Journal of Pure and Applied Algebra 213 (2009) 60–70

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Descent in $*$-autonomous categories

Bachuki Mesablishvili *

Razmadze Mathematical Institute, Tbilisi 0193, Georgia

ARTICLE INFO

Article history:
Received 23 October 2007
Received in revised form 11 March 2008
Available online 26 June 2008
Communicated by G. Rosolini

MSC:
18B35
18C15
18C20
18D10
18D20

ABSTRACT

We extend the result of Joyal and Tierney asserting that a morphism of commutative algebras in the $*$-autonomous category of sup-lattices is an effective descent morphism for modules if and only if it is pure, to an arbitrary $*$-autonomous category $\mathcal{V}$ (in which the tensor unit is projective) by showing that any $\mathcal{V}$-functor out of $\mathcal{V}$ is precomonadic if and only if it is comonadic.

1. Introduction

General descent theory, as originally developed by Grothendieck [5] in the abstract setting of fibred categories, is an invaluable tool in algebraic geometry, which one can also apply to various situations in Galois theory, topology and topos theory. A general aim of descent theory is to give characterizations of the so-called (effective) descent morphisms, which in the case of a fibred category satisfying the Beck–Chevalley condition reduces to monadicity of a suitable functor.

A basic example of Grothendieck's descent theory involves modules over commutative rings. Consider a homomorphism of commutative rings $i : A \to B$ and the corresponding *extension-of-scalars* functor $i_* = B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B$. It is well known that this functor admits as a right adjoint the underlying functor $i^* : \mathrm{Mod}_B \to \mathrm{Mod}_A$. The problem of Grothendieck's descent theory for modules is concerned with the characterization of those B-modules M for which there exists an A-module N and an isomorphisms $i_*(N) \simeq M$ of B-modules. To be more specific, let M be a B-module and let $\theta_M : M \otimes_A B \to B \otimes_A M$ be a homomorphism of $B \otimes_A B$-modules, where $B \otimes_A B$ acts on $M \otimes_A B$ by $(b_1 \otimes b_2)(m \otimes b) = b_1 m \otimes b_2 b$ and on $B \otimes_A M$ by $(b_1 \otimes b_2)(b \otimes m) = b_1 b \otimes b_2 m$. Define

$$\begin{aligned}
&(\theta_M)_1 : B \otimes_A M \otimes_A B \to B \otimes_A B \otimes_A M,\\
&(\theta_M)_2 : M \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M,\\
&(\theta_M)_3 : M \otimes_A B \otimes_A B \to B \otimes_A M \otimes_A B
\end{aligned}$$

by tensoring θ_M with 1_B in the first, second and third positions respectively. *Descent data* on a B-module M is an isomorphism $\theta_M : M \otimes_A B \to B \otimes_A M$ of $B \otimes_A B$-modules such that $(\theta_M)_2 = (\theta_M)_1 \cdot (\theta_M)_3$.Des$(i)$ denotes the category of pairs (M, θ_M), θ_M descent data on a B-module M, where morphisms $(M, \theta_M) \to (M', \theta_{M'})$ are morphisms $f : M \to M'$ of B-modules that commute with descent data in the obvious way. For any A-module N, there is an isomorphism $\theta_{i_*(N)} : N \otimes_A B \otimes_A B \to B \otimes_A N \otimes_A B$, arising from

$$(i_1)_*(i_*(N)) = (i_1 i)_*(N) = (i_2 i)(N) = (i_2)_*(i_*(N)),$$

* Corresponding address: Mathematical Institute of the Georgian Academy of Sciences, M. Alexidze Str. 1, 380093 Tbilisi, Georgia.
E-mail address: bachi@rmi.acnet.ge.

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2008.05.009

where $i_1, i_2 : B \to B \otimes_A B$ are the maps defined by $i_1(b) = 1 \otimes_A b$ and $i_2(b) = b \otimes_A 1$. Explicitly $\theta_{i_*(N)}$ is the map given by

$$n \otimes_A b_1 \otimes_A b_2 \longmapsto b_1 \otimes_A n \otimes_A b_2.$$

Thus one has a functor

$$K_i : \mathrm{Mod}_A \to \mathrm{Des}(i).$$

One says that $i : A \to B$ is an (effective) descent homomorphism of commutative rings if the functor K_i is full and faithful (an equivalence of categories). So that, when i is an effective descent morphism, to specify an A-module is to specify a B-module M together with descent data $\theta_M : M \otimes_A B \to B \otimes_A M$ of $B \otimes_A B$-modules. The descent theory for modules is thus the study of which homomorphisms of commutative rings $i : A \to B$ are (effective) descent morphisms. Grothendieck [5] proved that faithfully flat extensions of commutative rings are effective. A full characterization of effective descent morphisms for modules was given by A. Joyal and M. Tierney (unpublished, but see [12]) and by Olivier [13]: a morphism $i : A \to B$ of commutative rings is an effective descent morphism iff i is a pure morphism of A-modules. For example, if i is a split monomorphism of A-modules, then it is effective for descent (see, for example, [6]).

According to the theorem of Bénabou and Roubaud [1] and J. Beck (unpublished), the category $\mathrm{Des}(i)$ is equivalent to the Eilenberg–Moore category of $\mathbf{G}_i$-coalgebras, where $\mathbf{G}_i$ is the comonad on the category Mod_B generated by the adjunction $i_* \dashv i^* : \mathrm{Mod}_B \to \mathrm{Mod}_A$. Modulo this equivalence, the functor $K_i : \mathrm{Mod}_A \to \mathrm{Des}(i)$ can be identified with the comparison functor $K_{\mathbf{G}_i} : \mathrm{Mod}_A \to (\mathrm{Mod}_B)_{\mathbf{G}_i}$, and thus to say that i is an (effective) descent morphism is to say that the extension-of-scalars functor $i_* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is precomonadic (comonadic).

Since purity of any homomorphism of commutative rings is equivalent to precomonadicity of the corresponding extension-of-scalars functor, Grothendieck's descent theory for modules over commutative rings can be conceived by interpreting a descent result as a statement asserting that for a given homomorphism of commutative rings, precomonadicity of the corresponding extension-of-scalars functor implies (and hence is equivalent to) comonadicity.

Identifying the homomorphism $i : A \to B$ with the algebra B in the monoidal category Mod_A and considering the monad $\mathbf{T}_i$ on Mod_A given by tensoring with B, the category Mod_B can be seen as the Eilenberg–Moore category of $\mathbf{T}_i$-algebras and the functor i_* as the comparison functor $K_{\mathbf{T}_i} : \mathrm{Mod}_A \to (\mathrm{Mod}_A)^{\mathbf{T}_i}$. Thus the problem of effectiveness of i is equivalent to the one of the comonadicity of the functor i_*. This motivates to call a monad $\mathbf{T}$ on a category $\mathcal{A}$ to be of (effective) descent type if the free $\mathbf{T}$-algebra functor $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ is precomonadic (comonadic). Hence, in the language of monads, the Joyal–Tierney theorem can be paraphrased as follows: For any pure homomorphism of commutative rings $i : A \to B$, the monad $\mathbf{T}_i$ is an effective descent morphism iff it is of descent type.

The question whether a given morphism is effective for descent has been investigated for various categories. An important example of wide applicability of Grothendieck's descent theory is Joyal–Tierney's result [8] on a descent theory for open maps of locales. In [8], A. Joyal and M. Tierney looked at descent theory in the context of the $*$-autonomous category $\mathcal{CL}$ of sup-lattices and indicated that there was a useful analogy with the descent theory for modules over commutative rings. A main result of [8] asserts that a morphism of commutative algebras in $\mathcal{CL}$ is effective for descent iff it is pure. This is used to show that open surjections are effective in the category of locales, leading to a representation theorem for Grothendieck topoi (over a base topos) in terms of localic groupoids, which can be considered as a generalization of the fundamental theorem of Galois theory.

