Analysis and synthesis of discrete-time systems

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Abstract This paper presents an introduction to the analysis and synthesis of sampled-data (discrete-time) systems, i.e. systems in which some or all signals can change values only at discrete values of time. The description of these systems is presented using state-space concepts. Following an introduction to linear discrete-time systems including systems where two or more samplers operate at different frequencies, here is a brief introduction to non-linear systems including the study of stability using the Second Method of Lyapunov. The latter part of the paper describes pulse-width modulated discrete systems. The final section considers the synthesis of systems designed to reach equilibrium states in the minimum number of sampling periods. The concepts discussed in the paper are illustrated with a large number of examples.

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Foreword—A tribute to Lotfi A. Zadeh

The following text is a chapter from the book “Modern Systems Theory”, edited by C.T. Leondes, published by McGraw-Hill Book Company in 1965. It was written in 1964, about 47 years ago, when digital computers were in their infancy. Not much later, such systems would be called “digital control systems”. State-space methods were introduced to the control systems community by Professor Zadeh in 1962 (see [1]). However, these methods were not widely adopted until the following year (1963) when the book “Linear Control Systems: The State Space Approach” by Zadeh and Desoer was published.

The Zadeh and Desoer book made a dramatic difference to my own work. The chapter that follows was written about a year after the publication of Zadeh and Desoer, and would not have been possible without it. Many years later, Professor Zadeh’s work on fuzzy systems completely changed my view of determinism and uncertainty. My engineering career was deeply influenced by Professor Zadeh and I am glad to have this opportunity to express my appreciation for his inspiration and friendship over the years.

1. Introduction

If the signals at one or more points of a system can change only at discrete values of time, the system is known as a “discrete” or “sampled-data” system. Such systems generally contain elements operating on continuous signals, elements operating on discrete signals, devices for transforming continuous to discrete information (usually known as samplers), and devices for transforming discrete to continuous information (usually known as hold circuits). The sampling operations are often periodic, but may be arbitrary.

The analysis of sampled-data systems with linear elements has been accomplished in the past largely by the use of a special form of the Laplace transform known as the “z-transform”. The use of such transforms is generally limited to linear systems with periodic sampling and negligible sampling times, but can be extended to certain other cases.

In keeping with current trends in control system theory, discrete systems have recently been studied by means of state-space concepts. As with continuous systems, the concepts of state and state transformation make possible the systematic formulation of a large class of problems, including those with arbitrary sampling patterns and nonlinear operations.

The purpose of this chapter is to review some of the characteristics and methods of analysis of linear sampled-data systems, to present the state-variable formulation of sampled-data problems, and to discuss the solution of certain nonlinear problems. The use of Lyapunov’s second method for study of the asymptotic stability of discrete systems is also presented.
The chapter is not intended to be a complete survey of analysis and synthesis techniques. Rather, it is a discussion of a few selected problems which illustrate the use of state-space concepts in the analysis and synthesis of sampled-data systems.

2. Sampling process

The transition from continuous to discrete information is performed by means of a sampling switch or "sampler". The operation of a sampler may be viewed as the modulation of a train of pulses, $p(t)$, by a continuous, information-carrying signal, $x(t)$, as indicated in Figure 1.

If $x(t)$ is used to modulate the amplitude of the pulses, the process is termed Pulse Amplitude Modulation (PAM). If each pulse has equal width, $h$, and unit amplitude, and the pulses are periodic with period, $T$, the pulse train can be described by:

$$
p(t) = \sum_{k=-\infty}^{+\infty} [u(t - kT) - u(t - kT - h)],
$$

where $u(t)$ is the unit step function. The output pulse train in Figure 1 is then given by:

$$
x^*(t) = x(t)p(t).
$$

If the sampling pulse width is small, i.e. $h \ll T$, the output of the sampler may simply be considered the number sequence $\{x(kT^+), k\}$. We shall designate this type of periodic sampling with negligible pulse width as ordinary or conventional sampling. It can be that the ordinary sampler is a time-varying amplifier, and that ordinary sampling is a linear operation, i.e:

$$
[a_1x_1(t) + a_2x_2(t)]^* = a_1x_1^*(t) + a_2x_2^*(t),
$$

where $x_1(t)$ and $x_2(t)$ are continuous signals, and $a_1$ and $a_2$ are arbitrary constants. In accordance with convention, the asterisk (*) denotes sampled signals.

If the modulation pattern is signal-dependent, the sampler becomes a nonlinear device. Pulse-Width Modulation (PWM) and Pulse-Frequency Modulation (PFM) are examples of nonlinear sampling. We shall consider the analysis and stability of such systems in later sections of this paper.

2.1. Data reconstruction

Reconstruction of sampled signals is performed by clamping and extrapolation devices. The simplest data reconstruction device is the zero-order hold, which produces an output:

$$
x_h(t) = x(kT), \quad kT \leq t < (k + 1)T,
$$

i.e. the output is held constant between samples at the last sampled value. More complex devices can be used to extrapolate from sampled values with an arbitrary polynomial.

2.2. Quantization

If the sampled signal is to be used in a digital computer, it cannot assume any arbitrary value but can only take on a finite sequence of amplitudes dependent on the register length in the computer. Thus, the sampled signal must also be quantized in amplitude. Quantization is a nonlinear operation, which may or may not be negligible depending on the resolution available in the computer. In this chapter, we shall ignore the effect of quantization.

2.3. Frequency characteristics

The unit pulse train $p(t)$ of Eq. (1) can be represented by a Fourier series in the form:

$$
p(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jw_k t},
$$

where $w_k = 2\pi / T$ is the sampling frequency and the coefficients $c_k$ are:

$$
c_k = \frac{1 - e^{-jw_k h}}{jkw_k T},
$$

where again $h$ is pulse width. With the aid of Eq. (5), the sampler output (Eq. (2)) can be written as:

$$
x^*(t) = x(t) \left( \frac{h}{T} + \frac{2h}{T} \sum_{k=1}^{\infty} \frac{\sin(k\pi h/T)}{k\pi h/T} \cos (w_k t - \phi) \right),
$$

where $\phi = h_n / 2$. Eq. (7) shows that pulse-amplitude modulation involves multiplication by a function which contains an infinite number of harmonics of the sampling frequency. Consequently, the output of the sampler contains not only the original signal frequencies, $w_k$, but also an infinite number of sideband frequencies, $w_k \pm kw_s$, $k = 1, 2, \ldots$

2.4. Impulse modulation

The mathematical representation of the ordinary sampling process can be simplified if the sampler is replaced by an idealized sampler called an impulse modulator. The sampling pulse train, $p(t)$, is then replaced by:

$$
p(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),
$$

which is a train of impulses. If the input to the idealized sampler is a continuous signal, $x(t)$, defined for $t \geq 0$, then the sampler output is:

$$
x^*(t) = \sum_{k=0}^{\infty} x(kT) \delta(t - kT),
$$

which is a modulated impulse train. Eq. (9) can be viewed as a limiting case of Eq. (2), if the finite pulse-width sampler is assumed to have a gain of $1/h$, so that a particular output pulse is given by:

$$
h^{-1} [u(t - kT) - u(t - kT - h)] x(kT),
$$

for $h \ll T$, and we allow $h$ to approach zero. Unfortunately, the problems raised by the use of generalized functions, such as $\delta(t)$ have only been treated heuristically in the literature on ordinary sampled-data systems [2–4], but the process can be defined rigorously [5].

2.5. Stationarity

Assume that signals in a continuous time-invariant linear system are sampled. Since the sampler is a time-varying amplifier, the resulting signals will be nonstationary. The sequence $\{x(kT)\}$ however will be stationary.
Consider the system of Figure 2 where the output of an idealized sampler (impulse modulator) is used as the input to a linear time-invariant system described by its weighting function, \( g(t) \). Since the input to the plant is a train of impulses, output \( y(t) \) can be obtained by summing the impulse responses, i.e.:

\[
y(t) = \sum_{k=0}^{\infty} g(t - kT)x(kT),
\]

where \( x(t) = 0 \) for \( t < 0 \). At the \( n \)th sampling instant:

\[
y(nt) = \sum_{k=0}^{n} g(nt - kT)x(kT),
\]

which is known as the convolution summation. The sequence \( \{g(kT)\} \) by its analogy with weighting function \( g(t) \) is called the weight sequence. Eq. (12) is a linear relation between the input sequence \( \{x(kT)\} \) and the output sequence \( \{y(kT)\} \), and consequently characterizes a discrete system. It can be noted that such a relationship among discrete values of continuous signals does not require the actual presence of samplers in the system. The summation of Eq. (11) gives the output or response of the system for all values of time. If sampling instants are denoted by \( t_k \), \( 0 \leq k \leq n \), then Eq. (12) can be written as:

\[
y(t_n) = \sum_{k=0}^{n} g(t_n - t_k)x(t_k),
\]

which is valid even if the sampling is not periodic.

