N_R(e)$ or if $\{y_3, y_4, y_5\} \subseteq N_R(f)$ then $\{b, c, y_1, y_2, y_3, y_4, y_5\}$ spans a blue graph containing H_2.
- If $\{y_3, y_4, y_6\} \subseteq N_R(e)$ or if $\{y_3, y_4, y_6\} \subseteq N_R(f)$ then $\{b, c, y_1, y_2, y_3, y_4, y_6\}$ spans a blue graph containing H_2.
- If $\{y_3, y_5, y_6\} \subseteq N_R(e)$ or if $\{y_3, y_5, y_6\} \subseteq N_R(f)$ then $\{b, d, y_1, y_2, y_3, y_5, y_6\}$ spans a blue graph containing H_2.
- If $\{y_4, y_5, y_6\} \subseteq N_R(e)$ or if $\{y_4, y_5, y_6\} \subseteq N_R(f)$ then $\{b, d, y_1, y_2, y_4, y_5, y_6\}$ spans a blue graph containing H_2.
- If $\{y_1, y_2, y_7\} \subseteq N_R(e)$ or if $\{y_1, y_2, y_7\} \subseteq N_R(f)$ then $\{b, c, y_1, y_2, y_3, y_4, y_7\}$ spans a blue graph containing H_2.

Altogether this means $|N_R(e) \cap \{y_3, y_4, y_5, y_6\}| \leqslant 2$, $|N_R(e) \cap \{y_1, y_2, y_7\}| \leqslant 2$, $|N_R(f) \cap \{y_3, y_4, y_5, y_6\}| \leqslant 2$ and $|N_R(f) \cap \{y_1, y_2, y_7\}| \leqslant 2$.

Suppose $\{y_1, y_7\} \subseteq N_R(e)$. Considering the fact that $|N_R(e) \cap \{y_3, y_4, y_5, y_6\}| \leqslant 2$ we distinguish the following cases:

- If $\{y_5, y_6\} \nsubseteq N_R(e)$ then $\{a, d, e, y_2, y_5, y_6, y_7\}$ spans a blue graph containing H_2.
- If $\{y_3, y_4\} \nsubseteq N_R(e)$ then $\{b, c, e, y_2, y_3, y_4, y_7\}$ spans a blue graph containing H_2.
- If $\{y_3, y_5\}, \{y_3, y_6\}, \{y_4, y_5\}$ or $\{y_4, y_6\} \nsubseteq N_R(e)$ then Y spans a blue $K_7 - (P_4 \cup P_2)$.

We obtain similar results if we suppose $\{y_1, y_7\} \subseteq N_R(f)$, $\{y_2, y_7\} \subseteq N_R(e)$ or $\{y_2, y_7\} \subseteq N_R(f)$. Hence we conclude that if $y_7 \in N_R(e)$ $(N_R(f))$ then $y_1, y_2 \notin N_R(e)$ $(N_R(f))$. We also have to avoid a blue $K_6 - P_2$, therefore $y_1, y_2 \notin N_R(e)$ $(N_R(f))$ implies $N_R(e) \cap \{y_3, y_4\} \geqslant 2$ and $N_R(e) \cap \{y_5, y_6\} \geqslant 2$ $(N_R(f) \cap \{y_3, y_4\} \geqslant 2$ and $N_R(f) \cap \{y_5, y_6\} \geqslant 2)$, contradicting $|N_R(e) \cap \{y_3, y_4, y_5, y_6\}| \leqslant 2$ $(|N_R(f) \cap \{y_3, y_4, y_5, y_6\}| \leqslant 2)$. Thus we may assume $y_7 \notin N_R(e) \cup N_R(f)$. Avoiding a blue $K_7 - (P_4 \cup P_2)$, possibly spanned by $\{b, c, e, f, y_3, y_4, y_7\}$, we conclude $y_3, y_4 \in N_R(e)$. Now the fact that $|N_R(e) \cap \{y_3, y_4, y_5, y_6\}| \leqslant 2$ gives $y_5, y_6 \notin N_R(e)$ which implies the existence of a $K_7 - (P_3 \cup P_2)$, spanned by $\{a, d, e, f, y_5, y_6, y_7\}$. $\square$