Our aim is to generalize the descent theorem of Joyal–Tierney for sup-lattices by showing that for any $*$-autonomous category $\mathcal{V}$ with an injective dualizing object, a $\mathcal{V}$-functor is precomonadic iff it is comonadic.

The paper is organized as follows. Section 2 rather technical, and is devoted to extend classical monadicity results to the enriched setting. Section 3 contains some criteria for comonadicity that are used in the next section to generalize the theorem of Joyal–Tierney to $*$-autonomous categories.

2. Preliminaries

We let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ denote a symmetric monoidal category, where $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$ is the tensor product of $\mathcal{V}$ and where I is the tensor unit. A $\mathcal{V}$-functor will be called a functor if it is understood that the domain and codomain are $\mathcal{V}$-categories. Similarly a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors will be called simply a natural transformation. For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, $[\mathcal{A}, \mathcal{B}]_0$ will denote the ordinary category of functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them. In our paper we follow the notation from [9], which is our general reference for enriched category theory.

We start by recalling the basic facts about $\mathcal{V}$-monads; all can be found in [3,4].

Given $\mathcal{V}$-category $\mathcal{A}$, a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{A}$ consists of a functor $T : \mathcal{A} \to \mathcal{A}$ together with natural transformations $\eta : 1_{\mathcal{A}} \to T$ and $\mu : T^2 \to T$ satisfying the usual three axioms.

As in the ordinary case, every $\mathcal{V}$-adjunction $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ induces a monad on $\mathcal{A}$ by letting $\mathbf{T} = (UF, \eta, U\epsilon F)$.

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a $\mathcal{V}$-category $\mathcal{A}$. Then clearly $\mathbf{T}_0 = (T_0, \eta_0, \mu_0)$ is an ordinary monad on $\mathcal{A}_0$, and one has the Eilenberg–Moore category $\mathcal{A}_0^{\mathbf{T}_0}$ and the free-forgetful adjunction

$$F^{\mathbf{T}_0} \dashv U^{\mathbf{T}_0} : \mathcal{A}_0^{\mathbf{T}_0} \to \mathcal{A}_0$$

determining the monad $\mathbf{T}_0$.

When the category $\mathcal{V}_0$ has equalizers (at least of pairs with a common left inverse), $\mathcal{A}_0^{\mathbf{T}_0}$ has a $\mathcal{V}$-category structure $\mathcal{A}^{\mathbf{T}}$: Given $\mathbf{T}_0$-algebras (a, θ) and (a', θ'), one defines

$$U^{\mathbf{T}}_{(a,\theta),(a',\theta')} : \mathcal{A}^{\mathbf{T}}((a, \theta), (a', \theta')) \to \mathcal{A}(a, a')$$

to be the equalizer in $\mathcal{V}_0$ of the following pair of morphisms:

$$\begin{tikzcd} & \mathcal{A}(T(a), T(a')) \arrow[dr, "\mathcal{A}(T(a),\theta')"] & \\ \mathcal{A}(a, a') \arrow[ur, "T_{a,a'}"] \arrow[rr, "\mathcal{A}(\theta,a')"'] & & \mathcal{A}(T(a), a'). \end{tikzcd}$$

Moreover, one can give $F^{\mathbf{T}_0}$ and $U^{\mathbf{T}_0}$ the structure of $\mathcal{V}$-functors in such a way that the above adjunction is enriched in $\mathcal{V}$; in other words, there are $\mathcal{V}$-functors $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ and $U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ underlying $F^{\mathbf{T}_0} : \mathcal{A}_0 \to \mathcal{A}_0^{\mathbf{T}_0}$ and $U^{\mathbf{T}_0} : \mathcal{A}_0^{\mathbf{T}_0} \to \mathcal{A}_0$, respectively, such that $F^{\mathbf{T}}$ is left $\mathcal{V}$-adjoint to $U^{\mathbf{T}}$.

$\mathcal{A}^{\mathbf{T}}$ is called the Eilenberg–Moore category for the monad $\mathbf{T}$.

Given a $\mathcal{V}$-adjunction $\eta, \varepsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$, let $\mathbf{T}$ be the monad on $\mathcal{A}$ induced by (U, F), and let $\eta^{\mathbf{T}}, \varepsilon^{\mathbf{T}} : F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ be the adjunction associated to $\mathbf{T}$ as described above. Then there is a canonical functor $K_{\mathbf{T}} : \mathcal{B} \to \mathcal{A}^{\mathbf{T}}$, called the *comparison functor*, for which $U^{\mathbf{T}} \circ K_{\mathbf{T}} = U$ and $K_{\mathbf{T}} \circ F = F^{\mathbf{T}}$. (Recall that $K_{\mathbf{T}}$ is given on objects by $K_{\mathbf{T}}(b) = (U(b), U\epsilon_a)$.)

One says that the $\mathcal{V}$-functor U is *(pre)monadic* if $K_{\mathbf{T}}$ is an equivalence of $\mathcal{V}$-categories (a fully faithful $\mathcal{V}$-functor).

From now on we consider only monoidal categories $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ with $\mathcal{V}_0$ admitting all equalizers.

Let $\mathcal{A}$ be a $\mathcal{V}$-category. Recall that a diagram

$$a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{h} c$$

in $\mathcal{A}_0$ is a coequalizer diagram in $\mathcal{A}$ (or just a $\mathcal{V}$-coequalizer diagram) if each

$$\mathcal{A}(c, x) \xrightarrow{\mathcal{A}(h,x)} \mathcal{A}(b, x) \underset{\mathcal{A}(g,x)}{\overset{\mathcal{A}(f,x)}{\rightrightarrows}} \mathcal{A}(a, x)$$

is an equalizer diagram in $\mathcal{V}_0$. Recall also that a map $h : b \to c$ in $\mathcal{A}_0$ is a regular epimorphism in $\mathcal{A}$ if it is a $\mathcal{V}$-coequalizer of a pair of morphisms.

Theorem 2.1. *A right adjoint $\mathcal{V}$-functor $U : \mathcal{B} \to \mathcal{A}$ is (pre)monadic if and only if the underlying ordinary functor $U_0 : \mathcal{B}_0 \to \mathcal{A}_0$ is so, provided that $\mathcal{B}$ has coequalizers of reflexive pairs.*

Corollary 2.2. *Let $\eta, \varepsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-adjunction and suppose that $\mathcal{B}$ has coequalizers of reflexive pairs. Then U is premonadic if and only if each $\varepsilon_b : FU(b) \to b$ is a regular epimorphism in the ordinary category $\mathcal{B}_0$.*

Proof. It is well known (see, for example, [10]) that an ordinary functor with left adjoint is premonadic iff each component of the counit of the adjunction is a regular epimorphism. ∎

Since any regular epimorphism in $\mathcal{B}$ is of course a regular epimorphism in $\mathcal{B}_0$, we have:

Proposition 2.3. *Let $\mathcal{B}$ admit coequalizers of reflexive pairs and let $U : \mathcal{B} \to \mathcal{A}$ be a right adjoint functor with left adjoint $F : \mathcal{A} \to \mathcal{B}$ and counit $\varepsilon : FU \to 1$. If each $\varepsilon_b : FU(b) \to b$ is a regular epimorphism in $\mathcal{B}$, then U is premonadic.*

Proposition 2.4. *Let $\mathcal{B}$ be a $\mathcal{V}$-category admitting coequalizers of reflexive pairs and let $U : \mathcal{B} \to \mathcal{A}$ be a premonadic $\mathcal{V}$-functor. Then any morphism in $\mathcal{B}$ whose image under U is a split epimorphism is a regular epimorphism in $\mathcal{B}_0$.*

Proof. Since $\mathcal{B}$ is assumed to have coequalizers of reflexive pairs, it follows from Theorem 2.1 that U is premonadic iff U_0 is so. The conclusion now follows from Corollary 2.2. ∎

We shall need the following result of Janelidze and Tholen [6]:

Theorem 2.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ordinary categories with coequalizers and let $W : \mathcal{X} \to \mathcal{Y}$ be a right adjoint functor with left adjoint $V : \mathcal{Y} \to \mathcal{X}$ and counit $\sigma : VW \to 1_{\mathcal{X}}$. If the natural transformation σ is a split epimorphism, W is monadic.*

Theorem 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories admitting coequalizers and let $\eta, \varepsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction. If the ordinary natural transformation $\varepsilon_0 : F_0U_0 \to 1_{\mathcal{B}_0}$ is a split epimorphism, then U is monadic.*

Proof. Since $\varepsilon_0 : F_0U_0 \to 1_{\mathcal{B}_0}$ is a split epimorphism, U_0 is monadic by Theorem 2.5 and since $\mathcal{B}$ admits coequalizers, the result follows from Theorem 2.1. ■

A corollary follows immediately:

Corollary 2.7. *In the situation of Theorem 2.6, suppose that the natural transformation $\varepsilon : FU \to 1$ is a split epimorphism in $[\mathcal{B}, \mathcal{B}]_0$, then the functor U is monadic.*

3. Some criteria for comonadicity

Let $\mathbf{T}$ be a monad on a $\mathcal{V}$-category $\mathcal{A}$ with associated Eilenberg–Moore category $\mathcal{A}^{\mathbf{T}}$ of $\mathbf{T}$-algebras, and canonical free-forgetful adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$.$\mathbf{T}$ is called *of descent type* if $F^{\mathbf{T}}$ is precomonadic, and $\mathbf{T}$ is called *of effective descent type* if $F^{\mathbf{T}}$ is comonadic.

As an immediate consequence of the dual of Theorem 2.1 we observe that

Theorem 3.1. *Let $\mathbf{T}$ be a monad on a $\mathcal{V}$-category $\mathcal{A}$. If $\mathcal{A}$ admits coequalizers of reflexive pairs, then $\mathbf{T}$ is of (effective) descent type iff $\mathbf{T}_0$ is.*

Since the functor $U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ detects all limits that exist in $\mathcal{A}$, it follows from (the dual of) Corollary 2.7 that

Proposition 3.2. *Let $\mathcal{A}$ be a $\mathcal{V}$-category admitting equalizers and let $\mathbf{T} = (T, \eta, \mu)$ be monad on $\mathcal{A}$. If $\eta : 1 \to T$ is a split monomorphism in $[\mathcal{A}, \mathcal{A}]_0$, then $\mathbf{T}$ is a monad of effective descent type.*

Let $\mathcal{A}$ be a $\mathcal{V}$-category and (E, ν) be a pointed endofunctor on $\mathcal{A}$. (Recall that a pointed endofunctor on $\mathcal{A}$ is a pair (E, ν) where $E : \mathcal{A} \to \mathcal{A}$ is a functor and $\nu : 1 \to E$ is a natural transformation.) For an object Q of $\mathcal{A}$, we get from E a functor

$$\mathcal{A}(E(-), Q) : \mathcal{A}^{op} \to \mathcal{V},$$

and we can consider the natural transformation

$$\mathcal{A}(\nu_-, Q)\colon \mathcal{A}(E(-), Q) \to \mathcal{A}(-, Q).$$

Proposition 3.3. *The natural transformation $\mathcal{A}(\nu_-, Q)$ is a split epimorphism in $[\mathcal{A}^{op}, \mathcal{V}]_0$ if and only if the morphism $\nu_Q : Q \to E(Q)$ is a split monomorphism in $\mathcal{A}_0$.*

Proof. Suppose that $\nu\colon \mathcal{A}(-, Q) \to \mathcal{A}(E(-), Q)$ is a natural transformation such that $\mathcal{A}(\nu_{(-)}, Q) \cdot \tau = 1$. Then the diagram

$$\begin{tikzcd} I \arrow[r, "i_Q"] \arrow[drr, "i_Q"'] & \mathcal{A}(Q, Q) \arrow[r, "\tau_Q"] & \mathcal{A}(E(Q), Q) \arrow[d, "\mathcal{A}(\nu_Q, Q)"] \\ & & \mathcal{A}(Q, Q) \end{tikzcd} ,$$

where $i_Q = \ulcorner 1_Q \urcorner\colon I \to \mathcal{A}(Q, Q)$ is the "name" of the unit morphism $1_Q : Q \to Q$, commutes. Hence $\tau_Q \cdot i_Q\colon I \to \mathcal{A}(E(Q), Q)$ is the name of a map $E(Q) \to Q$ in $\mathcal{A}_0$ that splits ν_Q.

Conversely, suppose that $t \cdot \nu_Q = 1$ where $t\colon E(Q) \to Q$. By the Yoneda lemma, there is a unique natural transformation

$$\tau\colon \mathcal{A}(-, Q) \to \mathcal{A}(E(-), Q)$$

for which $\tau_Q\colon \mathcal{A}(Q, Q) \to \mathcal{A}(E(Q), Q)$ composed with the morphism $i_Q\colon I \to \mathcal{A}(Q, Q)$ is the name $\ulcorner t \urcorner\colon I \to \mathcal{A}(E(Q), Q)$ of t. Now the composite

$$\mathcal{A}(-, Q) \xrightarrow{\tau} \mathcal{A}(E(-), Q) \xrightarrow{\mathcal{A}(\nu_-, Q)} \mathcal{A}(-, Q)$$

is the identity, since the composite of its Q-component with i_Q is $\mathcal{A}(\nu_Q, Q)(\ulcorner t \urcorner) = \ulcorner (t \cdot \nu_Q) \urcorner = \ulcorner 1_Q \urcorner = i_Q$. ■

Recall that an object Q of a $\mathcal{V}$-category $\mathcal{A}$ is a regular cogenerator for $\mathcal{A}$ if the functor

$$\mathcal{A}(-, Q)\colon \mathcal{A}^{op} \to \mathcal{V}$$

has a left adjoint and the unit of this adjunction is componentwise a regular monomorphism in $\mathcal{A}$.

Theorem 3.4. *Let $\mathcal{A}$ be a $\mathcal{V}$-category admitting equalizers, and let $\eta, \varepsilon\colon F \dashv U\colon \mathcal{B} \to \mathcal{A}$ be an adjunction. Suppose that there exists a regular cogenerator Q for $\mathcal{A}$. If the morphism $\eta_Q\colon Q \to UF(Q)$ is a split monomorphism in $\mathcal{A}_0$, then F is a precomonadic functor. If, in addition, Q is an injective object in $\mathcal{A}_0$, then the converse also holds.*

Proof. If the morphism $\eta_Q : Q \to UF(Q)$ is a split monomorphism, then the natural transformation

$$\mathcal{A}(\eta_-, Q) \colon \mathcal{A}(UF(-), Q) \to \mathcal{A}(-, Q)$$

is a split epimorphism in $[\mathcal{A}^{op}, \mathcal{V}]_0$ (see Proposition 3.3). Hence, for any $a \in \mathcal{A}$, the morphism

$$\mathcal{A}(\eta_a, Q) \colon \mathcal{A}(UF(a), Q) \to \mathcal{A}(a, Q)$$

is a split epimorphism. Since Q is a regular cogenerator for $\mathcal{A}$ and since $\mathcal{A}$ admits equalizers by hypothesis, it follows from Proposition 2.3 that the functor $\mathcal{A}(-, Q)$ is premonadic. And applying Proposition 2.4 to this premonadic functor, we see that each $\eta_a \colon a \to UF(a)$ is a regular monomorphism in $\mathcal{A}_0$. Then that F is precomonadic follows from the dual of Corollary 2.2.

