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Further Reductions of Normal Forms for Dynamical Systems

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We propose in this paper a method for obtaining a significant refinement of normal forms for dynamical systems or vector fields, with concrete and interesting applications. We use lower order nonlinear terms in the normal form for the simplifications of higher order terms. Our approach is applicable for both the non nilpotent and the nilpotent cases. For dynamical systems of dimensions 2 and 3 we give an algorithm that leads to interesting finite order normal forms which are optimal (or unique) with respect to equivalence by formal near identity transformations. We can compute at the same time a formal diffeormorphism that realizes the normalization. Comparisons with other methods are given for several examples.

1. INTRODUCTION

Let K be a commutative field of characteristic zero (which, in practice, is $\mathbf{Q}$, $\mathbf{R}$). We denote by $K[[x]]$ the algebra of formal power series in n variables with coefficients belonging to the field K. In this paper we consider the normal form problem for a dynamical system (or its associated vector field)

$$\dot{x} = \frac{dx}{dt} = F(x) \tag{1}$$

where $x = (x_1, ..., x_n)^t$, and $F(x) = (f_1(x), ..., f_n(x))^t$ with $f_j(x) \in K[[x]]$ and $F(0) = 0$. One can write $F(x) = \sum_{k \geqslant 1} F^k(x)$, where $F^k(x)$ is a vector of homogeneous polynomials of degree k and $F^1(x) = Ax$, $A \in \mathrm{gl}(n, K)$, is

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the linear part of the system, where $\mathrm{gl}(n, K)$ denotes the set of $n \times n$ matrices with entries in K.

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be the eigenvalues of A and suppose that A is in the Jordan canonical form with $\lambda_1, ..., \lambda_n$ on the diagonal. Let $q = (q_1, ..., q_n) \in \mathbf{N}^n$ with

$$|q| = q_1 + \cdots + q_n \geqslant 2.$$

We define $x^q = x_1^{q_1} \cdots x_n^{q_n}$. A monomial $a_q x^q$ in $f_j(x)$ is called resonant of order $|q|$ if $\lambda_j = \langle q, \lambda \rangle = q_1 \lambda_1 + \cdots + q_n \lambda_n$.

The Poincaré–Dulac normal form theorem states the following

Theorem (Poincaré–Dulac). *Consider a dynamical system of the form* (1) *where $F(x) = Ax + \cdots$ is a vector of formal power series, and A is in the Jordan canonical form as above. Then* (1) *can be reduced to a normal form $\dot{y} = Ay + h(y)$ by a near identity change of variables of the form $x = y + \varphi(y)$, where $\varphi(y)$ is a vector of formal power series without constant term or linear part and $h(y)$ contains only resonant monomials.*

In practice one uses the Poincaré–Dulac Theorem to get a normal form up to some finite order, say N, to obtain

$$\dot{y} = Ay + z(y) + O(\|y\|^{N+1})$$

where $z(y)$ is a polynomial of total degree less than or equal to N without constant or linear terms and $O(\|y\|^{N+1})$ represents terms of order greater than or equal to $N+1$.

A fundamental improvement of the normal form is given by Takens in [20] which shall be studied and improved upon in this paper.

Systematic procedures for constructing normal forms have been given in many ways. A method using Lie brackets is given in [13, 20, 22], a method by considering an inner product in the space of homogeneous polynomials is given in [2, 12, 15], a method by direct computation is given in [5], and a method using Carleman linearization is given in [21]. The nilpotent case (A is a nilpotent matrix) is treated in [14] using representation theory of $\mathrm{sl}_2(\mathbf{R})$ and in [9, 10] using Carleman linearization.

All of the above works use the Jordan canonical form of the leading matrix A. But it is well known that handling eigenvalues and Jordan canonical forms is a very difficult problem in computer algebra systems. We propose in this paper to give further reductions of the classical normal forms for dynamical systems. Our approach is rational in the sense that if the coefficients of the system are in K (which is $\mathbf{Q}$, $\mathbf{R}$ in practice), so is the normal form and all the computations are done in K. We do not need to compute the Jordan canonical form of A nor its eigenvalues. Our method is applicable in both the nilpotent and the non nilpotent cases. For dynamical systems of dimensions 2 and 3 we give an algorithm that leads to interesting results which improve several normal forms given by Ushiki in [22] by increasing their orders. The normal forms obtained in these cases are unique with respect to equivalence by near identity changes of variables up to the given order. Uniqueness results on some semisimple examples are given in [3]. Our method can also be used to find degeneracy conditions and to compute the corresponding degenerate normal forms. We refer the reader to [8] for a general algorithm in any dimension using the Carleman linearization procedure.

We begin with a presentation of the classical normal form theorem in Section 2.1. We recall in Section 2.2 the Takens' normal form of vector fields in dimension 2 which is used in Section 3. And we then give a refinement of the classical normal form in Section 2.3. In Section 3 we give an algorithm for the case of dimension 2 by a detailed study of the homological equations. In Section 4 we study dynamical systems in dimension 3. We have a unified treatment for the nilpotent and the non nilpotent cases. We obtain interesting normal forms for both the nilpotent and the non nilpotent cases. In particular, we get further reductions of normal forms of Takens.