### 3.1. Difference equations

The input and output sequences of a discrete dynamic system are related by difference equations, analogous to the differential equations relating continuous dynamic systems. The difference equations will have constant coefficients for periodic sampling. An \( n \)th order system is described by an equation that expresses the output at any sampling instant, in terms of input and output values at the \( n \) past sampling instances.

**Example 1.** Let the system of Figure 2 be a double integration defined by:

\[
g(t) = t.
\]

Then applying Eq. (13),

\[
y(t_n) = \sum_{k=0}^{n} (n-k)T x(t_k),
\]

where the sampling is periodic. Consequently,

\[
y(t_{n+1}) = \sum_{k=0}^{n} (n-k)T x(t_k) + T \sum_{k=0}^{n} x(t_k),
\]

and:

\[
y(t_{n+2}) = \sum_{k=0}^{n} (n-k)T x(t_k) + 2T \sum_{k=0}^{n} x(t_k) + Tx(t_{n+1}).
\]

Combining Eq. (15) through (17):

\[
y(t_{n+2}) - 2y(t_{n+1}) + y(t_{n}) = Tx(t_{n+1}).
\]

This is a second-order linear difference equation, which relates the input and output of the system at the sampling instants.

### 4. Frequency domain analysis of conventional linear sampled-data systems

Consider the simple system of Figure 2 again. In accordance with the definition of the impulse modulator, the sampled signal, \( X^*(t) \), is given by:

\[
X^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT).
\]

The Laplace transform of this signal is given by:

\[
X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-ksT}.
\]

It can be shown that an equivalent representation is:

\[
X^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(s + nj\omega_s) + \frac{1}{2} x(0).
\]

where \( x(0) \) is the initial value of the time function and \( X(s) \) is its Laplace transform. Eqs. (20) and (21) show that the sampled function is periodic in the frequency domain, i.e.:

\[
X^*(j\omega) = X^*(j\omega + nj\omega_s).
\]

The Laplace transform of the system output is obtained from Eq. (20):

\[
Y(s) = X^*(s)G(s),
\]

where:

\[
G(s) = \mathcal{L}[g(t)].
\]

By direct application of the above relationships, transform of the sampled output is:

\[
Y^*(s) = X^*(s)G^*(s).
\]

This expression is periodic in \( \omega_s \). To avoid the difficulties connected with the evaluation of the infinite series of Eq. (25), the \( s \) plane is commonly mapped into a new complex plane, called the \( z \) plane, by the transformation:

\[
z = e^{sT}.
\]

Periodicity is eliminated by this transformation, since horizontal strips of the left half of the \( s \) plane, \( \omega_s \) rad/s in height, overlie the inside of the unit circle in the \( z \) plane [2,3].

Applying the transformation of Eq. (26) into Eq. (20), we have the following definition:

**Definition 1.** The \( z \)-transform of function \( x(t) \) is the function, \( X(z) \), of the complex variable, \( z \), defined by:

\[
X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}.
\]

If we now define \( G(z) \) as the \( z \)-transform of the weighting sequence \( \{g(nt)\} \), Eq. (25) becomes:

\[
Y(z) = X(z)G(z).
\]
G(z) is known as the “pulse transfer function”. Inversion of Eq. (28) gives information on the behavior of the output signal only at the sampling instants. It should be noted that the definition (Eq. (27)) is ambiguous, if \(x(t)\) has discontinuities at any of the sampling instants. We therefore require that if \(X(z)\) is to exist, and \(x(t)\) has any discontinuities at the sampling instants, then, \(x(nT^-)\) and \(x(nT^+)\) must exist, and Eq. (27) is written:

\[
X(z) = \sum_{k=0}^{\infty} x(kT^+)z^{-k}.
\]

The above relationships are the basis of the so-called “z-transform method” of analysis of conventional sampled-data systems. The method was introduced by Hurewicz [6] in the US, Barker [7] in England and Tsyupkin [8] in the USSR. The application of z-transforms to the analysis and design of sampled-data systems is treated extensively in several texts [2–4, 9].

### 4.1. Other applications of z-transforms

While the z-transform method is primarily applicable to systems with conventional sampling, and with all sampling operations synchronous, it can be extended to certain other problems. By appropriately delaying or advancing the functions to be sampled, response between sampling instants can be obtained. This variation is called the “modified z-transform” [2–4, 7]. Systems with several samplers, where the sampling periods are staggered but equal, and systems where certain samplers operate at different frequencies (generally multiples of one another), can also be analyzed by this method, but the labor involved can be considerable. If the sampling period varies in length periodically, z-transforms can still be applied [3, 10, 11]. However, even specialized techniques become inapplicable in the nonlinear case.

### 4.2. Relation of z-transforms and difference equations

The z-transform equation of Eq. (28) is a relationship between the sequences \(\{x(nt)\}\) and \(\{y(nt)\}\), and consequently it may be expected that a very close relationship exists between this equation and the corresponding difference equation. It is easy to show from the definition (Eq. (27)) that if:

\[
Z(x(nt)) = X(z),
\]

then:

\[
Z(x(n+1)T) = zX(z).
\]

This relationship immediately establishes a relationship between corresponding terms in a z-transform expression and in a difference equation.

**Example 2.** Consider the system of Example 1 again. By applying the definition (or consulting tables), if \(g(t) = t\),

\[
G(z) = \frac{Tz}{(z - 1)^2} = \frac{Y(z)}{X(z)}.
\]

This expression can be expanded to yield:

\[
z^2Y(z) - 2zY(z) + Y(z) = TZX(z).
\]

Applying Eq. (29) to each term in Eq. (31), one obtains:

\[
y(n + 2)T - 2yn + 1)T + y(nt) = Tx(n + 1)T,
\]

which is identical with Eq. (18). Thus z-transforms are a useful way of obtaining the difference equation in a linear discrete system.

### 5. The state-space formulation of discrete-time problems

The concepts of state and state transformation [12] provide a unifying framework for the description of discrete-time systems. The use of state-space concepts in the analysis of discrete-time systems in this country is due in large part to the work of Kalman and Bertram [13–16] and Bellman [17]. The state-space approach makes it possible to formulate both linear and nonlinear problems in a uniform and concise manner without restriction to conventional sampling methods. The resulting equations are well suited to digital computer solution. The concepts of vector spaces will be used in this chapter in a formal and heuristic manner; for more rigorous treatment, the reader is urged to consult the references.

#### 5.1. Intuitive definition of the state of a dynamic system

Assume that the initial conditions, which describe a dynamic system at time \(t_0\) are known. Then, the state of the system at any time, \(t > t_0\), is a minimum set of quantities (called the state variables), sufficient to describe the present and future outputs of a system, provided the inputs to the system are known. (For a more rigorous definition, see [1, 12].) For a continuous system characterized by an nth order differential equation, it is clear that such a set of quantities is the system output, \(y(t)\), and its \((n - 1)\) time derivatives:

\[
y^{(1)}(t), y^{(2)}(t), \ldots, y^{(n-1)}(t).
\]

Thus, these quantities determine the state of the system at time \(t\), and can be considered the elements of a vector, \(y(t)\), the state vector.

In Section 3, it was shown that an unforced nth order discrete system can be represented by means of a difference equation, which relates the value of the output signal at the \(k\)th sampling instant and at \(n\) past sampling instants. Thus, quantities \(y(t_k), y(t_{k-1}), \ldots, y(t_{k-n})\) represent the state of the discrete system.

#### 5.2. State transition equations of continuous dynamic systems

To illustrate these concepts, consider first a linear continuous system characterized by the vector differential equation:

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0,
\]

where \(x\) is an \(n\) vector (the state vector), \(u\) is an \(m\)-dimensional input vector, \(A\) is an \(n \times n\) matrix, and \(B\) is an \(n \times m\) matrix. The solution of this equation is given by [18, 19]:

\[
x(t) = \Phi(t - t_0)x(t_0) + \int_{t_0}^{t} \Phi(t - r)Bu(r)dr,
\]

where \(\Phi\) is the state transition matrix of the system (Eq. (33)), defined by:

\[
\Phi(t) = e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k.
\]

The solution can also be obtained by taking Laplace transforms of each term in Eq. (33). This procedure yields:

\[
X(s) = (sI - A)^{-1}x(t_0) + (sI - A)^{-1}Bu(s),
\]

where \(I\) is the unit matrix and \(X(s) = L\{x(t)\}\). By comparing Eqs. (34) and (36), the state transition matrix for this system can be defined as:

\[
\Phi(t - t_0) = L^{-1}[(sI - A)^{-1}].
\]
Example 3. Let the linear constant-coefficient system be described by the differential equation:
\[ y' + a_1 y + a_0 y = b u(t), \]
\[ y(0) = c_0, \quad y'(0) = c_1, \]  
(38)
where constants \( a_1, a_0 \) and \( b \) are not zero. Then if we let:
\[ x_1 = y, \quad x_2 = y'. \]  
(39)
Eq. (38) can be written in the form of the vector differential equation (33), with the matrices of coefficients \( A \) and \( B \) being given by:
\[ A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \]  
(40)
and the state of the system is given by:
\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}. \]  
(41)
The solution can be obtained using Eq. (34). Let us turn now to the discrete case.