If $F : \mathcal{A} \to \mathcal{B}$ is precomonadic, then since $\mathcal{A}$ is assumed to have equalizers, it follows from the dual of Corollary 2.2 that each component $\eta_a : a \to UF(a)$ of $\eta : 1 \to UF$ is a regular monomorphism in $\mathcal{A}_0$; in particular, so is $\eta_Q : Q \to UF(Q)$. Now, if Q is injective in $\mathcal{A}_0$, then η_Q is evidently a split monomorphism in $\mathcal{A}_0$. ■

Let now $\mathcal{V}$ be a $*$-autonomous category in the sense of Barr [2]. Thus, $\mathcal{V}$ is a symmetric monoidal closed category together with a so-called dualizing object Q such that the $\mathcal{V}$-adjunction

$$\mathcal{V}(-, Q) \dashv \mathcal{V}(-, Q) \colon \mathcal{V}^{op} \to \mathcal{V} \tag{1}$$

is an equivalence.

Theorem 3.5. *Let $\mathcal{V}$ be a $*$-autonomous category with dualizing object Q, and let (E, ν) be a pointed $\mathcal{V}$-endofunctor on $\mathcal{V}$. Then the morphism $\nu_Q \colon Q \to E(Q)$ is a split monomorphism in $\mathcal{V}_0$ if and only if the natural transformation $\nu : 1 \to E$ is a split monomorphism in $[\mathcal{V}, \mathcal{V}]_0$.*

Proof. One direction is clear, so suppose that the morphism $\nu_Q \colon Q \to E(Q)$ is a split monomorphism in $\mathcal{V}_0$. Then, by Proposition 3.3, the natural transformation

$$\mathcal{V}(\nu_{(-)}, Q) \colon \mathcal{V}(E(-), Q) \to \mathcal{V}(-, Q)$$

is a split epimorphism in $[\mathcal{V}, \mathcal{V},]_0$, and hence

$$\mathcal{V}(\mathcal{V}(\nu_{(-)}, Q), Q) \colon \mathcal{V}(\mathcal{V}(-, Q), Q) \to \mathcal{V}(\mathcal{V}(E(-), Q), Q)$$

is a split monomorphism in $[\mathcal{V}, \mathcal{V},]_0$. But since (2) is an equivalence of categories, $\mathcal{V}(\nu_{(-)}, Q) \simeq \nu$, and therefore $\nu \colon 1 \to E$ is a split monomorphism in $[\mathcal{V}, \mathcal{V},]_0$. ■

Theorem 3.6. *Let $\mathcal{V}$ be a $*$-autonomous category with dualizing object Q and $\eta, \varepsilon \colon F \dashv U \colon \mathcal{A} \to \mathcal{V}$ be a $\mathcal{V}$-adjunction such that the morphism $\eta_Q \colon Q \to UF(Q)$ is a split monomorphism in $\mathcal{V}_0$. Then F is a comonadic functor.*

Proof. Let us first recall that for any tensored $\mathcal{V}$-category $\mathcal{A}$, $\mathcal{V}$-equalizers are the same things as equalizers in $\mathcal{A}_0$; this is true in particular when $\mathcal{A} = \mathcal{V}$, since the $\mathcal{V}$-category $\mathcal{V}$, being closed, is tensored. So to say that $\mathcal{V}_0$ admits all equalizers is to say that $\mathcal{V}$ admits all $\mathcal{V}$-equalizers. We now apply Theorem 3.5 to the pointed endofunctor $(UF, \eta : 1 \to UF)$ to see that η is a split monomorphism in $[\mathcal{V}, \mathcal{V},]_0$. And since $\mathcal{V}$ admits all equalizers by hypothesis, it follows from Corollary 2.7 that F is comonadic. ■

Now we are ready to prove:

Theorem 3.7. *Let $\mathcal{V}$ be a $*$-autonomous category with dualizing object Q and let $\mathcal{V}_0$ admit all equalizers. For a given $\mathcal{V}$-adjunction $\eta, \varepsilon : F \dashv U : \mathcal{A} \to \mathcal{V}$, let us consider the following statements:*

(i) *each component of the natural transformation $\eta : 1 \to UF$ is a regular monomorphism;*
(ii) *the morphism $\eta_Q \colon Q \to UF(Q)$ is a regular monomorphism;*
(iii) *the morphism $\eta_Q \colon Q \to UF(Q)$ is a split monomorphism;*
(iv) *the natural transformation $\eta \colon 1 \to UF$ is a split monomorphism in $[\mathcal{V}, \mathcal{V}]_0$;*
(v) *the functor F is precomonadic;*
(vi) *the functor F is comonadic;*
(vii) *the ordinary functor F_0 is precomonadic;*
(viii) *the ordinary functor F_0 is comonadic;*
(ix) *the ordinary natural transformation $\eta_0 : 1 \to U_0F_0$ is a split monomorphism.*

Then one always has the implications

$$\begin{tikzcd}
 & \text{(iii)} \arrow[r, Leftrightarrow] & \text{(iv)} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{(ix)} & \\
 & & \text{(vi)} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(viii)} \arrow[d, Rightarrow] & \\
 & \text{(i)} \arrow[r, Leftrightarrow] & \text{(v)} \arrow[r, Rightarrow] & \text{(vii)} \arrow[r, Rightarrow] & \text{(ii)}.
\end{tikzcd}$$

Moreover, all nine are equivalent when Q is an injective object in $\mathcal{V}_0$.

Proof. The implications (iv) $\Rightarrow$ (ix), (vi) $\Rightarrow$ (v), (vi) $\Rightarrow$ (viii), (viii) $\Rightarrow$ (vii) (v) $\Rightarrow$ (vii), and (ix) $\Rightarrow$ (iii) are trivial. Since (iii) $\Longleftrightarrow$ (iv) by Theorem 3.5, it follows that (iii), (iv) and (ix) are equivalent. (iv) $\Rightarrow$ (vi) by the dual of Corollary 2.7, while (vii) $\Rightarrow$ (ii) by the dual of Corollary 2.2. Moreover, Since $\mathcal{V}$ admits all equalizers and since $\mathcal{V}$-equalizers are the same things as equalizers in $\mathcal{V}_0$, it follows from the dual of Proposition 2.3 that (i) and (v) are equivalent.

Finally, assume that Q is injective in $\mathcal{V}_0$. Then the morphism $\eta_Q : Q \to T(Q)$ is a split monomorphism if and only if it is a regular monomorphism. Thus, in this case (ii) $\Longleftrightarrow$ (iii) and hence the nine conditions are equivalent. ∎

Theorem 3.8. *Let $\mathcal{V}$ be a $*$-autonomous category with dualizing object Q which is injective in $\mathcal{V}_0$. If $\mathbf{T} = (T, \eta, \mu)$ is a $\mathcal{V}$-monad on $\mathcal{V}$, then the following are equivalent:*

(i) *the natural transformation $\eta : 1 \to T$ is componentwise a regular monomorphism;*
(ix) *the natural transformation $\eta_0 : 1 \to T_0$ is a split monomorphism;*
(iii) *the monad $\mathbf{T}$ is of descent type;*
(iv) *the monad $\mathbf{T}$ is of effective descent type.*

Proposition 3.9. *A dualizing object Q of a $*$-autonomous category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is injective in $\mathcal{V}_0$ if and only if the tensor unit I is projective in $\mathcal{V}_0$.*