Proof of Theorem 1(c). We assume that there is a two-colouring of K_{13} which avoids both a blue H_3 and a red triangle. As in the proof of Theorem 1 (a) and (b) we conclude that each two-colouring contains a blue F_1 (Lemma 1), a blue F_2 (Lemma 2) or an induced blue $K_6 - 2P_2$. Since $H_3 \subseteq F_2 \subseteq F_1$ suppose $X := \{a,b,c,d,e,f\}$ spans a $K_6 - 2P_2$ in $\langle B \rangle$ where the edges (ab) and (cd) are red. Let $Y := V(G) \backslash X$. If there is a $y \in Y$ with $N_R(y) \cap \{e,f\} = \emptyset$, then the set $X \cup \{y\}$ spans a blue $K_7 - P_5$. Hence we may assume $|N_R(y) \cap \{e,f\}| \geqslant 1$ for all $y \in Y$. Avoiding a blue K_6 we conclude $|N_R(e)| \leqslant 5$ and $|N_R(f)| \leqslant 5$. Let $|N_R(e)| \geqslant |N_R(f)|$ and consider the following cases:

1. $|N_R(e)| = 5$: Since there is no red triangle either $N_R(e) \cup \{c\}$ or $N_R(e) \cup \{d\}$ spans a blue $K_6 - P_3$. Following Lemma 4 this contradicts our assumption.
2. $|N_R(e)| = 4$ and $|N_R(f)| = 3$: Let without loss of generality $N_R(e) := \{y_1, y_2, y_3, y_4\}$, $N_R(f) := \{y_5, y_6, y_7\}$ and $|N_R(a) \cap \{y_1, y_2, y_3, y_4\}| \geqslant |N_R(b) \cap \{y_1, y_2, y_3, y_4\}|$. Avoiding a red triangle we obtain $|N_R(b) \cap \{y_1, y_2, y_3, y_4\}| \leqslant 2$. Again we find a blue $K_6 - P_3$, this time spanned by $\{b, f, y_1, y_2, y_3, y_4\}$.
3. $|N_R(e)| = 4$ and $|N_R(f)| = 4$: Let without loss of generality $N_R(e) := \{y_1, y_2, y_3, y_4\}$ and $N_R(f) := \{y_4, y_5, y_6, y_7\}$. Now the following holds:
   (a) $N_R(x) \cap \{y_1, y_2, y_3\} \neq \emptyset$ for all $x \in \{a,b,c,d\}$, since otherwise $\{x, f, y_1, y_2, y_3, y_4\}$ would span a blue $K_7 - P_3$.
   (b) Analogously we conclude $N_R(x) \cap \{y_5, y_6, y_7\} \neq \emptyset$ for all $x \in \{a,b,c,d\}$.
   (c) Avoiding a blue F_3 we conclude $|N_R(x_i) \cup N_R(x_j) \cap \{y_1, y_2, y_3\}| \geqslant 2$ and $|N_R(x_i) \cup N_R(x_j) \cap \{y_5, y_6, y_7\}| \geqslant 2$ for $x_i \in \{a,b\}$, $x_j \in \{c,d\}$.

   From (a)–(c) we infer that a, b, c or d has two red neighbours in $\{y_1, y_2, y_3\}$. Say $(ay_1), (ay_2)$ and (by_3) are red. Without loss of generality it also follows from (c) that (cy_1) and (dy_2) are red edges. Analogously one vertex of $\{a,b,c,d\}$ has two red neighbours in $\{y_5, y_6, y_7\}$.
   - If this is the vertex a, then $|N_R(a)| \geqslant 5$ and together with either c or d $N_R(a)$ would span a blue $K_6 - P_3$.
   - If the vertex b has two red neighbours in $\{y_5, y_6, y_7\}$ then suppose $(ay_5), (by_6)$ and (by_7) are red and because of (c) let also (cy_6) and (dy_7) be red. Since at least one of the edges (ay_4) or (by_4) is blue we find a blue $K_7 - P_5$ spanned by $\{a, y_2, y_3, y_4, y_5, y_6, y_7\}$ or by $\{b, y_1, y_2, y_3, y_4, y_5, y_6\}$.
   - If one of the vertices c or d has two red neighbours in $\{y_5, y_6, y_7\}$ then suppose without loss of generality that $(cy_5), (cy_6), (dy_7), (ay_5)$ and (by_6) are red. Now either $\{a, y_1, y_3, y_4, y_5, y_6, y_7\}$ spans a blue $K_7 - C_5$ or $\{b, y_1, y_2, y_3, y_4, y_5, y_6\}$ spans a blue $K_7 - P_5$.

Altogether we find in each case a blue graph containing H_3 which contradicts our assumption. $\square$

Proof of Theorem 2(a). Suppose, to the contrary, that there is a two-colouring (R,B) of K_{14} which avoids a blue H_4 and a red triangle. By Lemma 1 a blue K_6 gives in particular a blue H_4. Hence we conclude $|N_R(v)| \leqslant 5$ for all $v \in V(K_{14})$.