5.3. State transition equations of linear discrete systems

We begin by considering linear sampled-data systems with conventional sampling and assume that we are interested only in the system behavior at sampling instants. In many systems, the sampling process can be idealized sufficiently, so that the system can be described by a finite set of quantities, e.g. the values of the input and output at time \( t_k \) and at \( n \) past sampling instants. (The difference equation formulation of Eq. (27) is an example.) Thus by the definition above, these quantities can be considered state variables and they constitute the components of a state vector, \( x(t_k) \). The dynamic behavior of such systems can then be described by vector difference equations of the form:
\[ x(t_{k+1}) = Ax(t_k) + Bu(t_k), \]
\[ x(t_0) = x_0, \]  
(42)
for \( k = 0, 1, 2, \ldots \). As before, \( x_0 \) represents the initial state of the system. Eq. (42) corresponds to vector differential equation (33) for a linear continuous system. (It should be noted that a system need not in fact be sampled for Eq. (42) to apply. If the behavior of a continuous system is observed or measured once every \( T \) seconds, it can be considered a discrete system.)

Example 4. Let a linear discrete system be described by the difference equation:
\[ x(t_{k+2}) + a_1 x(t_{k+1}) + a_0 x(t_k) = u(t_k), \]  
(43)
(where \( u(t_k) \) is the input, and initial conditions are given) or by the z-transform relationship:
\[ X(z) = \frac{1}{z^2 + a_1 z + a_0}, \]  
(44)
If we let:
\[ x_1(t_k) = x(t_k), \]
\[ x_2(t_k) = x_1(t_{k+1}). \]  
(45)
Then Eq. (43) can be written as the equivalent set:
\[ x_1(t_{k+1}) = x_2(t_k), \]
\[ x_2(t_{k+1}) = -a_0 x_1(t_k) - a_1 x_2(t_k) + u(t_k). \]  
(46)
This set of first-order scalar difference equations can be written as the vector difference equation:
\[ x(t_{k+1}) = Ax(t_k) + Bu(t_k), \]
\[ x(t_0) = x_0, \]  
(47)
where:
\[ x(t_k) = \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix}, \]  
(48)
is the state vector and \( x_0 \) represents the initial state. The constant matrices, \( A \) and \( B \), are given by:
\[ A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  
(49)
The solution of linear difference equation (47) is analogous to that of the continuous case. Taking z-transforms of Eq. (47), we obtain:
\[ zX(z) = AX(z) + BU(z) - z x(t_0), \]  
(50)
where the transform of each term of \( x(t_k) \) is obtained. Solving for \( X(z) \):
\[ X(z) = (zI - A)^{-1} x(t_0) + (zI - A)^{-1} BU(z). \]  
(51)
Since \( A \) is a constant matrix, the inverse transform of Eq. (51) can be evaluated to yield:
\[ x(t_k) = \Phi(t_k) x(t_0) + \sum_{n=0}^{k-1} \Phi(t_{k-n} - t_0) Bu(t_n), \]  
(52)
where the state transition matrix is given by:
\[ \Phi(t_k) = Z^{-1} \left\{ (zI - A)^{-1} z \right\}, \]  
(53)
and the second term on the right-hand side is the matrix form of the convolution summation.

The state transition matrix can also be obtained by considering the unforced system, and this approach yields considerable insight into the meaning of the state transitions. Consider the system of Eq. (47) without a forcing term, i.e.:
\[ x(t_{k+1}) = Ax(t_k). \]  
(54)
If the initial state of the system is denoted by \( x(t_0) \), then at the first sampling instant \( t_1 = t_0 + T \)
\[ x(t_1) = Ax(t_0), \]  
(55)
which is a transformation of the initial state. At the second sampling instant,
\[ x(t_2) = Ax(t_1) = A^2 x(t_0), \]  
(56)
and at the \( k \)th sampling instant,
\[ x(t_k) = A^k x(t_0). \]  
(57)
Consequently, this vector difference equation represents a successive series of transformations of the initial state. Comparison of Eq. (57) with Eq. (52) shows that the state transition matrix can also be written as:
\[ \Phi(t_k) = A^k, \]  
(58)
for the linear sampled-data system with conventional sampling.

Vector difference Eq. (47) can be viewed as a recurrence relation, which describes the state at time \( t_{k+1} \), given the state and the input at time \( t_k \). Such expressions are conveniently solved on digital computers.
Example 5. As a detailed example of the formulation and solution of the vector difference equation for a linear discrete system, consider the block diagram of Figure 3. Let \( h(t) \) represent the output of the hold circuit, and let:

\[
\begin{align*}
x_1 &= y, \\
x_2 &= \dot{y}.
\end{align*}
\]

Then we can write directly:

\[
\begin{align*}
h(t_k) &= u(t_k), \\
x_2(t_{k+1}) &= x_2(t_k) + Th(t_k), \\
x_1(t_{k+1}) &= x_1(t_k) + Tx_2(t_k) + T^2/2h(t_k).
\end{align*}
\]

Making appropriate substitutions, these equations can be written:

\[
\begin{align*}
x(t_{k+1}) &= Ax(t_k) = Bu(t_k), \\
x(t_0) &= C,
\end{align*}
\]

where \( x(t_k) \) is the state vector at time \( t_k \),

\[
x(t_k) = \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix},
\]

and:

\[
A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T^2/2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},
\]

\( c_1 \) and \( c_2 \) are constants. The complete solution of this equation can be obtained from Eq. (52). We first obtain the state transition matrix using Eq. (53).

\[
\begin{align*}
[zI - A]^{-1} &= \begin{bmatrix} 1 & T \\ z & z-1 \end{bmatrix}, \\
\Phi(t_k) &= Z^{-1} \{[zI - A]^{-1}z\} = \begin{bmatrix} 1 & kT \\ 0 & 1 \end{bmatrix}.
\end{align*}
\]

It is easy to verify, using Eq. (53), that \( \Phi(t_k) = A^k \). Consequently, the complete solution may be written:

\[
x(t_k) = \begin{bmatrix} 1 & kT \\ 0 & 1 \end{bmatrix} x(t_0) + \sum_{n=0}^{k-1} \left( T^2/2 + (k - 1 - n)T^2 \right) u(t_n).
\]

This expression describes the state of the system at time \( t_k = kt \), in terms of the control input, \( u(t) \), and the initial state, \( x(t_0) \).

### 5.4. Closed-loop systems

The techniques described above apply directly to the closed-loop case. If the control input is the closed-loop error, as in Figure 4, it is clear that:

\[
u(t_k) = r(t_k) - x_1(t_k),
\]

where \( r(t) \) is the reference input. With this additional relationship, to supplement the open-loop difference equations of the previous paragraphs, the closed-loop system can be described completely.

### 5.5. Discrete systems with nonconventional sampling

One advantage of the state-space formulation is that it provides a unified approach to the study of discrete-time problems [14], even when the sampling is non-conventional, in cases where:

(a) Response between sampling instants is desired,
(b) Sampling period \( T_k \) is not constant, but is a periodic function of \( k \),
(c) Sampling operations in the system are not synchronized,
(d) Multirate sampling is present,
(e) Finite pulse width or non-instantaneous sampling is present.

If the plant is linear, the vector difference equation obtained in these cases will also be linear, but in general will have time-varying coefficients. If the plant is nonlinear, or when the sampling depends on the state of the system (such as pulse-width modulated discrete systems), the resulting nonlinear difference equations may be written:

\[
x(t_{k+1}) = f(x(t_k), u(t_k)).
\]

where \( f \) is a vector-valued vector function describing the functional relationship. To illustrate the applicability of the state-space formulation, three examples will be used; two linear and one nonlinear.