Proof. Suppose that I is projective in $\mathcal{V}_0$. Since the $\mathcal{V}$-functor

$$\mathcal{V}(-, Q) : \mathcal{V}^{op} \to \mathcal{V}$$

is an equivalence of categories, the functor

$$\mathcal{V}(-, Q)_0 : \mathcal{V}_0^{op} \to \mathcal{V}_0$$

preserves all types of limits that exist in $\mathcal{V}_0$; in particular, it preserves regular epimorphisms; and that Q is injective in V_0 now follows from the fact that there is an isomorphism of ordinary functors:

$$\mathcal{V}_0(-, Q) \simeq \mathcal{V}_0(I, \mathcal{V}(-, Q)_0).$$

Conversely, suppose that Q is injective in $\mathcal{V}_0$ and consider a diagram

$$\begin{tikzcd}
 & I \arrow[d, "x"] \\
V \arrow[r, "f"'] & V'
\end{tikzcd} \tag{2}$$

in $\mathcal{V}_0$ with f a regular epimorphism, and let

$$\begin{tikzcd}
\mathcal{V}(V', Q)_0 \arrow[r, "\mathcal{V}(f, Q)_0"] \arrow[d, "\mathcal{V}(x, Q)_0"'] & \mathcal{V}(V, Q)_0 \\
\mathcal{V}(I, Q)_0 \simeq Q &
\end{tikzcd} \tag{3}$$

be the image of this diagram under the functor

$$\mathcal{V}(-, Q)_0 : \mathcal{V}_0^{op} \to \mathcal{V}_0.$$

Since the functor $\mathcal{V}(-, Q)_0$ preserves regular monomorphisms, the morphism $\mathcal{V}(f, Q)$ is a regular monomorphism; and from the assumption on Q we conclude that there is a morphism $\mathcal{V}(V, Q) \to \mathcal{V}(I, Q) \simeq Q$ making diagram (4) commute. Applying the functor $\mathcal{V}(-, Q)_0$ to this commutative diagram and using that $\mathcal{V}(\mathcal{V}(-, Q)_0, Q)_0 \simeq 1_{\mathcal{V}_0}$, we get a completion of diagram (3) to a commutative one. Hence I is projective in $\mathcal{V}_0$, as desired. ∎

Combining Theorem 3.7 and Proposition 3.9 gives the following result:

Theorem 3.10. *Let $\mathcal{V}$ be a $*$-autonomous with dualizing object Q, and suppose that $\mathcal{V}_0$ admits all equalizers and that the tensor unit I is projective in $\mathcal{V}_0$. For a $\mathcal{V}$-adjunction $\eta, \varepsilon : F \dashv U : \mathcal{A} \to \mathcal{V}$, the following are equivalent:*

(i) *each component of the natural transformation $\eta : 1 \to UF$ is a regular monomorphism;*
(ii) *the $\mathcal{V}$-natural transformation $\eta : 1 \to UF$ is a split monomorphism in $[\mathcal{V}, \mathcal{V}]_0$;*
(iii) *the functor F is precomonadic;*
(iv) *the functor F is comonadic.*

Corollary 3.11. *Let $\mathcal{V}$ be a $*$-autonomous with dualizing object Q and suppose that $\mathcal{V}_0$ admits all equalizers and that the tensor unit I is projective in $\mathcal{V}_0$. For a $\mathcal{V}$-monad $\mathbf{T} = (T, \eta, \mu)$, the following are equivalent:*

(i) *each component of the natural transformation $\eta : 1 \to T$ is a regular monomorphism;*
(ii) *the natural transformation $\eta : 1 \to T$ is a split monomorphism in $[\mathcal{V}, \mathcal{V}]_0$;*
(iii) *the monad $\mathbf{T}$ is of descent type;*
(iv) *the monad $\mathbf{T}$ is of effective descent type.*

Given a $\mathcal{V}$-category $\mathcal{A}$, recall that a $\mathcal{V}$-monad $\mathbf{T}$ on $\mathcal{A}$ is called an *enrichment* of an ordinary monad T on $\mathcal{A}_0$ when the underlying ordinary monad $\mathbf{T}_0$ of $\mathbf{T}$ is T.

Corollary 3.12. *Let $\mathcal{V}$ be a $*$-autonomous category with dualizing object Q and let $T = (T, \eta, \mu)$ be an (ordinary) monad on $\mathcal{V}_0$ admitting an enrichment to a $\mathcal{V}$-monad on $\mathcal{V}$. If I is projective in $\mathcal{V}_0$, then the following are equivalent:*

(i) *each component of the natural transformation $\eta : 1 \to T$ is a regular monomorphism;*
(ii) *the natural transformation $\eta : 1 \to T$ is a split monomorphism in $[\mathcal{V}, \mathcal{V}]_0$;*
(iii) *T is of descent type;*
(iv) *T is of effective descent type*

4. Applications

Fix a symmetric monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ with a symmetry $c_{U,V} : U \otimes V \to V \otimes U$. Recall that an algebra in $\mathcal{V}$ (or $\mathcal{V}$-algebra) is an object R of $\mathcal{V}_0$ equipped with a multiplication $m_R : R \otimes R \to R$ and a unit $i_R : I \to R$ subject to the condition that the following diagrams commute:

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "m_R \otimes R"] \arrow[d, "R \otimes m_R"'] & R \otimes R \arrow[d, "m_R"] \\ R \otimes R \arrow[r, "m_R"'] & R, \end{tikzcd}$$

$$\begin{tikzcd} R \simeq I \otimes R \arrow[r, "i_R \otimes R"] \arrow[dr, equal] & R \otimes R \arrow[d, "m_R"] & R \otimes I \simeq R \arrow[l, "R \otimes i_R"'] \arrow[dl, equal] \\ & R & \end{tikzcd}$$

For a $\mathcal{V}$-algebra $R = (R, i_R, m_R)$ in $\mathcal{V}$, one defines a $\mathcal{V}$-monad $\mathbf{T}^R = (T^R, \eta^R, \mu^R)$ on $\mathcal{V}$ by

- $T^R_{(-)} = R \otimes -$;
- $\eta^R_{(-)} = i_R \otimes - : 1_{\mathcal{V}} \simeq I \otimes - \to R \otimes -$;
- $\mu^R_{(-)} = m_R \otimes - : (R \otimes R) \otimes - \to R \otimes -$.

The $\mathcal{V}$-algebra structure on R gives the required identities for $\mathbf{T}^R$ to be a $\mathcal{V}$-monad on $\mathcal{V}$. We refer to $\mathbf{T}^R$ as the $\mathcal{V}$-monad induced by the $\mathcal{V}$-algebra $R = (R, i_R, m_R)$.

Applying Corollary 3.11 to this situation gives the following

Theorem 4.1. *Let $\mathcal{V}$ be a $*$-autonomous category with dualizing object Q, and suppose that I is projective in $\mathcal{V}_0$. For an algebra $R = (R, i_R, m_R)$ in $\mathcal{V}$, the following statements are equivalent:*

(i) *the morphism $i_R : I \to R$ is a split monomorphism in $\mathcal{V}_0$;*
(ii) *the morphism $i_R : I \to R$ pure; that is, for any $V \in \mathcal{V}_0$, the morphism*

$$i_R \otimes V : V \simeq I \otimes V \to R \otimes V$$

is a regular monomorphism;

(iii) *the monad* $\mathbf{T}^R$ *is of descent type;*
(iv) *the monad* $\mathbf{T}^R$ *is of effective descent type.*

Let $R = (R, i_R, m_R)$ be an algebra in $\mathcal{V}$. Recall that a left R-module is an object $M \in \mathcal{V}_0$ equipped with a (left) action $\lambda_M : R \otimes M \to M$ making commutative the following diagrams:

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r, "m_R \otimes M"] \arrow[d, "R \otimes m_R"'] & R \otimes M \arrow[d, "\lambda_M"] \\ R \otimes M \arrow[r, "\lambda_M"'] & M, \end{tikzcd} \qquad \begin{tikzcd} M \simeq I \otimes M \arrow[r, "i_R \otimes M"] \arrow[dr, equal] & R \otimes M \arrow[d, "\lambda_M"] \\ & M. \end{tikzcd}$$

The category of left R-modules is denoted by ${}_R\mathcal{V}$. The category of right R-modules $\mathcal{V}_R$ is defined similarly.

Note that the evident forgetful functor

$${}_RU : {}_R\mathcal{V} \to \mathcal{V}$$

is right adjoint, with the left adjoint sending $V \in \mathcal{V}_0$ to $R \otimes V \in {}_R\mathcal{V}$, where $R \otimes V$ is an object in ${}_R\mathcal{V}$ via the morphism $m_R \otimes V : R \otimes R \otimes V \to R \otimes V$.