First we assume that there is a vertex $v \in V(K_{14})$ with at most 4 red and thus at least 9 blue neighbours. It is $r(K_3, K_4) = 9$ and hence we know that the set $N_R(v)$ contains a blue K_4, spanned by $Y := \{y_1, y_2, y_3, y_4\}$. Together with the vertex v this gives a blue clique of order five. Let $Z := V(K_{14}) \backslash (X \cup Y \cup \{v\}) = \{z_1, z_2, z_3, z_4, z_5\}$. Since there is no blue K_6 there is at least one red edge in Z, say $(z_1 z_2) \in \langle R \rangle$. To avoid a blue graph H_4 – possibly spanned by $Y \cup \{v, z_1, z_2\}$ – at least one of the vertices z_1

and z_2 has three red neighbours in Y. And since there is no red triangle either, this in particular implies that $Y \cup \{v, z_2\}$ spans a blue $K_6 - P_2$. Following Lemma 3 there exists a blue F_2 and hence a blue H_4.

Hence we conclude $|N_R(v)| = 5$ and thus $|N_B(v)| = 8$ for all $v \in V(K_{14})$. We start with the vertex $a \in V(K_{14})$ and let $X := N_R(a)$ and $Y := N_B(a)$. Since each vertex has exactly five red neighbours and since the set $N_R(a)$ spans a clique in $\langle B \rangle$ there are exactly $4 \times 5 = 20$ red edges between X and Y. By Lemma 3 we have to avoid a blue $K_6 - P_2$, hence we conclude $|N_R(y)| \geqslant 2$ for all $y \in Y$. Altogether there is a vertex $y \in Y$ with two red neighbours in X and three red neighbours in Y. Without loss of generality let $(y_5 y_6), (y_5 y_7), (y_5 y_8), (y_5 x_1)$ and $(y_5 x_2) \in \langle R \rangle$. Also let y_2, y_3 and y_4 be the further red neighbours of x_2 in Y. Since there is no red triangle at least two of these three vertices are also red neighbours of x_1, say $(x_1 y_3), (x_1 y_4) \in \langle R \rangle$. Now we consider the vertex y_1: Avoiding a blue $K_6 - P_2$ it has at least two red neighbours in each K_5:

- y_1 has two red neighbours in $N_R(x_2)$, in particular in the set $\{y_2, y_3, y_4\}$. Therefore and since there is no red triangle we conclude $(x_1 y_1) \notin \langle R \rangle$.
- y_1 has two red neighbours in $N_R(y_5)$, in particular in the set $\{y_6, y_7, y_8\}$ and
- y_1 has two red neighbours in X, in particular in the set $\{x_3, x_4, x_5\}$.

Altogether this implies $|N_R(y_1)| \geqslant 6$ and we obtain a contradiction to our assumption. □

Proof of Theorem 2(b)–(e). We assume, to the contrary, that there is a two-colouring of K_{14} which avoids both a red triangle and a blue H_i where $i = 5$ in Theorem 2(b), $i = 6$ in Theorem 2(c), $i = 7$ in Theorem 2(d) and $i = 8$ in Theorem 2(e). As in the proof of Theorem 1 let $X := \{a, b, c, d, e, f\}$ span a blue K_6, a blue $K_6 - P_2$ or a blue $K_6 - 2P_2$. By Lemma 1 and Lemma 3 a blue K_6 or an induced blue $K_6 - P_2$ gives a blue F_1 or F_2 and thus also a blue H_5, H_6, H_7 and H_8. Hence consider the remaining case that X spans an induced blue $K_6 - 2P_2$ and assume that (ab) and (cd) are red edges ($r(K_3, K_6 - 2P_2) = 13$ [4]). Let $Y := V(K_{14}) \setminus X = \{y_1, \ldots, y_8\}$, $|Y| = 8$. Avoiding a red triangle we get $|N_R(y) \cap \{a, b\}| \leqslant 1$ and $|N_R(y) \cap \{c, d\}| \leqslant 1$ for all $y \in Y$. If there is a vertex $y \in Y$ with $N_R(y) \cap \{e, f\} = \emptyset$ then the set $\{y\} \cup X$ spans a blue $K_7 - P_5$. Hence we may assume that $N_R(y) \cap \{e, f\} \neq \emptyset$ for all $y \in Y$, which means $|Y_R(e)| + |Y_R(f)| \geqslant 8$. Also let $|Y_R(e)| \geqslant |Y_R(f)|$. $|Y_R(v_i)| \geqslant 6$ implies a blue K_6 and following Lemma 1 also a blue F_1. Thus we have $5 \geqslant |Y_R(e)| \geqslant |Y_R(f)| \geqslant 3$. Now we have to consider the following possibilities for the choice of $|Y_R(e)|$ and $|Y_R(f)|$:

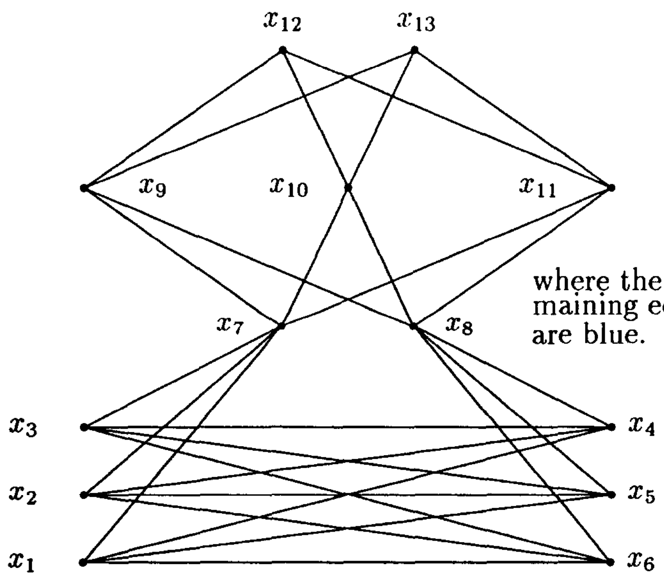
1. $|Y_R(e)| = 5$ and $|Y_R(e) \cap Y_R(f)| \leqslant 1$ which means $|Y_R(f)| = 3$ or $|Y_R(f)| = 4$: This and $Y_R(a) \cap Y_R(b) = \emptyset$ imply $|Y_R(x) \cap Y_R(e)| \leqslant 2$ for $x = a$ or $x = b$. Thus $Y_R(e) \cup \{x\} \cup \{f\}$ spans either a blue $K_7 - P_4$ or a blue $K_7 - (P_3 \cup P_2)$.
2. $|Y_R(e)| = 5$ and $|Y_R(f)| = 5$. Let $Y_R(e) := \{y_1, \ldots, y_5\}$ and $Y_R(f) := \{y_4, \ldots, y_8\}$. We distinguish two cases:
   (a) $|Y_R(x) \cap \{y_4, y_5\}| = 2$ for at least one $x \in \{a, b, c, d\}$ – say $x = a$. This means that $|Y_R(b) \cap \{y_4, y_5\}| = 0$. To avoid that either $\{b\} \cup \{e\} \cup Y_R(f)$ or $\{b\} \cup \{f\} \cup Y_R(e)$ spans a $K_7 - (P_3 \cup P_2)$ we get $|Y_R(b) \cap \{y_1, y_2, y_3\}| \geqslant 2$ and $|Y_R(b) \cap \{y_6, y_7, y_8\}| \geqslant 2$. Let $(by_1), (by_2), (by_6)$ and (by_7) be red. But then the set $\{y_1, \ldots, y_7\}$ spans a blue $K_7 - P_3$.

(b) $|Y_R(x) \cap \{y_4, y_5\}| \leqslant 1$ for all $x \in \{a,b,c,d\}$. In particular, this is true for $x=a$ and $x=b$. Also at least one of a or b has at most one red neighbour in $\{y_1, y_2, y_3\}$ – say a. Now the set $\{a, f, y_1, y_2, y_3, y_4, y_5\}$ spans a $K_7 - P_5$ in $\langle B \rangle$.

3. $|Y_R(e)| = 4$ and $|Y_R(f)| = 4$. Let $Y_R(e) := \{y_1, \ldots, y_4\}$ and $Y_R(f) := \{y_5, \ldots, y_8\}$. Without loss of generality let $|Y_R(e) \cap Y_R(x)| \leqslant 2$ for $x=a$ and $x=c$. Avoiding a blue $K_7 - P_4$ and a blue $K_7 - (P_3 \cup P_2)$ we get $|Y_R(e) \cap Y_R(x)| = 2$ for $x=a$ and $x=c$ and for symmetric reasons for $x=b,d$ as well. Analogously we observe that $|Y_R(f) \cap Y_R(x)| = 2$ for $x \in \{a,b,c,d\}$. Considering these red neighbours we have four more different cases:
   (a) $|Y_R(a) \cap Y_R(c) \cap Y_R(e)| = 2$ and $|Y_R(a) \cap Y_R(c) \cap Y_R(f)| = 2$: Now $Z := \{b,d,f\} \cup Y_R(a)$ spans a blue $K_7 - P_3$.
   (b) $|Y_R(a) \cap Y_R(c) \cap Y_R(e)| = 1$ and $|Y_R(a) \cap Y_R(c) \cap Y_R(f)| = 2$: This time Z spans a blue $K_7 - (P_3 \cup P_2)$.
   (c) $|Y_R(a) \cap Y_R(c) \cap Y_R(e)| = 1$ and $|Y_R(a) \cap Y_R(c) \cap Y_R(f)| = 1$: This gives a blue $K_7 - P_5$, again spanned by Z.
   (d) $|Y_R(a) \cap Y_R(c) \cap Y_R(e)| = 0$: This implies $|Y_R(b) \cap Y_R(c) \cap Y_R(e)| = 2$ and according to either (a) or (b), we find a blue $K_7 - P_3$ or a blue $K_7 - (P_3 \cup P_2)$. □