2. FUNDAMENTAL THEOREMS ON NORMAL FORMS FOR DYNAMICAL SYSTEMS

2.1. *Classical Normal Form Theorem*

Consider a dynamical system of dimension n of the form (1), i.e.,

$$\dot{x} = F(x) = Ax + F^2(x) + F^3(x) + \cdots, \qquad x \in K^n, \tag{2}$$

where A is an $n \times n$ matrix with coefficients in K and F is a vector of formal power series with coefficients in K. The notation F^k is used to denote the homogeneous part of degree k of F, $F^k \in H_k^n$, where H_k is the space of homogeneous polynomials of degree k in n variables with coefficients in K.

Consider at first a linear transformation

$$x = \varphi^1(y) = Py \tag{3}$$

where P is an invertible matrix with coefficients in K. We obtain a new system in the same form: $\dot{y} = P^{-1}APy + \cdots$. It is clear that one can choose a matrix P such that A is in some canonical form. In the classical methods one chooses the Jordan canonical form. In the following we shall use, as in [8], the Frobenius canonical form whose coefficients remain in the field K.

Consider next a formal transformation (a near identity change of coordinates)

$$x = y + \varphi(y) \tag{4}$$

where $\varphi(y)$ has no constant or linear terms. By substituting this in (2), we get:

$$\dot{y} = (I + \partial_y \varphi)^{-1} F(y + \varphi(y)) \tag{5}$$

where $\partial_y \varphi$ denotes the Jacobian matrix of φ with respect to $(y_1, ..., y_n)$. Therefore one can write the new system in the form

$$\dot{y} = Ay + G(y).$$

We shall say that this new system is equivalent to the original system (2) by a formal near identity change of variables.

The goal is to determine changes of coordinates (3) and (4) such that the transformed system will be in the simplest form possible in a sense to be specified later on. The desired simplification of (2) will be obtained, up to terms of a specified order, by performing inductively a sequence of near identity changes of coordinates of the form $x = y + \varphi^k(y)$ where $\varphi^k(y)$ is a vector of homogeneous polynomials of degree $k \geqslant 2$. We have

$$(I + \partial_y \varphi^k(y))^{-1} = I - \partial_y \varphi^k(y) + O(\|y\|^{2k-2})$$

where $O(\|y\|^m)$ denotes terms of order greater than or equal to m. Substituting it into (5) we obtain

$$\dot{y} = Ay + \cdots + F^{k-1}(y) + \{F^k(y) - [\partial_y \varphi^k(y)\, Ay - A\varphi^k(y)]\} + O(\|y\|^{k+1}). \tag{6}$$

One then has the equation

$$F^k(y) - [\partial_y \varphi^k(y)\, Ay - A\varphi^k(y)] = G^k(y) \tag{7}$$

which is called the homological equation (of order k). In order to obtain a G^k that contains as few monomials as possible we have to choose a suitable $\varphi^k(y)$. We introduce for each $k \geqslant 2$ a linear operator $L_A^k\colon H_k^n \to H_k^n$ defined by

$$(L_A^k \varphi^k)(y) = \partial_y \varphi^k(y)\, Ay - A\varphi^k(y), \qquad \varphi^k \in H_k^n.$$

Then (6) can be expressed as

$$\dot{y} = Ay + \cdots + F^{k-1}(y) + [F^k(y) - L_A^k \varphi^k(y)] + O(\|y\|^{k+1}).$$

Let $\mathscr{R}^k$ be the range of L_A^k in H_k^n and $\mathscr{C}^k$ be any supplementary subspace to $\mathscr{R}^k$ in H_k^n. We have

$$H_k^n = \mathscr{R}^k \oplus \mathscr{C}^k. \tag{8}$$

The fundamental idea of the classical normal form theory is in the following theorem (due to Takens [20]) which is an improvement of the Poincaré–Dulac theorem (see also [1, 11, 12]).

THEOREM 2.1 (Takens). *Consider a dynamical system of the form* (2). *Let the notations be as above. Suppose that the decomposition* (8) *is given for* $k = 2, ..., N$. *Then there exists a sequence of near identity changes of variables* $x = y + \varphi^k(y)$ *where* $\varphi^k(y) \in H_k^n$ *such that the dynamical system* (2) *is transformed into*

$$\dot{y} = Ay + G^2(y) + \cdots + G^N(y) + O(\|y\|^{N+1})$$

where $G^k \in \mathscr{C}^k$ *for* $k = 2, ..., N$.

The method in [15] using an inner product in H_k^n and the method in [14] using the representation theory of $\mathrm{sl}_2(\mathbf{R})$ lead to the above normal form of Takens. Their methods construct a basis of a supplementary subspace $\mathscr{C}^k$. In general this basis consists of homogeneous polynomials (not always monomials). See also [2, 8, 10–12, 21] for other methods.

2.2. *An Example of Takens' Normal Form*

In this subsection we suppose $K = \mathbf{R}$. For notations and definitions we refer the reader to [20].