Example 6 (Discrete System with Nonsynchronized Sampling). Consider the system of Figure 5 where both samplers are synchronized, but Sampler \( S_2 \) lags behind Sampler \( S_1 \) by \( r \) seconds. (This example is based on one published by Bertram [16].) The formulation of the difference equation is facilitated by careful selection of state variables and division of the sampling interval into subintervals. The output can always be selected as one of the state variables, as indicated in Figure 5. It should be noted that two kinds of state transitions take place in this system:

(a) Transitions of sample-and-hold elements, which occur at the sampling instants,
(b) Transitions of the continuous elements, which occur during the intervals between samples.

Consequently, we divide interval \( T = t_{k+1} - t_k \) into four subintervals, and consider the following relationships:

\[
\begin{align*}
I & \quad x(t_k + T^-) = \Phi_1 x(t_k^-), \\
II & \quad x(t_k + T^+) = D_2 x(t_k + T^-), \\
III & \quad x(t_{k+1}^-) = \Phi_2 x(t_k + T^+), \\
IV & \quad x(t_{k+1}^+) = D_1 x(t_{k+1}^-).
\end{align*}
\]

Relations I and III represent the transitions of the continuous elements, and Relations II and IV those of the discrete (sample-and-hold) elements. The matrices, \( D_1 \) and \( D_2 \), represent the
The effects of sampling of samplers $S_1$ and $S_2$, respectively, and are given by:

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  \hfill (70)

(The input $r(t)$ is assumed equal to zero.) The transition of the system during the interval $(t_k^-, t_k^+)$ is given by:

$$\Phi_1(t) = \begin{bmatrix} 1 & 0 & \tau & 0 \\ 0 & e^{-\tau} & 0 & (1 - e^{-\tau}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  \hfill (71)

and it can be seen that:

$$\Phi_2 = \Phi_1(T - \tau).$$  \hfill (72)

Expressions (69) can be combined to yield the vector difference equation:

$$x(t_{k+1}) = A(x(t_k),$$  \hfill (73)

where the state transition matrix is given by:

$$A(t) = D_1 \Phi_1(T - \tau) D_2 \Phi_1(t).$$  \hfill (74)

As is customary, sampling instant $t_k$ in Eq. (73) is understood as $t_k^+$.  

**Example 7** *(Multirate Sampled-data System)*. The analysis of systems, where two or more samplers operate at different sampling frequencies, can be carried out in the frequency domain by means of the modified $z$-transform [2-4,9]. The state-space formulation of such problems proceeds in a manner analogous to that of the previous example, where the largest sampling interval is subdivided into several subintervals.

Consider the system illustrated in Figure 6, where we again assume $r(t) = 0$. Sampler $S_1$ is periodic with a period of 2 s. Sampler $S_2$ is aperiodic having a period which is alternately 0.5 and 1.5 s. Therefore, the sequence of sampling operations is: at time $t_k$, both $S_1$ and $S_2$ sample; at time $t_k + 0.5$, only $S_2$ samples and; at time $t_{k+1} = t_k + 2.0$ s both $S_1$ and $S_2$ sample.

We describe the system by the following state transitions:

I. $x(t_k + 0.5^-) = \Phi_1 x(t_k^+)$,

II. $x(t_k + 0.5^+) = E_1 x(t_k + 0.5)$,

III. $x(t_{k+1}^-) = \Phi_2 x(t_k + 0.5^+)$,

IV. $x(t_{k+1}^+) = E_2 x(t_{k+1}^-)$.

Again, Relations I and III represent continuous transitions, while II and IV represent discrete transitions. The discrete (sample-and-hold) transitions are given by:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$  \hfill (75)

and during the interval $(t_k^+)$, $t_k + 0.5^+$ is described by Relation (75)-I where:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  \hfill (76)

The behavior of the system during the interval $(t_k^+, t_{k+1}^-)$ is described by Relation (75)-I where:

$$\Phi_1 = \begin{bmatrix} 1 & \left(\frac{1}{2} + e^{-\frac{t}{2}}\right) & \left(1 - e^{-\frac{t}{2}}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$  \hfill (77)

and during the interval $(t_k + 0.5^+, t_{k+1}^+)$ is described by Relation (75)-III where:

$$\Phi_2 = \begin{bmatrix} 1 & \left(\frac{1}{2} + e^{-\frac{t}{2}}\right) & \left(1 - e^{-\frac{t}{2}}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$  \hfill (78)
Consequently, the vector difference equation describing the system at the sampling instants of the slower sampler is:

\[ x(t_{k+1}) = \Phi_1 x(t_k), \]  

(79)

where the overall state transition matrix is defined by:

\[ \Phi_1 = E \Phi_0 E \Phi_1. \]  

(80)

**Example 8 (Discrete System with Nonlinear Gain).** Consider the system of Figure 7 which adds a nonlinear gain (e.g., a saturating amplifier) to the system previously considered in Example 3. Now, transform techniques fail, but recurrence relations can be formulated in a straightforward manner. For this simple case, they can be written by inspection as:

\[ x_1(t_{k+1}) = x_1(t_k) + T x_2(t_k) + \frac{T^2}{2} g(u(t_k)), \]

\[ x_2(t_{k+1}) = x_2(t_k) + T g(u(t_k)), \]  

(81)
or in vector form:

\[ x(t_{k+1}) = f(x(t_k), u(t_k)). \]  

(82)

6. Stability of discrete time systems

6.1. Definitions of Stability

We concentrate in this section on the unforced system (i.e., \( u(t_k) = 0 \) for all \( t_k \)) represented by equation:

\[ x(t_{k+1}) = f(x(t_k)). \]  

(83)

Let us assume that the system has an equilibrium state denoted by \( x_e \). Then mathematically the system is in its equilibrium state, if:

\[ x_e = f(x_e). \]  

(84)

That is, the equilibrium state has the property that if initial state \( x(t_0) = x_e \), then repeated iterations of the transformation (Eq. (83)) do not result in any change of state. The question of engineering interest, however, is whether the system will return to the equilibrium state if disturbed. Intuitively, if the system remains near the equilibrium state, it is called stable. If it is stable and tends to the equilibrium state, as \( k \to \infty \), it is called asymptotically stable. If the system is asymptotically stable, regardless of the magnitude of the disturbance from equilibrium, it is called asymptotically stable in the large, or globally asymptotically stable.

Let us now formulate these definitions more precisely [13,15]. Assume that equilibrium is at the origin (this can always be accomplished by a translation of coordinates), i.e., \( x_e = 0 \). Let \( \|x\| \) denote the Euclidean norm of the vector \( x \):

\[ \|x\| = (x'x)^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}. \]  

(85)

**Definition 2.** The equilibrium solution, \( x_e \), is stable if given any \( \varepsilon > 0 \), there exists a \( \delta (\varepsilon) > 0 \), such that for all initial states \( x_0 \), in the sphere of radius \( \delta \)

\[ \|x_0 - x_e\| \leq \delta, \]

the solution for all \( k, x(t_k) \), remains within a sphere of radius \( \|x(t_k) - x_e\| < \varepsilon \).

If the equilibrium solution is stable and if in addition;

\[ \lim_{k \to \infty} \|x(t_k) - x_e\| = 0, \]

the solution is asymptotically stable. If in addition radius \( \delta \) of the initial disturbance can be arbitrarily large, the solution is asymptotically stable in the large.

It is important to note that stability is defined, here, in terms of the motion of the state of the system in state space, not in terms of the system output.

6.2. Stability of linear sampled-data systems

Unforced linear sampled-data systems can be represented by:

\[ x(t_{k+1}) = A x(t_k), \]  

(86)

where \( A \) is a constant (time-invariant) matrix (see Example 5). The solution is of the form:

\[ x(t_k) = A^k x(t_0), \]  

(87)
i.e. it represents \( n \) successive transformations of the initial state. The null solution, \( x_e = 0 \), is asymptotically stable in the large, if and only if every element of \( A^k \) tends to zero uniformly with \( k \) as \( k \to \infty \). It can be shown [12,14,19] that this statement implies that the system is globally asymptotically stable, if and only if the roots, \( \lambda_1, \lambda_2, \ldots, \lambda_n \), of the characteristic equation:

\[ \det(A - \lambda I) = 0, \]

(88)

(eigenvalues of \( A \)) satisfy the condition:

\[ |\lambda_i| < 1, \quad i = 1, 2, \ldots, n. \]

This statement is equivalent to the statement that the poles of the closed-loop pulse transfer function of the system must lie inside the unit circle in the \( z \) plane.