It is well known that the functor ${}_RU$ is monadic and that the monad on $\mathcal{V}$ generated by the adjunction $R \otimes - \dashv {}_RU$ is just the monad $\mathbf{T}^R$. In other words, the category $\mathcal{V}^{\mathbf{T}^R}$ is just the category of left R-modules and the free $\mathbf{T}^R$-algebra functor $\mathcal{V} \to \mathcal{V}^{\mathbf{T}^R}$ is just the functor $R \otimes - : \mathcal{V} \to {}_R\mathcal{V}$.

It follows from Theorem 3.10 that

Theorem 4.2. *Let $\mathcal{V}$ be a $*$-autonomous category whose tensor unit I is projective in $\mathcal{V}_0$, and let $R = (R, i_R, m_R)$ be an $\mathcal{V}$-algebra. Then the following statements are equivalent:*

(i) *the functor $R \otimes - : \mathcal{V} \to {}_R\mathcal{V}$ is precomonadic;*
(ii) *the functor $R \otimes - : \mathcal{V} \to {}_R\mathcal{V}$ is comonadic;*
(iii) *the morphism $i : I \to R$ is a pure morphism in $\mathcal{V}$; that is, for any $v \in \mathcal{V}$, the morphism $i \otimes v : I \otimes v \to R \otimes v$ is a regular monomorphism in $\mathcal{V}_0$;*
(iv) *the morphism $i : I \to R$ is a split monomorphism in $\mathcal{V}_0$;*
(v) *the monad $\mathbf{T}^R$ is of effective descent type.*

Assume now that $\mathcal{V}_0$ has coequalizers and that all functors $V \otimes - : \mathcal{V}_0 \to \mathcal{V}_0$ as well as $- \otimes V : \mathcal{V}_0 \to \mathcal{V}_0$ preserve these coequalizers, as they surely do when $\mathcal{V}$ is closed. Then there is a well-defined functor

$$- \otimes_R - : \mathcal{V}_R \times {}_R\mathcal{V} \to \mathcal{V}_0$$

which assigns to any pair (N, M) $(N \in \mathcal{V}_R$ and $M \in {}_R\mathcal{V})$ the coequalizer of the two morphisms

$$N \otimes R \otimes M \rightrightarrows N \otimes M$$

induced by the actions of R on N and M.

Recall that a $\mathcal{V}$-algebra $R = (R, i_R, m_R)$ is called commutative if the multiplication morphism is unchanged when composed with the symmetry (i.e. if $m_R \cdot c_{R,R} = m_R$).

For any commutative $\mathcal{V}$-algebra $R = (R, i_R, m_R)$, the assignment

$$(M, \lambda_M) \longmapsto (M, \lambda_M \cdot \beta_{M,R})$$

defines a functor

$${}_R\mathcal{V} \longmapsto \mathcal{V}_R \tag{4}$$

that identifies the category ${}_R\mathcal{V}$ of left R-modules with the category $\mathcal{V}_R$ of right R-modules, and we simply speak of R-modules. In this case, the tensor product over R of two R-modules is another R-module and tensoring over R makes ${}_R\mathcal{V}$ (as well as $\mathcal{V}_R$) into a monoidal category, and the symmetry c induces a symmetry on ${}_R\mathcal{V}$. Hence ${}_R\mathcal{V}$ is a symmetric monoidal category. Moreover, if $\mathcal{V}$ is closed (i.e. there is a right adjoint functor $[V, -] : \mathcal{V}_0 \to \mathcal{V}_0$ for every functor $V \otimes - : \mathcal{V}_0 \to \mathcal{V}_0$), then ${}_R\mathcal{V}$ is also closed: The internal Hom object ${}_R[M, N]$ of two R-modules M and N is the equalizer of two morphisms

$$[M, N] \rightrightarrows [R \otimes M,\ N],$$

where the first morphism is induced by the action of R on M, while the second morphism is the composition of

$$(R \otimes -)_{M,N} : [M, N] \to [R \otimes M, R \otimes N]$$

followed by the morphism induced by the action of R on N. Note that, for any $V \in \mathcal{V}_0$, the object $[R, V]$ becomes a left R-module and that the assignment $V \longmapsto [R, V]$ defines a functor $[R, -] : \mathcal{V} \to {}_R\mathcal{V}$ which is a right adjoint of the underlying functor ${}_RU : {}_R\mathcal{V} \to \mathcal{V}$. See [14] for a reference on algebras and modules in a symmetric monoidal category.

Theorem 4.3. *Let $\mathcal{V}$ be a $*$-autonomous category. Then, for any commutative algebra $R = (R, i_R, m_R)$ in $\mathcal{V}$, ${}_R\mathcal{V}$ is a $*$-autonomous category. Moreover, if tensor unit I is projective in $\mathcal{V}_0$, then R is projective in ${}_R\mathcal{V}$.*

Proof. First, since R is assumed to be commutative, ${}_R\mathcal{V}$ is a symmetric monoidal closed category by the discussion preceding the theorem.

Next, it is easy to check by using the adjunction ${}_RU \dashv [R, -] : \mathcal{V} \to {}_R\mathcal{V}$ that if Q is a dualizing object for $\mathcal{V}$, then $[R, Q]$ is a dualizing object for ${}_R\mathcal{V}$.

Finally, since the functor ${}_RU : {}_R\mathcal{V} \to \mathcal{V}$ admits as a left adjoint the functor $R \otimes - : \mathcal{V} \to {}_R\mathcal{V}$, there is a natural isomorphism

$$ {}_R\mathcal{V}(R \otimes V, M) \simeq \mathcal{V}_0(V, {}_RU(M)), \quad \text{for all } V \in \mathcal{V} \text{ and } M \in {}_R\mathcal{V}\,. $$

Putting $V = I$ and using the fact that ${}_RU$, being right adjoint, preserves regular epimorphisms, we see that R is projective in ${}_R\mathcal{V}$, provided that I is projective in $\mathcal{V}_0$. ■

For a commutative $\mathcal{V}$-algebra $R = (R, i_R, m_R)$, an R-algebra is defined to be an algebra in the symmetric monoidal category ${}_R\mathcal{V}$. It is well known that specifying an R-algebra structure on an object $S \in \mathcal{V}_0$ is the same as giving S a $\mathcal{V}$-algebra structure together with a morphism $i : R \to S$ of $\mathcal{V}$-algebras which is central in the sense that the diagram

$$\begin{tikzcd} R \otimes S \arrow[r, "\beta_{R,S}"] \arrow[d, "i \otimes S"'] & S \otimes R \arrow[r, "S \otimes i"] & S \otimes S \arrow[d, "m_S"] \\ S \otimes S \arrow[rr, "m_S"] & & S \end{tikzcd}$$

is commutative. (Quite obviously, any morphism of commutative $\mathcal{V}$-algebras is central.) Thus, if $i : R \to S$ is a central morphism of $\mathcal{V}$-algebras, then S can be viewed as an object of ${}_R\mathcal{V}$ via

$$ R \otimes S \xrightarrow{i \otimes S} S \otimes S \xrightarrow{m_S} S\ , $$

or, equivalently (since i is central), via

$$ R \otimes S \xrightarrow{\beta_{R,S}} S \otimes R \xrightarrow{S \otimes i} S \otimes S \xrightarrow{m_S} S\ , $$

and then the multiplication $m_S : S \otimes S \to S$ factors through the quotient $S \otimes S \to S \otimes_R S$ (that is, there is some (necessarily unique) morphism $m'_S : S \otimes_R S \to S$ making the diagram

$$\begin{tikzcd} S \otimes S \arrow[r] \arrow[dr, "m_S"'] & S \otimes_R S \arrow[d, "m'_S"] \\ & M \end{tikzcd}$$

commute), and it is easy to see that the triple $S^R = (S, i, m'_S)$ is a commutative algebra in the symmetric monoidal category ${}_R\mathcal{V}$.