A dynamical system of the form (1) can be seen as a vector field in $\mathbf{R}^n$ of the form $X = \sum_{i=1}^n f_i(x)(\partial/\partial x_i)$ with a singularity at the origin, i.e., $X(0) = 0$. We define X_1 to be the vector field in $\mathbf{R}^n$ which has the same 1-jet in 0 as X and whose coefficient functions are linear, i.e. the linear part of X.

$[X_1, -]\colon H_k^n \to H_k^n$ is the linear map which assigns to each $Y \in H_k^n$ the Lie product $[X_1, Y]$ which is again in H_k^n. For X_1 fixed, we define a splitting of $H_k^n = \mathscr{R}^k \oplus \mathscr{C}^k$ such that $\mathscr{R}^k = \mathrm{Im}([X_1, -])$ and $\mathscr{C}^k$ is some supplementary subspace of $\mathscr{R}^k$ in H_k^n. In a similar way Theorem 2.1 can be stated in terms of vector fields (see [20]): For any integer $k \geqslant 2$, there is a formal diffeomorphism $\varphi\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that

$$\varphi^*(X) = X_1 + g_2 + \cdots + g_k + R_k,$$

where $g_i \in \mathscr{C}^i$, $i = 2, ..., k$, and R_k is a vector field, the component functions of which all have zero k-jet.

We shall apply this theorem to vector fields in $\mathbf{R}^2$ with a linear part having only one-block Frobenius canonical form, i.e.,

$$X_1 = x_2 \frac{\partial}{\partial x_1} + (c_0 x_1 + c_1 x_2) \frac{\partial}{\partial x_2}$$

with $c_0, c_1 \in \mathbf{R}$. The range $\mathscr{R}^k$ of $[X_1, -]$ in H_k^n is determined by the formulae:

$$\left[X_1, x_1^k \frac{\partial}{\partial x_1} \right] = k x_1^{k-1} x_2 \frac{\partial}{\partial x_1} - c_0 x_1^k \frac{\partial}{\partial x_2}, \tag{9}$$

$$\left[X_1, x_1^k \frac{\partial}{\partial x_2} \right] = -x_1^k \frac{\partial}{\partial x_1} + (k x_1^{k-1} x_2 - c_1 x_1^k) \frac{\partial}{\partial x_2}, \tag{10}$$

and for $0 \leqslant j \leqslant k-1$,

$$\begin{aligned}\left[X_1, x_1^j x_2^{k-j} \frac{\partial}{\partial x_1} \right] &= -c_0 x_1^j x_2^{k-j} \frac{\partial}{\partial x_2} \\ &\quad + (j x_1^{j-1} x_2^{k-j+1} + (k-j)\, c_1 x_1^j x_2^{k-j} + (k-j)\, c_0 x_1^{j+1} x_2^{k-j-1}) \frac{\partial}{\partial x_1},\end{aligned} \tag{11}$$

$$\begin{aligned}\left[X_1, x_1^j x_2^{k-j} \frac{\partial}{\partial x_2} \right] &= -x_1^j x_2^{k-j} \frac{\partial}{\partial x_1} \\ &\quad + (j x_1^{j-1} x_2^{k-j+1} + (k-j-1)\, c_1 x_1^j x_2^{k-j} + (k-j)\, c_0 x_1^{j+1} x_2^{k-j-1}) \frac{\partial}{\partial x_2}.\end{aligned} \tag{12}$$

Now we prove that H_k^n decomposes into $H_k^n = \mathscr{R}^k + \mathscr{B}^k$ where the subspace $\mathscr{B}^k$ is spanned by $x_1^k(\partial/\partial x_1)$ and $x_1^k(\partial/\partial x_2)$. In fact, we prove, by downward induction on j, that $x_1^j x_2^{k-j}(\partial/\partial x_1)$ and $x_1^j x_2^{k-j}(\partial/\partial x_2)$ belong to $\mathscr{R}^k + \mathscr{B}^k$ for all $j = k, ..., 0$. Since $x_1^k(\partial/\partial x_i) \in \mathscr{B}^k$ for $i = 1, 2$, it is clear from (9) and (10) that

$$x_1^{k-1} x_2 \frac{\partial}{\partial x_i} \in \mathscr{R}^k + \mathscr{B}^k \qquad \text{for} \quad i = 1, 2.$$

Suppose now that this is true for $\ell \geqslant 1$. Equations (11) and (12) for $j = \ell$ imply that

$$x_1^{\ell-1} x_2^{k-\ell+1} \frac{\partial}{\partial x_i} \in \mathscr{R}^k + \mathscr{B}^k \qquad (i = 1, 2).$$

This proves the following:

PROPOSITION 2.1. *Let X be a vector field in $\mathbf{R}^2$ whose 1-jet in 0 equals the 1-jet of $x_2(\partial/\partial x_1)+(c_0x_1+c_1x_2)(\partial/\partial x_2)$ with $c_0, c_1 \in \mathbf{R}$. Then for any integer $k \geqslant 2$ there is a formal diffeomorphism φ: $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that*

$$\varphi^*(X)=x_2\frac{\partial}{\partial x_1}+(c_0x_1+c_1x_2)\frac{\partial}{\partial x_2}+\sum_{\ell=2}^{k}\left(a_\ell x_1^\ell\frac{\partial}{\partial x_1}+b_\ell x_1^\ell\frac{\partial}{\partial x_2}\right)+R_k$$

where the k-jet of R_k is zero.