**Example 9.** Consider the simple system of Figure 4 again, where we assume \( T = 1 \) for simplicity. The pulse transfer function is given by:

\[ G(z) = \frac{Y(z)}{U(z)} = \frac{1 - e^{-T}}{s^2} = \frac{T^2(z - 1)}{2(z - 1)^2}. \]  

(88)

The denominator of the closed-loop pulse transfer function (characteristic equation) is (for \( T = 1 \)):

\[ 1 + G(z) = z^2 - 3/2z + 3/2 = 0, \]  

(89)

and since for roots \( z_i, |z_i| > 1, \) the system is unstable.

The difference equations are written, as in Example 5, with the additional feature that the system is closed-loop; consequently, Eq. (67) applies. For \( r(t) = 0 \), the vector difference equation is Eq. (86), where:

\[ A = \begin{bmatrix} 1/2 & 1 \\ -1 & 1 \end{bmatrix}, \]  

(90)

from which the characteristic equation is obtained as:

\[ |A - \lambda I| = \begin{bmatrix} (1/2 - \lambda) & 1 \\ 1 & (1 - \lambda) \end{bmatrix} = 0, \]  

(91)

which is identical with Eq. (89), if \( \lambda \) is substituted for \( z \).

6.3. Stability of linear systems with inputs

The definitions of asymptotic stability given above are based on the free or unforced behavior of the state in state space. If bounded input vectors are applied to an asymptotically stable
system, the state vector remains bounded. This statement can be made more precise in the following theorem [12], given here without proof:

**Theorem 1.** If a linear discrete system is described by:
\[ x(t_{k+1}) = Ax(t_k) + Bu(t_k), \]  
and the eigenvalues of \( A \) are in the open disk, \( |\lambda| < 1 \), then for all initial states, any bounded input vector sequence, \( [u(t_k)] \), produces a bounded state vector, \( x(t_k) \).

In other words, if \( \|u(t_i)\| < M, i = 0, 1, 2, \ldots, \) where \( M \) is a real positive number, then, it is possible to find a real positive number, \( C \), such that \( \|x(t_k)\| < C\|x(t_0)\| \) for all \( k \), where \( x(t_0) \) is the initial state.

6.4. Applications of the second method of Lyapunov to discrete systems

The so-called “direct” or “second method” of Lyapunov for determining the asymptotic stability of nonlinear differential equations has become quite important in recent years. The method is based on finding a scalar function of the state variables of the system that satisfies certain conditions. If such a function, called a Lyapunov function, does indeed exist, then the null solution of the differential equation is asymptotically stable in the large. The importance of the method is based on the fact that the stability information is obtained without having to solve the differential equation. Much less literature is available on the use of the Lyapunov method for determining asymptotic stability (either global or local) of difference equations. The basic references are those of Hahn [20] and Kalman and Bertram [13,15]. From these references, the following stability theorem can be stated (for proof, see [13]):

**Theorem 2.** If for the vector difference equation:
\[ x(t_{k+1}) = f(x(t_k)), \]
there exists a scalar function of the state variables, \( V(x) \), such that \( V(0) = 0 \) and:

(i) \( V(x) > 0 \) when \( x \neq 0 \),
(ii) \( V(x(t_{k+1})) < V(x(t_k)) \) for \( k > K, K \) finite,
(iii) \( V(x) \) is continuous in \( x \),
(iv) \( \lim V(x) = \infty \) when \( \|x\| \to \infty \),

then the equilibrium solution, \( x = 0 \), is asymptotically stable in the large and \( V(x) \) is a Lyapunov function for this system. (It should be noted that this is only a sufficient condition.)

As with continuous systems, considerable ingenuity is required to find appropriate Lyapunov functions. For example, Bertram [15] discusses the application of functions of the form:
\[ V_1(x) = \sum_{i=1}^{n} c_i|x_i|, \]  
(93)

(where the \( c_i \) are constants) to the study of stability of sampled systems with nonlinear gain elements. For similar systems with finite pulse width, Kadota [21] uses functions of the form:
\[ V_2(x) = \sum_{i=1}^{n} e^{\lambda_i t} x_i^2. \]  
(94)

Clearly, both functions are positive definite in the whole space. Examples of the use of the Second Method in the study of asymptotic stability of nonlinear discrete-time systems will be found in the following sections.

7. Pulse frequency modulated sampled data systems

In all systems previously considered, information was transmitted using pulse-amplitude modulation, and the sampling intervals were assumed either fixed or periodically time-varying. If, however, the sampling intervals are functions of the state variables, the system becomes nonlinear. An illustration of such a system is presented in this section [22].

7.1. Difference equations of the system

Consider the example illustrated in Figure 8. The \( n \)th sampling interval is defined as:
\[ T_n = T_{n+1} - T_n. \]  
(95)

Note that the hold periods will be variable, as well as the sampling periods. Let a control law governing the variable sample and hold device be given as:
\[ T_n = \frac{\alpha}{|e(t_n)| + 1}, \]  
(96)

with the result that a large error results in an increase in the sampling frequency.

By using the techniques of previous sections, system equations at the sampling instants may be written as:
\[ x(t_n) = e(t_n), \]  
(97)

\[ y(t_{n+1}) = y(t_n) \exp(-2T_n) + Ke(t_n) (1 - \exp(-2T_n)) \]  
(98)

where \( T_n \) is given by Eq. (96). Eq. (97) can be combined to yield a single expression:
\[ y(t_{n+1}) = -y(t_n) [K(1 - K) \exp(-2a/|y(t_n) - y(t_0)|)] \]  
(99)

This is a nonlinear difference equation, which can be solved by sample and hold, if the initial state, \( y(t_0) \), is known and the input, \( r(t_n) \), is specified for \( t_n > t_0 \).

7.2. Determination of asymptotic stability

Let us now try to find a Lyapunov function for the unforced system, to determine a sufficient condition for asymptotic stability in the large. If \( r(t) = 0 \), Eq. (98) reduces to:
\[ y(t_{n+1}) = Ky(t_n) + (1 + K)g[y(t_n)], \]  
(100)

where:
\[ g(y(t_n)) = y(t_n) \exp(-2a/|y(t_n)| + 1). \]  
(101)

This is a nonlinear vector difference equation, where the state vector \( y(t_n) \) has only one component.
Let us pick, for a prospective Lyapunov function, the square of the Euclidean norm:

\[ V_1(y(t_n)) = ||y(t_n)||^2 = y(t_n)^2. \]  

(101)

The first difference of \( V \) is:

\[ \Delta V_1(y) = y(t_{n+1})^2 - y(t_n)^2. \]  

(102)

Since \( V_n(y_n) \) is, by inspection, positive-definite, continuous in \( y_n \) and tends to \( \infty \) as \( ||y_n|| \to \infty \), all that remains to be shown is that \( \Delta V_1(y_n) \) is negative-definite. This implies the existence of a region in the \( a, K \) parameter space in which:

\[ y_n^2 > y_{n+1}^2 \quad \text{for all } n. \]  

(103)

If Eq. (96) is combined with Eq. (100) to obtain \( g \) as a function of \( T_n \), and the result substituted in Eq. (99), Inequality (103) becomes:

\[ y_n^2[(1 + K)^2 \exp(-4T_n) - 2K(1 + K) \exp(-2T_n) + K^2] < y_n^2. \]  

(104)

where:

\[ T_n = T_n(a, y_n), \]

according to Eq. (96). Note that \( T_n > 0 \) and \( K > 0 \) from physical considerations. Since \( y_n^2 \geq 0 \), inequality (104) is equivalent to:

\[ (1 + K)^2 \exp(-4T_n) - 2K(1 + K) \exp(-2T_n) + K^2 < 1. \]  

(105)

This inequality can be used to determine bounds on parameters \( a \) and \( K \). If such bounds can be found, \( V_1(y_n) \) will qualify as a Lyapunov function, and the system is asymptotically stable in the large.

As an illustration, Let \( K = 2 \) to simplify the arithmetic. Then Eq. (105) becomes:

\[ e^{-2T_n}(9e^{-2T_n} - 12) < -3, \]  

(106)

or since \( e^{-2T_n} \) is greater than zero for all \( T_n > 0 \), we must have:

\[ 3e^{-2T_n} - 4 < - \frac{1}{e^{-2T_n}}. \]  

(107)

If we interpret the two sides of this inequality as equations defining two functions, \( f_1(e^{-2T_n}) \) and \( f_2(e^{-2T_n}) \), we can obtain a graphical interpretation of the stability requirement by plotting \( f_1 \) and \( f_2 \) vs. \( e^{-2T_n} \), as in Figure 9.