Theorem 4.4. *Let $\mathcal{V}$ be a $*$-autonomous category whose tensor unit is projective in $\mathcal{V}_0$, let R be a commutative $\mathcal{V}$-algebra, and let $i : R \to S$ be a central morphism of $\mathcal{V}$-algebras. Then the following are equivalent:*

(i) *the functor $S \otimes_R - : {}_R\mathcal{V} \to_S \mathcal{V}$ is precomonadic;*
(ii) *the functor $S \otimes_R - : {}_R\mathcal{V} \to_S \mathcal{V}$ is comonadic;*
(iii) *i is a pure morphism of (left) R-modules, i.e. for any (left) R-module M, the morphism $i \otimes_R M : M \simeq R \otimes_R M \to S \otimes_R M$ is a regular monomorphism;*
(iv) *i is a split monomorphism of (left) R-modules.*

Proof. We begin by noticing that, since R is a commutative $\mathcal{V}$-algebra and since I is assumed to be projective in $\mathcal{V}_0$, ${}_R\mathcal{V}$ is a $*$-autonomous category whose tensor unit R is projective (see Proposition 3.9). Next, writing $S^R = (S, i : R \to S, m'_S)$ for the R-algebra corresponding to the central morphism i, it is easy to see that the category ${}_{(S^R)}({}_S\mathcal{V})$ can be identified with the category ${}_S\mathcal{V}$, and that, modulo this identification, $S \otimes_R - : {}_R\mathcal{V} \to {}_S\mathcal{V}$ corresponds to $S^R \otimes_R - : {}_R\mathcal{V} \to {}_{(S^R)}({}_S\mathcal{V})$. Now, applying Theorem 4.2 gives that (i)–(iv) are equivalent. ■

Now let $\mathcal{V}$ be a symmetric monoidal category whose underlying ordinary category $\mathcal{V}_0$ is locally small, complete, and cocomplete. Then recall that, for any small $\mathcal{V}$-category $\mathcal{A}$, there exists the so-called $\mathcal{V}$-functor category $[\mathcal{A}, \mathcal{V}]$, which is an enrichment of the ordinary category $[\mathcal{A}, \mathcal{V}]_0$ over $\mathcal{V}$, and that $[\mathcal{A}, \mathcal{V}]$ is a $\mathcal{V}$-tensored category in the sense that any representable functor

$$ [F, -] = [\mathcal{A}, \mathcal{V}](F, -) : [\mathcal{A}, \mathcal{V}] \to \mathcal{V} $$

is right adjoint: The left adjoint to the functor $[F, -]$ is the functor

$$F \otimes - : \mathcal{V} \to [\mathcal{A}, \mathcal{V}],$$
$$v \longrightarrow [a \to F(a) \otimes v].$$

Recall also that $F : \mathcal{A} \to \mathcal{V}$ is *small projective* if the functor $[F, -]$ preserves all small (indexed) colimits. Fix such a functor F, and write $\mathbf{T}_F$ for the $\mathcal{V}$-monad on $\mathcal{V}$ generated by the adjunction $F \otimes - \dashv [F, -]$. Since F is assumed to be small projective, it follows that $T_F(v) = v \otimes [F, F]$ for all $v \in \mathcal{V}_0$. Thus, tensoring with the $\mathcal{V}$-algebra $[F, F]$ is isomorphic to the monad $\mathbf{T}_F$; in other words, the monads $\mathbf{T}_F$ and $\mathbf{T}^{[F,F]}$ are isomorphic. So the category $\mathcal{V}^{\mathbf{T}_F}$ is (isomorphic to) the category of right $[F, F]$-modules, $\mathcal{V}_{[F,F]}$, and the free $\mathbf{T}_F$-algebra functor $\mathcal{V} \to \mathcal{V}^{\mathbf{T}_F}$ is (isomorphic to) the functor $- \otimes [F, F] : \mathcal{V} \to \mathcal{V}_{[F,F]}$. Since $\mathcal{V}$ has all equalizers, and since any category with all equalizers is Cauchy complete (in the sense that every idempotent endomorphism e has a factorization $e = ij$ with $ji = 1$), Theorem 3.20 in [11] and Theorem 2.1 together give the following:

Theorem 4.5. *Let $\mathcal{A}$ be a small $\mathcal{V}$-category. If $F : \mathcal{A} \to \mathcal{V}$ is small projective, then the functor*

$$F \otimes - : \mathcal{V} \to [\mathcal{A}, \mathcal{V}]$$

is (pre)comonadic if and only if the functor

$$- \otimes [F, F] : \mathcal{V} \to \mathcal{V}_{[F,F]}$$

is.

Combining this with Corollary 4.1 in [6], we conclude that

Corollary 4.6. *In the situation of the previous theorem, if the morphism*

$$\ulcorner 1_F \urcorner : I \to [F, F]$$

is a split monomorphism, then the functor

$$F \otimes - : \mathcal{V} \to [\mathcal{A}, \mathcal{V}]$$

is comonadic.

Let $\mathcal{E}$ be an elementary topos with the subobject classifier Ω, and let $\mathcal{CL}(\mathcal{E})$ denote the category of internal sup-lattices (i.e. internally complete lattices and sup-preserving morphisms) in $\mathcal{E}$. It is well known (see, [8]) that $\mathcal{CL}(\mathcal{E})$ is a symmetric monoidal closed category, in fact $*$-autonomous. We recall that the unit object for the tensor product in $\mathcal{CL}(\mathcal{E})$ is Ω, while the dualising object is Ω^{op}, the sup-lattice provided with the partial order opposite to that one of Ω.

Recall also that $\mathcal{CL}(\mathcal{E})$ is a monadic category over $\mathcal{E}$: If we consider the covariant power set functor $\mathcal{P} : \mathcal{E} \to \mathcal{E}$, then it is a monad via the morphisms $f : X \to \mathcal{P}(X)$ of taking singletons and $\cup : \mathcal{P}\mathcal{P}(X) \to \mathcal{P}(X)$ of taking the internal union, respectively; an object $X \in \mathcal{E}$ is an algebra for this monad precisely if X is an internal sup-lattice in $\mathcal{E}$, and thus $\mathcal{CL}(\mathcal{E})$ is (isomorphic to) the category of $\mathcal{P}$-algebras. Moreover, since the forgetful functor $\mathcal{CL}(\mathcal{E}) \to \mathcal{E}$ takes epimorphisms to split epimorphisms [8], the free $\mathcal{P}$-algebras (=the free complete lattices) are projective in $\mathcal{CL}(\mathcal{E})_0$ [15]; in particular, Ω is projective in $\mathcal{CL}(\mathcal{E})_0$, since $\mathcal{P}(1) = \Omega$ – here 1 is the terminal object of $\mathcal{E}$ – and hence the dualizing object Ω^{op} is injective in $\mathcal{CL}(\mathcal{E})_0$ by Proposition 3.9. Finally, since $\mathcal{CL}(\mathcal{E})$ is monadic on $\mathcal{E}$, all small limits (and hence all small colimits) exist in $\mathcal{CL}(\mathcal{E})$ provided they exist in $\mathcal{E}$.

Recall that a morphism $i : R \to S$ of commutative $\mathcal{CL}$-algebras is an *effective descent morphism for modules* if the (ordinary) functor $S \otimes_R - : {}_R\mathcal{CL}(\mathcal{E}) \to {}_S\mathcal{CL}(\mathcal{E})$ is comonadic.