Of course, some of the a_ℓ and b_ℓ may turn out to be zero.

In particular with $X_1=x_2(\partial/\partial x_1)$ we obtain the Takens' nilpotent normal form (see [20])

$$\varphi^*(X)=x_2\frac{\partial}{\partial x_1}+\sum_{i=1}^{k}\left(a_i x_1^i\frac{\partial}{\partial x_1}+b_i x_1^i\frac{\partial}{\partial x_2}\right)+R_k,$$

where the k-jet of R_k is zero.

Ushiki in [22] has obtained further reductions of normal forms up to some finite orders in some cases of dimensions 2 and 3. Baider in [3] and Gaeta in [17] also have given further reduction for systems in dimension 2 with a semisimple linear part. Baider and Sanders in [4] and Gaeta in [17] also have obtained further reduction of the Takens normal form in the nilpotent case of dimension 2. But their reductions are not complete in the latter case.

Our aim in this paper is to give further reductions of the above normal forms up to some finite order. Our normal forms will be optimal (or unique) up to the given order with respect to equivalence by formal near identity transformations.

2.3. *A Refinement of the Classical Normal Form*

First we note that a normal form is not unique for a fixed A. In fact it depends on the choices of the supplementary subspaces $\mathscr{C}^k$ $(k=2, ..., N)$. It is not unique even with fixed $\mathscr{C}^k$. It is also important to note that G^ℓ and φ^ℓ $(2 \leqslant \ell \leqslant k)$ are unique if and only if $\mathscr{C}^\ell=\{0\}$. The basic idea is that if φ^ℓ $(\ell<k)$ is not unique then it may be used to simplify some terms of G^k. This idea is essential in our method and has been used by other authors (see [17, 22]). We will show that it is possible to make such a choice.

Let $\ell \geqslant 2$ be an integer such that $\mathscr{C}^\ell \neq \{0\}$. From the decomposition (8) one obtains that $\dim \operatorname{Ker}(L_A^\ell)=\dim \mathscr{C}^\ell$. Let $\varphi_1^\ell \in H_\ell^n$ such that

$$G^\ell(y)=F^\ell(y)-L_A^\ell(\varphi_1^\ell)(y) \in \mathscr{C}^\ell$$

is in a normal form as in Theorem 2.1. Then

$$L_A^\ell(\varphi_1^\ell+\psi^\ell)=L_A^\ell(\varphi_1^\ell) \qquad \text{for all} \quad \psi^\ell \in \operatorname{Ker}(L_A^\ell).$$

Consider a transformation of the form

$$x = y + \varphi(y) = y + \varphi^{\ell}(y) + \cdots + \varphi^{k}(y).$$

Then the original system is converted to

$$\begin{aligned}\dot{y} = {} & Ay + F^2(y) + \cdots + F^{\ell-1}(y) \\ & + F^{\ell}(y) \qquad + F^{\ell+1}(y) \qquad + \cdots + F^{k-1}(y) \qquad + F^{k}(y) \\ & - L_A^{\ell}(\varphi^{\ell})(y) - L_A^{\ell+1}(\varphi^{\ell+1})(y) - \cdots - L_A^{k-1}(\varphi^{k})(y) - L_A^{k}(\varphi^{k})(y) \\ & \qquad\qquad - T_F^{\ell+1}(\varphi)(y) \qquad - \cdots - T_F^{k-1}(\varphi)(y) - T_F^{k}(\varphi)(y)\end{aligned}$$

where $T_F^j(\varphi)$ are terms of order j. Then $T_F^j(\varphi)$ only depends on $F^2, ..., F^{j-1}$ and $\varphi^{\ell}, ..., \varphi^{j-1}$. Let $\varphi^{\ell} = \varphi_1^{\ell} + \psi^{\ell}$ where $\psi^{\ell} \in \mathrm{Ker}(L_A^k)$. Then

$$G^{\ell}(y) = F^{\ell}(y) - L_A^{\ell}(\varphi^{\ell})(y) \in \mathscr{C}^{\ell}.$$

According to Theorem 2.1, one can choose successively $\varphi^{\ell+1}, ..., \varphi^k$ (which may depend on ψ^{ℓ}) such that the transformed system is in a normal form, i.e.,

$$F^j - T_F^j(\varphi) - L_A^j(\varphi^j) \in \mathscr{C}^j$$

for $j = \ell + 1, ..., k$. One can write

$$F^k(y) - T_F^k(\varphi)(y) - L_A^k(\varphi^k)(y) = \tilde{G}^k - \hat{G}^k$$

where $\tilde{G}^k$ and $\hat{G}^k$ belong to $\mathscr{C}^k$ and where $\hat{G}^k$ contains all terms depending on ψ^{ℓ}.