The points of intersection are obtained from solution of equation \( f_1 = f_2 \). Therefore, \( \Delta V(y(t_n)) < 0 \) for:

\[ \frac{1}{3} < e^{-2T_n} < 1. \]  

(108)

The upper limit clearly cannot be exceeded, since for all:

\[ T_n > 0, \quad e^{-2T_n} < 1. \]  

The lower limit means that:

\[ -2T_n < \ln \frac{1}{3}, \quad T_n < \frac{1}{2} \ln 3. \]  

(109)

Using the “control law” for \( T_n \) as given by Eq. (96), we have:

\[ T_n = \frac{a}{||y(t_n)|| + 1} < \frac{\ln 3}{2}. \]  

(110)

This expression assumes its maximum value for \( ||y(t_n)|| = 0 \) and consequently:

\[ a = (\ln 3)/2. \]  

(111)

Therefore, provided that \( a \) stays below the limit of Eq. (111), the null solution of the nonlinear system is asymptotically stable in the large.

7.3. Integral pulse-frequency modulation

The system discussed above employed pulse amplitude modulation with an additional degree of freedom introduced by the adjustment of sampling frequency. A pure pulse-frequency modulator would produce a series of equal pulses, the frequency of which is dependent on the modulator input as illustrated in Figure 10. If the modulator input is denoted by \( e(t) \), then the \( n \)th interval between pulses \( T_n = T_{n+1} - T_n \) can be obtained from equation:

\[ \int_{t_n}^{t_{n+1}} e(t)dt = \pm K, \]  

(112)

where \( K \) is a design parameter. This equation indicates that if a pulse is produced at time \( t_n \), the next pulse occurs when the magnitude of the integral reaches \( K \). The pulse then carries the sign of the integral at that time. This type of modulation is known as “integral pulse-frequency modulation” and appears to be similar to a type of modulation occurring in the nervous system [23].
A detailed analysis of an Integral Pulse-Frequency (IPF) modulation attitude control system has been made [24]. This analysis shows by an extension of the second method of Lyapunov that the ultimate state of an IPF-controlled second-order plant is a limit cycle oscillation, to which the system converges asymptotically.

8. Pulse-width modulated discrete-time systems

One of the most interesting areas of study in discrete systems involves the analysis and design of PWM systems. These systems are inherently nonlinear. In this section, we shall review the work on PWM systems by Kadota and Bourne [25,26] and Nelson [27], since they illustrate the usefulness of the concepts discussed above.

8.1. Formulation of the difference equations

The system we consider is illustrated in Figure 11, using a pulse-width modulator, which provides control inputs to a linear continuous plant at time intervals, T. The outputs of the pulse-width modulator will be flat-top pulses of constant amplitude, M, and variable width given by:

\[ u(t) = \begin{cases} 
M \text{ sgn} e(kT), & kT \leq t < kT + h(kT) \\
0, & kT + h(kT) \leq t < (k + 1)T 
\end{cases} \tag{113} \]

where sgn is the signum function and \( h(kT) \) (to be denoted as \( h(k) \) henceforth for simplicity) is the width of the \( k \)th pulse,

\[ h(k) = T \text{ sat} \left( \frac{|e(kT)|}{\beta} \right) \tag{114} \]

where \( \beta \) is a positive constant and the saturation function, sat \( x \), is defined as:

\[ \text{sat } x = \begin{cases} 
+1 & x > 1 \\
x, & |x| \leq 1 \\
-1, & x < -1 
\end{cases} \tag{115} \]

Then \( u(t) \) represents the control input to the plant.

The plant can be described by the vector differential equation:

\[ \dot{x}(t) = Ax(t) + Bu(t), \tag{116} \]

where \( x(t) \) is the state vector of the plant at time \( t \).

Since the plant is assumed linear and invariant, \( A \) is an \( n \times n \) matrix with constant elements, and \( B \) is an \( n \) vector with constant elements. From Eq. (113), the input to the plant is either zero or constant at value \( \pm M \).

Then following the techniques of Section 5, for the interval of time when the input is equal to \( M \), \( t_0 \leq t \leq t_1 \), the solution of Eq. (116) is given by Eq. (34),

\[ x(t) = \Phi(t - t_0)x(t_0) + Mg(t - t_0), \quad t_0 \leq t \leq t_1, \tag{117} \]

where \( \Phi(t) \) is the fundamental matrix or state transition matrix of the differential equation (116)

\[ \Phi(t) = \exp At = \sum_{k=0}^{\infty} A^k t^k/k!, \tag{118} \]

and \( g(t) \) is the forcing vector given by:

\[ g(t) = \int_0^t \Phi(u)Bu \, du. \tag{119} \]

It can be seen that to satisfy Eq. (117) we must have:

\[ \Phi(0) = I, \quad g(0) = 0, \]

where \( I \) is the unit matrix of order \( n \) and \( 0 \) is the null matrix. Other useful properties of \( \Phi(t) \) and \( g(t) \) are [19,27]:

\[ \Phi(u + v) = \Phi(u)\Phi(v), \]

\[ g(-v) = -\Phi(-v)g(v), \tag{120} \]

which arise from the properties of linear systems.

Applying the above general results to the PWM problem, with \( u(t) \) given by Eq. (113), we can write the equation:

\[ x[kT + h(k)] = \Phi[h(k)]x[kT] + u(k)g[h(k)]. \tag{121} \]

for “input-on” time where in Eq. (117) we let:

\[ t_0 = kT, \quad t = kT + h(k), \]

and similarly,

\[ x[(k + 1)T] = \Phi \left( T - h(k) \right) x[kT + h(k)] + 0, \tag{122} \]

for the “input-off” time where:

\[ t_0 = kT + h(k), \quad t = (k + 1)T. \]

Eqs. (121) and (122) can be combined to give:

\[ x[(k + 1)T] = \Phi[T - h(k)] \Phi[h(k)]x[kT] + u(k)g[h(k)]. \tag{123} \]

Using property (Eq. (120)) of the state transition matrix, we can write:

\[ \Phi[T - h(k)]\Phi[h(k)] = \Phi(T), \tag{124} \]

and consequently obtain a single vector difference equation for the system

\[ x[(k + 1)T] = \Phi[T]x[kT] + u(k)\Phi[T - h(k)]g[h(k)]. \tag{125} \]

This equation can be used to study the time behavior of the system.

8.2. Stability of the PWM system

In order to apply the second Method of Lyapunov to the system of Eq. (125), let us first diagonalize matrix \( A \).

We consider the class of systems, where (for simplicity) poles \( a_i \) of the plant transfer function \( G(s) \) are all real and distinct. Then there exists a real nonsingular matrix, \( P \), such that:

\[ P^{-1}AP = J, \tag{126} \]

diagonal matrix, with poles \( a_i \) along its diagonal, i.e.:

\[ J = \begin{bmatrix} 
a_1 & & 0 & & 0 \\
0 & a_2 & & 0 & \\
0 & & \ddots & & \ddots \\
0 & & & \ddots & 0 \\
0 & & & & a_n 
\end{bmatrix}. \tag{127} \]

We then perform the transformation:

\[ x = Py, \]

which maps state space \( X \) onto state space \( Y \). In terms of the new state vector, \( y(kt) \), difference equation (125) can now be
written as:
\[
y[(K + 1)T] = E(T)y(kT) + u(k)E[T - h(k)]f[h(k)],
\]
where:
\[
E(T) = P^{-1} \Phi(T)P
\]
and:
\[
f[h(k)] = P^{-1}g[h(k)] = \begin{bmatrix}
\frac{1}{a_1} (1 - e^{a_1 h_k}) \\
\vdots \\
\frac{1}{a_n} (1 - e^{a_n h_k})
\end{bmatrix}.
\]

By taking advantage of the properties in Eq. (120), Expression (128) can be further simplified to the form:
\[
y[(k + 1)T] = E(T)y(kT) - u(k)[g[-h(k)]].
\]

We now choose, as a Lyapunov function, the square of the generalized Euclidean norm (For a discussion of the suitability of various norms of the state vector for Lyapunov functions, see [15]), of the state vector, \( y \):
\[
V = ||y||_p^2 = \sum_{i=1}^{n} c_i y_i^2,
\]
where \( c_i \) are positive constants and \( V \) is obviously positive definite. Then a sufficient condition for asymptotic stability in the large of the equilibrium solution, \( y_e = 0 \), is that \( \Delta V(y) < 0 \), for all \( y \). To prove this, we follow Kadota [25], and write \( \Delta V[y(kT)] \) in explicit form, using Eq. (131) and:
\[
\Delta V[y(kT)] = V[y(k + 1)T] - V[y(kT)] < 0,
\]
so that:
\[
\Delta V = \sum_{i=1}^{n} c_i \left[ e^{2a_i T} (y_i + u_i)^2 - y_i^2 \right],
\]
where \( y_i \) are the state variables (components of the state vector) and \( u_i \) are defined as:
\[
w_i(k) = \frac{u(k) (e^{-a_i h_k} - 1)}{a_i}.
\]

By manipulation of Eq. (134), it can be shown that Condition (133) reduces to finding constants \( c_i \), such that certain matrices (whose elements are determined by the \( c_i \) and the system parameters) have negative eigenvalues. Consequently, it is possible to find conditions on the system parameters, such that the pulsewidth modulated system of Figure 11 is asymptotically stable in the large. The conditions are then used to instrument the feedback control function, \( e(t) \), to insure stability. These concepts are best illustrated by means of a simple example, based on the work of Kadota [25,26].