Applying Theorem 3.4, we obtain the following result of Joyal and Tierney [8]:

Theorem 4.7. *Let $\mathcal{E}$ be an elementary topos with small limits. A morphism $i : R \to S$ of commutative algebras in $\mathcal{CL}(\mathcal{E})$ is an effective descent morphism for modules if and only if i is a split monomorphism of R-modules.*

We now look at the case where $\mathcal{E}$ = Sets, and write simply $\mathcal{CL}$ for the category $\mathcal{CL}$(Sets). It is well known (see, for example [8]) that, in $\mathcal{CL}$, (small) coproducts are biproducts; that is, the coproduct of $\{X_i, i \in I\}$ in $\mathcal{CL}$ is the same as the product $\prod_{i \in I} X_i$. We will write $\bigoplus_{i \in I} X_i$ for this biproduct. It is also a well-known fact (see [7]) that, for any small $\mathcal{CL}$-category $\mathcal{A}$, the functor $P = \bigoplus_{a \in \mathcal{A}_0} \mathcal{A}(a, -)$ is small projective, and thus, by Theorem 4.5, the functor

$$P \otimes - : \mathcal{CL} \to [\mathcal{A}, \mathcal{CL}]$$

is comonadic iff the functor

$$- \otimes [P, P] : \mathcal{CL} \to \mathcal{CL}_{[P,P]}$$

is. One easily sees that the underlying object of the $\mathcal{CL}$-algebra $[P, P]$ is $\bigoplus_{a_0, a_1 \in \mathcal{A}_0} \mathcal{A}(a_0, a_1)$, and that the unit morphism $i : I \to [P, P] = \bigoplus_{a_0, a_1 \in \mathcal{A}_0} \mathcal{A}(a_0, a_1)$ is given as follows:

$$(i(1))_{a_0,a_1} = \begin{cases} 1_{a_0}, & \text{if } a_0 = a_1, \\ 0, & \text{if } a_0 \neq a_1 \end{cases} \quad \text{and} \quad (i(0))_{a_0,a_1} = 0 \quad \text{for all } a_0, a_1 \in \mathcal{A}_0.$$

Theorem 4.8. *For any small $\mathcal{CL}$-category $\mathcal{A}$, the functor*

$$P \otimes - : \mathcal{CL} \to [\mathcal{A}, \mathcal{CL}],$$

where $P = \bigoplus_{a \in \mathcal{A}_0} \mathcal{A}(a, -)$, is comonadic if and only if there exists an object $a \in \mathcal{A}_0$ such that the morphism $\ulcorner 1_a \urcorner : I \to \mathcal{A}(a, a)$ is a split monomorphism.

Proof. We claim that the morphism $i : I \to [P, P] = \bigoplus_{a_0, a_1 \in \mathcal{A}_0} \mathcal{A}(a_0, a_1)$ is a split monomorphism in $\mathcal{CL}$ if and only if there exists an object $a \in \mathcal{A}_0$ such that the morphism $\ulcorner 1_a \urcorner : I \to \mathcal{A}(a, a)$ is a split monomorphism in $\mathcal{CL}$. Indeed, since one direction is clear, suppose that i is split by some $j : \bigoplus_{a_0, a_1 \in \mathcal{A}_0} \mathcal{A}(a_0, a_1) \to I$. Then, writing $e_a,\ a \in \mathcal{A}_0$, for the image of $1 \in I$ under the composite

$$I \xrightarrow{\ulcorner 1_a \urcorner} \mathcal{A}(a, a) \xrightarrow{i_{a,a}} \bigoplus_{a_0, a_1 \in \mathcal{A}_0} \mathcal{A}(a_0, a_1),$$

it is easy to see that $\vee_{a \in \mathcal{A}_0} e_a = i(1)$, and since $ji = 1$, this implies that $1 = ji(1) = j(\vee_{a \in \mathcal{A}_0} e_a) = \vee_{a \in \mathcal{A}_0} j(e_a)$. Now, if each $j(e_a) = 0$, then $1 = \vee_{a \in \mathcal{A}_0} j(e_a) = 0$, a contradiction. Therefore, there exists an object $\bar{a} \in \mathcal{A}_0$ with $j(e_{\bar{a}}) = 1$, which just means that the morphism $\ulcorner 1_{\bar{a}} \urcorner : I \to \mathcal{A}(\bar{a}, \bar{a})$ is a split monomorphism. The result now follows from Theorems 4.4 and 4.7. ∎

In order to understand the meaning of this theorem, we consider the comonad $\mathbf{G}_{\mathcal{A}}$ on the category $[\mathcal{A}, \mathcal{CL}]$ generated by the adjunction

$$P \otimes - \dashv [P, -] : [\mathcal{A}, \mathcal{CL}] \to \mathcal{CL}.$$

It is not hard to see that the functor-part of this comonad takes any $\mathcal{CL}$-functor $F : \mathcal{A} \to \mathcal{CL}$ to the $\mathcal{CL}$-functor $P \otimes (\bigoplus_{a \in \mathcal{A}_0} F(a))$. Now, we can interpret Theorem 4.8 as saying that $\mathbf{G}_{\mathcal{A}}$-coalgebras are exactly the $\mathcal{CL}$-functors of the form $P \otimes X$ with $X \in \mathcal{CL}$ iff there exists an object $a \in \mathcal{A}_0$ such that the morphism $\ulcorner 1_a \urcorner : I \to \mathcal{A}(a, a)$ is a split monomorphism.

Another way of interpreting the meaning of Theorem 4.8 is the following:

Theorem 4.9. *For any algebra in $\mathcal{CL}$ of the form $[P, P]$, where $P = \bigoplus_{a \in \mathcal{A}_0} \mathcal{A}(a, -)$ for a small $\mathcal{CL}$-category $\mathcal{A}$, the induced monad $\mathbf{T}^{[P,P]}$ on $\mathcal{CL}$ is of effective descent type if and only if there exists an object $a \in \mathcal{A}_0$ such that the morphism $\ulcorner 1_a \urcorner : I \to \mathcal{A}(a, a)$ is a split monomorphism.*

Acknowledgements

This research was supported by the research project "Algebraic and Topological Structures in Homotopical and Categorical Algebra, K-theory and Cyclic Homology", with the financial support of the grant GNSF/ST06/3-004.

References

[1] J. Bénabou, J. Roubaud, Monades et descente, Comptes Rendus de l'Academie des Sciences 270 (1970) 96–98.
[2] M. Barr, ∗-Autonomous Categories, in: Lecture Notes in Mathematics, vol. 752, Springer-Verlag, 1979.
[3] M. Bunge, Relative functor categories and categories of algebras, Journal of Algebra 11 (1969) 64–101.
[4] E. Dubuc, Kan Extensions in Enriched Category Theory, in: Lecture Notes in Mathematics, vol. 145, Springer-Verlag, 1970.
[5] A. Grothendieck, Technique de descente et théoremes d'existence en géometrie algébrique I, Séminaire Bourbaki 190 (1959).
[6] G. Janelidze, W. Tholen, Facets of descent, III: Monadic descent for rings and algebras, Applied Categorical Structures 12 (2004) 461–476.
[7] S.R. Johnson, Small Cauchy completions, Journal of Pure and Applied Algebra 62 (1989) 35–45.
[8] A. Joyal, M. Tierney, An extension of the Galois theory of Grothendieck, Memoirs of the American Mathematical Society (309) (1984).
[9] G.M. Kelly, Basic Concepts of Enriched Category Theory, Cambridge University Press, Cambridge, 1982.
[10] G.M. Kelly, A.J. Power, Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads, Journal of Pure and Applied Algebra 89 (1993) 163–179.
[11] B. Mesablishvili, Monads of effective descent type and comonadicity, Theory and Applications of Categories 16 (2006) 1–45.
[12] B. Mesablishvili, Pure morphisms of commutative rings are effective descent morphisms for modules — A new proof, Theory and Applications of Categories 7 (2000) 38–42.
[13] J.-P. Olivier, Descente par morphismes purs, Comptes Rendus de l'Academie des Sciences Paris Serie A–B 271 (1970) 821–823.
[14] B. Pareigis, Non-additive rings and module theory I. General theory of monoids, Publicationes Mathematicae (Debrecen) 24 (1977) 189–204.
[15] E. Vitale, On the characterization of monadic categories over *SET*, Cahiers de Topologie et Géométrie Différentielle Catégoriques 35 (1994) 351–358.