Define a nonlinear operator

$$N_F^{\ell,k}\colon \mathrm{Ker}(L_A^{\ell}) \to \mathscr{C}^k \qquad \text{by} \quad N_F^{\ell,k}(\psi^{\ell}) = \hat{G}^k.$$

Let $\mathscr{R}_2^k$ be a subspace contained in the range of the operator $N_F^{\ell,k}$ in $\mathscr{C}^k$ and $\mathscr{C}_2^k$ be a supplementary subspace to $\mathscr{R}_2^k$ in $\mathscr{C}^k$. Then

$$\mathscr{C}^k = \mathscr{R}_2^k \oplus \mathscr{C}_2^k. \tag{13}$$

The main theorem of this paper is the following, which is a refinement of the fundamental Theorem 2.1.

THEOREM 2.2. *Consider the dynamical system* (2) *and let the notations be as above. Assume that* $\mathrm{Ker}(L_A^{\ell}) \neq \{0\}$. *Suppose that there exists a non-trivial subspace* $\mathscr{R}_2^k$ *such that we have decomposition* (13). *Then there exist*

$\varphi^\ell, ..., \varphi^k$ *as above and* $\psi^\ell \in \mathrm{Ker}(L_A^\ell)$ *such that the dynamical system* (2) *is transformed into*

$$\dot{y} = Ay + G^2(y) + \cdots + G^k(y) + O(\|y\|^{k+1})$$

where $G^j \in \mathscr{C}^j$ *for* $2 \leqslant j \leqslant k-1$ *and* $G^k \in \mathscr{C}_2^k$.

Although the above normal form is in general not unique, in some cases of interest it turns out to be actually unique as we shall show in the next sections.

Note that Ushiki in [22], and similarly Gaeta in [17], used $\mathrm{Ker}(L_A^\ell)$ to give further reduction of normal forms in a different way. Ushiki has treated some examples in dimensions 2 and 3 including the nilpotent and some non nilpotent cases. Gaeta has treated many examples with semisimple linear part and also the nilpotent case of dimension 2. It seems that his normal form in the latter case is the same as that obtained by Elphick *et al.* in [15]. We shall compare our method with that of Ushiki on some examples in the next sections.

In the following we study applications of the above theorem to dynamical systems of dimensions 2 and 3. We give an algorithm that computes both the classical and our refined normal forms for dynamical systems of dimensions 2 and 3.

3. DYNAMICAL SYSTEMS OF DIMENSION 2

3.1. *Reduction to Classical Normal Forms*

Consider a dynamical system of dimension 2, i.e., $\dot{x} = F(x) = Ax + \cdots$ with $n = 2$. We shall consider a rational normal form using the Frobenius canonical form of A. Write

$$A = \begin{pmatrix} 0 & 1 \\ c_0 & c_1 \end{pmatrix}$$

where $c_0, c_1 \in K$. One can determine matrices P such that $P^{-1}AP = A$. There may exist arbitrary parameters in the matrix P. The linear transformation $x = Py$ will not change the linear part of the system but one can use the arbitrary parameters in P to simplify some higher order terms of the system.

Let k be an integer $\geqslant 2$ and

$$F(x) = Ax + \cdots + F^k(x) + \cdots$$

where F^k is the homogeneous part of degree k in F. Make a change of variables of the form $x = y + \varphi(y)$ in the system (1), where $\varphi(y)$ is homogeneous of degree k, then one can determine $\varphi(y)$ by solving the homological equation (7), which is

$$(\partial_y \varphi)\, Ay - A\varphi(y) = F^k(y) - G^k(y). \tag{14}$$

The problem is then to determine φ so that $G^k(y)$ contains the smallest possible number of monomials.

One writes

$$F^k(x) = \begin{pmatrix} \sum_{|q|=k} \alpha_q x^q \\ \sum_{|q|=k} \beta_q x^q \end{pmatrix}, \quad G^k(y) = \begin{pmatrix} \sum_{|q|=k} \alpha'_q y^q \\ \sum_{|q|=k} \beta'_q y^q \end{pmatrix}, \quad \varphi(y) = \begin{pmatrix} \sum_{|q|=k} \alpha_q y^q \\ \sum_{|q|=k} b_q y^q \end{pmatrix}$$

with $a_q, b_q, \alpha'_q, \beta'_q \in K$ to be determined.

The equations in (14) can be written as

$$\sum_{|q|=k} ((q_1+1)\, a_{q+e_1-e_2} + q_2 c_1 a_q - b_q + (q_2+1)\, c_0 a_{q-e_1+e_2})\, y^q$$

$$= \sum_{|q|=k} (\alpha_q - \alpha'_q)\, y^q,$$

$$\sum_{|q|=k} ((q_1+1)\, b_{q+e_1-e_2} - c_0 a_q + (q_2-1)\, c_1 b_q + (q_2+1)\, c_0 b_{q-e_1+e_2})\, y^q$$

$$= \sum_{|q|=k} (\beta_q - \beta'_q)\, y^q$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$ form the standard basis of K^2. We then need to solve the equations

$$(q_1+1)\, a_{q+e_1-e_2} + q_2 c_1 a_q - b_q + (q_2+1)\, c_0 a_{q-e_1+e_2} = \alpha_q - \alpha'_q,$$

$$(q_1+1)\, b_{q+e_1-e_2} - c_0 a_q + (q_2-1)\, c_1 b_q + (q_2+1)\, c_0 b_{q-e_1+e_2} = \beta_q - \beta'_q.$$

If we can solve these equations with $\alpha'_q = \beta'_q = 0$ for all q such that $|q| = k$ then $G^k = 0$. If not, then a normal form obtained in this way will contain non zero terms in the homogeneous part of degree k.