**Example 10.** Let the linear plant be defined by the transfer function:
\[
G(s) = \frac{1}{s - a}.
\]

The differential equation for the system during one sampling interval (pulse-on) is:
\[
\dot{x} = ax + u(t),
\]
where \( u(t) \) is defined by Eq. (113). The general solution of Eq. (137) is:
\[
x(t) = e^{a(t - t_0)} x(t_0) + \int_{t_0}^{t} e^{a(t - \tau)} u(\tau) d\tau.
\]
Let the width of the \( k \)th pulse \( h(k) \), \( k = 1 \), such that the pulsewidth modulated system of Figure 11 is and the system parameters) have negative eigenvalues. Consequently, to find conditions on the system parameters, such that the pulsewidth modulated system of Figure 11 is asymptotically stable in the large. The conditions are then used to instrument the feedback control function, \( e(t) \), to insure stability. These concepts are best illustrated by means of a simple example, based on the work of Kadota [25,26].
compensation of sampled-data systems. Frequency-domain
Figure 12, which illustrates two common approaches to the
9.1. Synthesis of discrete controllers using z-transform techniques
optimal discrete system is discussed briefly.
input signal. The state-space approach to the synthesis of
that achieves minimum settling time for particular classes of
parameters $a$, $T$, $M$, and $\beta$ must now be selected in order to satisfy
this relationship, and the system is asymptotically stable in the
large.
The minimum-time control of pulse-width modulated
sampled-data systems was discussed by Polak [28].

9. Synthesis of discrete-time systems

The previous sections of this chapter have been devoted
to the analysis of linear and nonlinear sampled-data systems.
The purpose of this section is to introduce the synthesis
problem, with major emphasis on linear systems capable of
reaching equilibrium states in the minimum possible number of
sampling periods. We shall begin by examining the use of
$z$-transform techniques for synthesizing discrete controllers
that achieve minimum settling time for particular classes of
input signal. The state-space approach to the synthesis of
optimal discrete systems is discussed briefly.

9.1. Synthesis of discrete controllers using $z$-transform techniques

Consider the unity feedback error-$z$-transform sampled systems of
Figure 12, which illustrate two common approaches to the
compensation of sampled-data systems. Frequency-domain
synthesis techniques used with linear continuous systems are
based on the following steps:
1. Performance criteria are formulated as a goal for the
synthesis procedure. These may be such conventional
factors as peak time or maximum overshoot to step inputs,
minimum mean squared error or the time required for the
system to reach an equilibrium state from arbitrary initial
conditions.
2. The closed-loop transfer function is determined from the
specifications of (1) and the plant transfer function.
3. The transfer function of a physically realizable controller is
computed from step (2).
4. The controller is synthesized exactly or approximately.
The steps listed above cannot be applied directly to the
synthesis of the controller, $N(s)$, in Figure 12(a). The closed-loop
pulse transfer function of the system of Figure 12(a) is given by:
$$K(z) = \frac{C(z)}{R(z)} = \frac{G_0 N G_p(z)}{1 + G_0 N G_p(z)}$$
where:
$$G_0 N G_p(z) = Z\{G_0(s)N(s)G_p(s)\}$$
and $G_0(s)$ is the transfer function of the hold circuit. It can be seen
that the controller characteristic, $N(s)$, cannot be isolated in
Eq. (150). Consequently, continuous compensation of
discrete-time systems generally requires the use of approximations
or cut-and-try techniques [2–4,9]. When a discrete
compensator is used, the closed-loop pulse transfer function is:
$$D(z) = \frac{1}{G(z)} \left(1 - \frac{K(z)}{1 + K(z)}\right)$$
$$D(z) = \frac{1}{G(z)} \left(1 - \frac{K(z)}{1 + K(z)}\right)$$
$$D(z) = \frac{1}{G(z)} \left(1 - \frac{K(z)}{1 + K(z)}\right)$$
Consequently, if the desired performance specification can be
incorporated into a closed-loop pulse transfer function, $K(z)$,
the discrete controller pulse transfer function can be obtained
from Eq. (154). Such controllers are commonly referred to
as “digital controllers” in the literature, since the difference
equation represented by $D(z)$ can be implemented on a digital
computer. (The phrase “discrete controller” is more accurate,
however, since $D(z)$ can also be implemented using analog
elements and sample-hold circuits without the amplitude
quantization present in a digital computer. In fact, the nonlinear
effect of quantization may have to be considered in the design of
strictly digital controllers.)
It remains to be shown how performance criteria can be incor-
porated in the selection of an overall pulse transfer func-
tion, $K(z)$. We shall concentrate on the following performance
criteria:
1. Zero steady-state error at the sampling instants.
2. Minimum settling time (i.e., the system error must become
zero in the minimum possible number of sampling periods).
Systems of this type are known as “minimal systems”. To
incorporate these performance specifications in $K(z)$, we
begin by writing the error signal transform:
$$E(z) = R(z)[1 - K(z)]$$
where $R(z)$ is the $z$-transform of the input signal, $r(t)$. If the
steady-state error is to be zero, the final value theorem of
$z$-transforms [2–4] is used:
$$\lim_{n \to \infty} e(nT) = \lim_{z \to 1} (1 - z^{-1}) R(z)[1 - K(z)] = 0.$$ (156)
Consider now polynomial inputs of the form $r(t) = t^m$ for
which:
$$R(z) = \frac{A(z)}{(1 - z^{-1})^m}$$ (157)
where \( A(z) \) is a polynomial in \( z \) with no roots at \( z = 1 \) and \( n = m + 1 \). Substituting in Eq. \( (156) \), it can be seen that a necessary condition for zero steady-state error is that \( 1 - K(z) \) contain the factor \((1 - z^{-1})^n\). Consequently, we must have:

\[
1 - K(z) = (1 - z^{-1})^n F(z),
\]

where \( F(z) \) is a polynomial in \( z^{-1} \), with no roots at \( z = 1 \). The desired closed-loop pulse transfer function is given by:

\[
K(z) = 1 - (1 - z^{-1})^n F(z),
\]

which guarantees zero steady-state error. To obtain minimum settling time, we substitute Eqs. \( (158) \) and \( (157) \) into Eq. \( (155) \), and note that:

\[
E(z) = A(z)F(z).
\]

For the error to settle in the minimum number of sampling periods, \( F(z) \) must be a polynomial of the lowest order possible. If \( G(z) \) has no poles at the origin and no zeros on or outside the unit circle, it is possible to choose \( F(z) = 1 \) and still meet physical realizability considerations. With these conditions, it can be seen that \( K(z) \) takes the following form for the “minimal system”:

- Step input \( K(z) = z^{-1} \),
- Ramp input \( K(z) = 2z^{-1} - z^{-2} \),
- Parabolic input \( K(z) = 3z^{-1} - 3z^{-2} + z^{-3} \),

and so forth. In general, if the input is of the form \( r(t) = t^m \), the minimal response system settles in \((m + 1)\) sampling periods.

The following assumptions have been tacitly made in the above development:

1. The initial conditions are equal to zero;
2. The plant has no poles on or outside the unit circle;
3. The plant contains enough integration to make zero steady-state error possible for the particular input selected.

### 9.2. Comments on minimal synthesis

Unfortunately, the method of synthesis outlined above is completely impractical for the following reasons:

1. Minimum settling time at the sampling instants does not guarantee zero ripple between the sampling instants.
2. The synthesis depends on the cancellation of poles and zeros of \( G(z) \), in order to obtain a closed-loop transfer function, \( K(z) \), with poles only at the origin. Such cancellation is not possible if \( G(z) \) has poles on or outside the unit circle.
3. Multiple poles of \( K(z) \) at the origin are undesirable from the standpoint of sensitivity.
4. Minimal systems are optimum only for the specific input for which they are designed, and in general are not satisfactory for other inputs.