Proof of Theorem 2(f). Grenda and Harborth [8] proved that in each two-colouring of a K_{14} there is either a red triangle or a blue graph of order seven missing only three edges. In particular, this means that each two-colouring where $\langle R \rangle$ is triangle-free, contains each graph in $\langle B \rangle$ the complement of which contains three isolated edges, a $P_3 \cup P_2$, a P_4, a K_3 and a $K_{1,3}$. The graph H_9 meets this condition. □

Proof of Theorem 3(c). To prove this theorem we consider the following two-colouring of a K_{13}:



where the remaining edges are blue.

Obviously, there is no red K_3, hence we have to show that there is no blue graph G either. Therefore we consider the set $X := \{x_1, \ldots, x_6\}$ and distinguish between the number of elements in X which possibly are vertices of such a graph G.

1. $|X \cap V(G)| \leqslant 2$: at least one vertex of $\{x_9, x_{10}, x_{11}\}$ is in $V(G)$. Since $d_G(x) \geqslant 4$ for all $x \in V(G)$ (which means that $|N_R(x) \cap V(G)| \leqslant 2$) at most – and because of $|V(G| = 7$ exactly – two vertices of $\{x_7, x_8, x_{12}, x_{13}\}$ are in $V(G)$. Thus a second vertex of $\{x_9, x_{10}, x_{11}\}$ is in $V(G)$, which means that there exists a C_4 in $\overline{G}$. Hence $\overline{G} \nsubseteq C_5$, $\overline{G} \nsubseteq C_6$ and $\overline{G} \nsubseteq C_7$.
2. $|X \cap V(G)| = 3$: either x_7 or x_8 are not in $V(G)$, since otherwise $d_G(x) \leqslant 3$ for at least one $x \in V(G)$ or $C_4 \subseteq \overline{G}$. Analogous to 1, we conclude that two vertices of $\{x_9, x_{10}, x_{11}\}$ and two vertices of $\{x_7, x_8, x_{12}, x_{13}\}$ are in $V(G)$ which gives a C_4 in $\overline{G}$. Again we conclude $\overline{G} \nsubseteq C_5$, $\overline{G} \nsubseteq C_6$ and $\overline{G} \nsubseteq C_7$.
3. $|X \cap V(G)| \geqslant 4$: this directly gives $d_G(x) \leqslant 3$ for at least one vertex $x \in V(G)$ or the existence of a C_4 in $\overline{G}$ and thus $\overline{G} \nsubseteq C_5$, $\overline{G} \nsubseteq C_6$ and $\overline{G} \nsubseteq C_7$. □

Proof of Theorem 4(a). This time we use that Grenda and Harborth [8] proved that in each two-colouring of a K_{17} there is either a red triangle or a blue graph of order seven missing only two edges. In particular, this means that each two-colouring where $\langle R \rangle$ is triangle-free, contains each graph in $\langle B \rangle$ the complement of which contains two adjacent edges as well as two isolated edges. The largest graphs meeting this condition are the $K_7 - P_4$ and the $K_7 - (P_3 \cup P_2)$. □

Proof of Theorem 6. [5] gives $r(K_3, K_6) = 18$. Thus Lemma 2 already proves this result. □

Remark. During the preparation of this paper in summer 1995 we were informed by Gunnar Brinkmann about the existence of the Master's thesis "Ramsey numbers involving a triangle: theory & algorithms" by Xia Jin finished in August 1993 at the Rochester Institute of Technology. Using efficient algorithms he computed all triangle-graph Ramsey numbers for connected graphs of order seven.

Unfortunately, there are some strange inconsistencies in the presentation of the 853 numbers implying that several of these numbers are not correct.

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