Let

$$\mathscr{M}_q = \begin{pmatrix} a_q \\ b_q \end{pmatrix}, \qquad \Lambda_q = \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}, \qquad \Lambda'_q = \begin{pmatrix} \alpha'_q \\ \beta'_q \end{pmatrix},$$

and let I denote the 2×2 identity matrix. Then the above equations can be written in the form

$$(q_1+1)\,\mathcal{M}_{q+e_1-e_2}+(q_2c_1I-A)\,\mathcal{M}_q+(q_2+1)\,c_0\mathcal{M}_{q-e_1+e_2}=\Lambda_q-\Lambda'_q. \quad (15)$$

It is to be understood that $\mathcal{M}_q=0$ if one of the components of q is strictly negative. We consider the lexicographical order for the set

$$\{q=(q_1,q_2)\in\mathbf{N}^2 : |q|=k\}.$$

For $q_1=0$, $q_2=k$ one has the equations:

$$\mathcal{M}_{(1,k-1)}+(kc_1I-A)\,\mathcal{M}_{(0,k)}=\Lambda_{(0,k)}-\Lambda'_{(0,k)}.$$

Let $\mathcal{M}_{(0,k)}$ be chosen arbitrarily. The above equations have a unique solution $\mathcal{M}_{(1,k-1)}$ for any value on the right-hand side. Hence one can find a solution $\mathcal{M}_{(1,k-1)}$ for $\Lambda'_{(0,k)}=0$.

For $q=(j,k-j)$ the above equations become

$$\begin{aligned}(j+1)\,\mathcal{M}_{(j+1,k-j-1)}&+((k-j)\,c_1I-A)\,\mathcal{M}_{(j,k-j)}\\&+(k-j+1)\,c_0\mathcal{M}_{(j-1,k-j+1)}=\Lambda_{(j,k-j)}-\Lambda'_{(j,k-j)}.\end{aligned}$$

By induction on j, we can compute $\varphi(y)$ such that $\Lambda'_{(j,k-j)}=0$ for all $0\leqslant j\leqslant k-1$, i.e.,

$$G^k(y)=\begin{pmatrix}\alpha'_{k,0}\,y_1^k\\ \beta'_{k,0}\,y_1^k\end{pmatrix}.$$

It is clear that this can be done for any $k\geqslant 2$. Then a truncated rational normal form for dynamical systems of dimension 2 is

$$\dot{y}=Ay+\begin{pmatrix}\sum_{k=2}^{N}\alpha'_{k,0}\,y_1^k\\ \sum_{k=2}^{N}\beta'_{k,0}\,y_1^k\end{pmatrix}+O(\|y\|^{N+1}),$$

which coincides with a normal form of Proposition 2.1.

We now give a detailed study of Eq. (15) to show that further reductions can be done.

Let $Z_j = \mathscr{M}_{(j,k-j)}$, then

$$Z_{j+1} = \frac{1}{j+1}((A-(k-j)\,c_1 I)\,Z_j - (k-j+1)\,c_0 Z_{j-1}$$

$$+\Lambda_{(j,k-j)} - \Lambda'_{(j,k-j)}).$$

Let Z_0 be given arbitrarily. One can determine $Z_1, ..., Z_k$ by the above formulae such that $\Lambda'_{(j,k-j)} = 0$ for $0 \leqslant j \leqslant k-1$. For example, for $j=0$, $Z_1 = (A - kc_1 I)\,Z_0 + \Lambda_{(0,k)}$. In particular, all the $Z_1, ..., Z_k$ depend linearly on Z_0. In fact, with

$$Z_j = E_{j-1} Z_0 + B_{j-1}, \qquad 1 \leqslant j \leqslant k,$$

we have $E_0 = A - kc_1 I$, $B_0 = \Lambda_{(0,k)}$,

$$E_1 = \tfrac{1}{2}([A-(k-1)\,c_1 I]\,E_0 - kc_0 I),$$

$$B_1 = \tfrac{1}{2}(\Lambda_{(1,k-1)} + [A-(k-1)\,c_1 I]\,B_0),$$

and for $j \geqslant 2$,

$$E_j = \frac{1}{j+1}([A-(k-j)\,c_1 I]\,E_{j-1} - (k-j+1)\,c_0 E_{j-2}),$$

$$B_j = \frac{1}{j+1}(\Lambda_{(j,k-j)} + [A-(k-j)\,c_1 I]\,B_{j-1} - (k-j+1)\,c_0 B_{j-2}).$$

Then for $j=k$ we have

$$\Lambda'_{(k,0)} = Z_{k+1} = \Lambda_{(k,0)} + AZ_k - c_0 Z_{k-1} = E_k Z_0 + B_k \tag{16}$$

where E_k is a 2×2 matrix and B_k is a vector of dimension 2, which can be determined by iteration of the above form.