It can be shown that, in order to overcome difficulties (1) and (2), it is necessary to modify the specifications on \( K(z) \), as follows:

(a) \( K(z) \) must be a finite polynomial in \( z^{-1} \),
(b) The zeros of \( K(z) \) must include all the zeros of \( G(z) \),
(c) The zeros of \( 1 - K(z) \) must include all the poles of \( G(z) \) on or outside the unit circle.

These synthesis concepts are best illustrated by means of a simple example.

### Example 11

Consider the system of Figure 12(b), with:

\[
G_p(s) = \frac{1}{s^2},
\]

and a zero-order hold. Minimal response to a ramp input is derived by synthesizing an appropriate discrete compensator. Then:

\[
G(z) = Z \left\{ \frac{1 - e^{-Tz}}{s^3} \right\} = T^2 z^{-1}(1 + z^{-1}) \frac{2}{(1 - z^{-1})^2}.
\]

For a ramp input we choose:

\[
1 - K(z) = (1 - z^{-1})^2 F(z),
\]

and begin by letting \( F(z) = 1 \). Then:

\[
K(z) = 2z^{-1} - z^{-2},
\]

and the compensator becomes:

\[
D(z) = \frac{1}{G(z) - K(z)} = \frac{2(2 - z^{-1})}{T^2(1 + z^{-1})}.
\]

It can be seen that the compensator attempts to cancel a zero of \( G(z) \) on the unit circle. As is well known, imperfect cancellation results in instability. Furthermore, even with perfect cancellation, ripple is present between sampling instants. Consider the system error:

\[
E(z) = R(z)[1 - K(z)] = T z^{-1},
\]

so that the system settles in two sampling instants. The output of the compensator is obtained from:

\[
E_2(z) = E(z)D(z) = \frac{2(2 - z^{-1})}{T^2(1 + z^{-1})} = 2z^{-1} - 3z^{-2} + 3z^{-3} - 3z^{-4} + \ldots.
\]

Thus the minimal system not only requires perfect zero cancellation, but also produces ripple. If we modify the design, by including the zeros of \( G(z) \) in \( K(z) \), we obtain:

\[
1 - K(z) = (1 - z^{-1})^2 F_1(z),
\]

\[
K(z) = (1 + z^{-1}) F_2(z),
\]

where the simplest polynomials to satisfy these relationships become:

\[
F_1(z) = 1 + 4/5z^1,
\]

\[
F_2(z) = (6/5z^1 - 4/5z^2),
\]

and the resulting discrete compensator is:

\[
D(z) = \frac{2(6/5 - 4/5z^1)}{T^2(1 + 4/5z^2)}.
\]

It is easy to show that this improved system requires three sampling periods to settle. Furthermore, \( E_2(z) \) is now a finite polynomial and consequently the system settles with zero ripple.

### 9.3. Introduction to synthesis of time-optimal systems

The synthesis of optimal sampled-data systems, using state-space formulation, has been the subject of intensive research in recent years [29–38]. While a detailed discussion of time-optimal synthesis is beyond the scope of this chapter, a simple formulation of the problem is presented here. Assume that
Consider now state 

\[ x(t+1) = Ax(t) + hu(t), \]  

(172)

where \( x(t) \) is the \( n \)-dimensional state vector, \( u(t) \) is a scalar input, \( A \) is an \( n \times n \) matrix and \( h \) is an \( n \)-dimensional vector. The state of the system at the first two sampling instants is given by:

\[ x(t_1) = Ax(t_0) + hu(t_0), \]
\[ x(t_2) = A^2x(t_0) + Ahu(t_0) + hu(t_1). \]  

(173)

Successive iteration results in:

\[ x(t_N) = A^N x(t_0) + A^{N-1} hu(t_0) + \cdots + hu(t_{n-1}). \]  

(174)

The time-optimal regulator problem is concerned with bringing the state of the system to the origin from an arbitrary initial condition, \( x(t_0) \). Consequently, if we desire \( x(t_N) = 0 \), the above equation can be pre-multiplied by \(-A^{-N}\) and written as:

\[ x(t_0) = A^{-1}hu(t_0) - A^{-2}hu(t_1) - \cdots - A^{-N}hu(t_{n-1}). \]  

(175)

If we collect \( u(t_0), u(t_1), \ldots, u(t_{n-1}) \) into a vector \( U(t_0) \) where:

\[ U(t_0) = u(t_{n-1}), \]  

(176)

we can state the objective of time-optimal synthesis as finding the minimum \( N \) and the corresponding \( U(t_0) \), which satisfy Eq. (175), i.e. which bring the state of the system to the origin of state space. In general, \( U(t_0) \) is constrained in magnitude, so that only those control vectors which satisfy the constraint are considered admissible.

To illustrate one approach to the problem, consider the following simple example [29].

**Example 12.** Let the system to be controlled be given by:

\[ G_p(s) = \frac{1}{s(s+1)}, \]  

(177)

and the input applied through a hold circuit, as in Figure 12(c). The vector difference equation describing the system can be formulated by the techniques of Section 5. If we let the output \( c(t) = x_1(t) \) and \( \dot{c}(t) = x_2(t) \), the system is described by Eq. (172), where:

\[ A(T) = \begin{bmatrix} 1 & (1 - e^{-T}) \\ 0 & e^{-T} \end{bmatrix}, \]
\[ h = \begin{bmatrix} T - 1 + e^{-T} \\ 1 - e^{-T} \end{bmatrix}. \]  

(178)

Now let \( x^{(1)}(t_0) \) be an initial state from which it is possible to reach the origin in exactly one sampling interval. Then from Eq. (175):

\[ x^{(1)}(t_0) = -u(t_0)A^{-1}(T)h(T), \]  

(179)

If we let \( A^{-1}(T)h(T) = v_1 \), we have:

\[ x^{(1)}(t_0) = -u(t_0)v_1. \]  

(180)

Consider now state \( x^{(2)}(t_0) \), from which the origin can be reached in two sampling intervals. Clearly, this is equivalent to being able to reach the state of Eq. (180) in one step. Then, applying Eqs. (175) and (180), we have:

\[ x^{(2)}(t_0) = A^{-1}(T)[-u(t_1)v_1 - u(t_0)h(T)] \]
\[ = -u(t_1)A^{-2}(T)h(T) - u(t_0)A^{-1}(T)h(T). \]  

(181)

Now letting \( A^{-2}h = v_2 \), Eq. (181) becomes:

\[ x^{(2)}(t_0) = u(t_1)v_2 - u(t_0)v_1. \]  

(182)

Now, it can be shown [29] that the two vectors, \( v_1 \) and \( v_2 \), are linearly independent for any value of \( T \). Since any \( n \)-dimensional vector can be represented by a linear combination of \( n \) linearly independent vectors, it follows that, since \( x(t_k) \) in the example is two dimensional, the origin can be reached in two steps. Consequently, Eq. (181) can be solved for \( u(t_1) \) and \( u(t_0) \):

\[ u(t_0) = a_1x_1(t_0) + a_2x_2(t_0), \]
\[ u(t_1) = a_2x_1(t_0) + a_2x_2(t_0). \]  

(183)

This is a sequence of values, which depends only on the initial state, \( x(t_0) \). The \( a_i's \) in Eq. (183) represent terms obtained from the solution of Eq. (182). But if one considers the system at time \( t_1 \), from which it is possible to reach the origin in one sampling interval, it is clear that the second of Eq. (183) can be written as:

\[ u(t_1) = a_1x_1(t_1) + a_2x_2(t_1). \]  

(184)

The significance of this result is that the optimal controller can be instrumented using linear time invariant feedback. In fact, the feedback controller for this case is simply:

\[ F(s) = a_{11} + a_{12}s. \]  

(185)

To obtain the controller coefficients, \( a_{11} \) and \( a_{12} \), in terms of the system parameters, we substitute in Eq. (183). This can be facilitated by noting that:

\[ A^{-1}(T) = AT(-JT), \]  

(186)

and one obtains:

\[ a_{11} = -e^{T} \quad a_{12} = \frac{-e^{2T} - e^{T} - T}{T(e^{T} - 1)^2}. \]  

(187)

The development of a linear discrete time-optimal controller for the same type of linear system is discussed by Mullin [38].

When restrictions are placed on the class of admissible control vectors, \( U(t_k) \), a possible approach to the problem is to divide the state space into regions, such that an optimum solution is known if the system is in one of these regions [29–33]. A similar approach can be extended to the synthesis of optimal pulse-width modulated discrete systems [28,39,40] where regions from which convergence to the origin in a minimum number of sampling instants are found. Another approach is based on finding a linear functional that completely specifies the solution [31,36,37]. It can also be shown [31] that under certain conditions, time-optimal discrete controls approach continuous time optimal control if the sampling period is allowed to approach zero.

**References**