There are three cases to be distinguished:

(a) If $\det E_k \neq 0$ then one can find Z_0 such that $\Lambda'_{(k,0)} = 0$. This means that one can eliminate all terms of order k in the dynamical system, i.e., the homogeneous part of degree k in the normal form is reduced to 0. With notations as in Theorem 2.1, $\mathscr{C}^k = \{0\}$ in this case.

(b) If $\operatorname{rank}(E_k) = 1$ then one can solve Eq. (16) in such a way that one element of $\Lambda'_{(k,0)}$ ($\alpha'_{(k,0)}$ or $\beta'_{(k,0)}$) vanishes. This means that one can eliminate all terms of order k except one monomial. Then the homogeneous part of degree k in the normal form is

$$\begin{pmatrix} \alpha'_{(k,0)}\, y_1^k \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \beta'_{(k,0)}\, y_1^k \end{pmatrix}.$$

In this case we have $\dim \mathscr{C}^k = \dim \mathrm{Ker}(L_A^k) = 1$.

(c) $E_k = 0$ then in general $\Lambda'_{(k,0)} \neq 0$. The homogeneous part of degree k in the normal form is

$$\begin{pmatrix} \alpha'_{(k,0)}\, y_1^k \\ \beta'_{(k,0)}\, y_1^k \end{pmatrix}.$$

In this case we have $\dim \mathscr{C}^k = \dim \mathrm{Ker}(L_A^k) = 2$.

For a given k, we are interested in the values of the determinants of the matrices E_k. We now give a detailed discussion for some values of k. The following computations have been done in Maple V, Release 4.

If $c_0 = 0$ then $E = 0$. Then we are in case (c).

We now consider the case where $c_0 \neq 0$. For $k = 2$ one has

$$\det(E_2) = -\tfrac{1}{4} c_0^2\,(2\,c_1^2 + 9\,c_0).$$

Then if $c_0 \neq -\frac{2}{9} c_1^2$ all terms of order 2 can be eliminated in the normal form of the dynamical system. If $c_0 = -\frac{2}{9} c_1^2$ then $\mathrm{rank}(E_2) = 1$. In fact, we have

$$E_2 = \begin{bmatrix} -\frac{2}{27} c_1^3 & \frac{1}{9} c_1^2 \\ -\frac{2}{81} c_1^4 & \frac{1}{27} c_1^3 \end{bmatrix}.$$

Then we are in case (b). The homogeneous part of degree 2 in the normal form can be chosen to be in the form

$$\begin{pmatrix} 0 \\ \beta'_{(2,0)}\, y_1^2 \end{pmatrix}.$$

For $k = 3$, $\det(E_3) = -\frac{4}{9} c_0^2 c_1^2 (3\,c_1^2 + 16\,c_0)$, and for $k = 4$,

$$\det(E_4) = -\tfrac{9}{64} c_0^2 (25 c_0 + 4 c_1^2)(-2 c_1^2 + c_0)^2;$$

we can give discussions similar to those above and continue with higher orders.

For the particular case where $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we can prove by induction that $E_0 = A$ and for $i \geqslant 1$,

$$E_{2i-1} = \frac{k-1}{2} \cdots \frac{k-2i+1}{2i}\, I, \qquad E_{2i} = \frac{k-1}{2} \cdots \frac{k-2i+1}{2i}\, A.$$

Therefore if $k=2m$ is even then $E_k=(2m-1)/2 \cdots (1/2m)\, A$. We have $\det(E_k) \neq 0$. In these cases the homogeneous part of degree k can be reduced to 0. If k is odd then $E_k=0$, and we will study the further reductions in the next subsection.

For the case where

$$A=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have $E_k=0$ and we are in case (c). The homogeneous part of degree k in the normal form contains two non-zero parameters. Note that Cushman and Sanders in [14] and Elphick *et al.* in [15] by another method obtained a normal form

$$\dot{x}_1=x_2, \qquad \dot{x}_2=x_1 x_2 P(x_1)+x_1^2 P_2(x_1),$$

where $P_1(x_1)$ and $P_2(x_1)$ are formal power series in x_1. In particular, the homogeneous part of degree k contains also two nonzero parameters.

The above algorithm leads to a classical normal form. In the following, we apply Theorem 2.2 to study further reductions of the normal forms in the cases (b) and (c).

3.2. *Further Reductions of the Classical Normal Forms*

If there remain elements of Z_0 undetermined for the computation of a normal form of a certain order then one may choose them in looking for normal forms of higher order. This idea is essential in our method. Now we explain the method on two typical examples. We study dynamical systems of dimension 2,

$$\frac{dx}{dt}=F(x)=\begin{pmatrix} x_2+f_1(x) \\ c_0 x_1+c_1 x_2+f_2(x) \end{pmatrix} \tag{17}$$

where

$$f_1(x)=\sum_{|q| \geqslant 2} \alpha_q x^q; \qquad f_2(x)=\sum_{|q| \geqslant 2} \beta_q x^q.$$

EXAMPLE 3.1. Consider at first a dynamical system of the above form with $c_0=c_1=0$, i.e., the matrix of the linear part is

$$A=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} u & u_0 \\ 0 & u \end{pmatrix} \tag{18}$$

with u, $u_0 \in K$, $u \neq 0$. Then P is invertible and $P^{-1}AP = A$.

Suppose that $\beta_{2,0} \neq 0$, then one takes $u = 1/\beta_{2,0}$ and $\beta_{2,0}$ is reduced to 1. So one can suppose without loss of generality that $\beta_{2,0} = 1$ and take $u = 1$ in P. We take $u_0 = 0$ since it is not used in our normal form.

To obtain a normal form of the dynamical system we apply the algorithm given in the previous section. We use superscripts $Z_j^{(k)}$ to show the dependence of the Z_j on k. For $k = 2$, let $Z_0^{(2)} = (U_1, U_2)^t$. We then have

$$Z_1^{(2)} = \begin{bmatrix} U_2 + \alpha_{0,2} \\ \beta_{0,2} \end{bmatrix}, \qquad Z_2^{(2)} = \begin{bmatrix} \frac{1}{2}\beta_{0,2} + \frac{1}{2}\alpha_{1,1} \\ \frac{1}{2}\beta_{1,1} \end{bmatrix}, \qquad Z_3^{(2)} = \begin{bmatrix} \alpha_{2,0} + \frac{1}{2}\beta_{1,1} \\ 1 \end{bmatrix}.$$

Hence one can perform the transformation $x \to x + \varphi(x)$ as given in Section 3.1. The homogeneous part of degree 2 in the normal form is (for arbitrary $Z_0^{(2)}$):

$$\begin{pmatrix} (\alpha_{2,0} + \frac{1}{2}\beta_{1,1})\, x_1^2 \\ x_1^2 \end{pmatrix}.$$

With the notations of Section 2.3, the subspace $\mathrm{Ker}(L_A^2)$ is spanned by

$$\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}.$$

Hence ψ^2 is in the form

$$\begin{pmatrix} U_1\, x_2^2 \\ U_2\, x_2^2 \end{pmatrix}.$$

Theorem 2.2 is applicable with $\ell = 2$. We shall choose suitable ψ^2 for further simplifications of higher order normal forms.

For $k = 3$, let $Z_0^{(3)} = (U_3, U_4)^t$, then one has for example

$$Z_1^{(3)} = \begin{bmatrix} U_4 + 2\,\alpha_{0,2}\, U_2 + \alpha_{0,3} + \alpha_{1,1}\, U_1 \\ U_1\, \beta_{1,1} + \beta_{0,3} + 2\,\beta_{0,2}\, U_2 \end{bmatrix}, \ldots, Z_4^{(3)} = \begin{bmatrix} u \\ v \end{bmatrix},$$

where

$$u = \tfrac{1}{6}\beta_{0,2}\,\beta_{1,1} + \alpha_{3,0} - U_2 - \tfrac{1}{3}\alpha_{0,2} - \tfrac{1}{3}\beta_{0,2}\,\alpha_{2,0} + \tfrac{1}{3}\beta_{2,1} + \tfrac{1}{6}\alpha_{1,1}\,\beta_{1,1}$$

and $v = -\beta_{1,1}\,\alpha_{2,0} + \beta_{3,0} + \alpha_{1,1}$. It is clear now that one can choose

$$U_2 = \tfrac{1}{6}\beta_{0,2}\,\beta_{1,1} + \alpha_{3,0} + \tfrac{1}{6}\alpha_{1,1}\,\beta_{1,1} - \tfrac{1}{3}\alpha_{0,2} - \tfrac{1}{3}\beta_{0,2}\,\alpha_{2,0} + \tfrac{1}{3}\beta_{2,1}$$

such that the first element of $Z_4^{(3)}$ is zero. Hence the homogeneous part of degree 3 in the normal form is

$$\begin{pmatrix} 0 \\ (\beta_{3,0} + \alpha_{1,1} - \beta_{1,1}\,\alpha_{2,0})\,x_1^3 \end{pmatrix}.$$

In this case with the notations of Theorem 2.2, $\mathscr{R}_2^3$ is the subspace generated by

$$\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}.$$

With $\mathscr{C}_2^3$ generated by

$$\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix},$$

one has

$$\mathscr{C}^3 = \mathscr{R}_2^3 \oplus \mathscr{C}_2^3.$$

Theorem 2.1 is applicable with $\ell = 2$ and $k = 3$.

If $\beta_{2,0} = 0$, then there are two parameters in the homogeneous part of degree 3 in the normal form but only one parameter in the homogeneous part of degree 2.

PROPOSITION 3.1. *Let the notations be as above. Assume that the matrix of the linear part of the system is* $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

(a) *If* $\beta_{2,0} = 1$ *and* $\gamma_1 = \alpha_{2,0} + \frac{1}{2}\beta_{1,1} \neq 0$, *then a normal form of order 9 of the dynamical system* (17) *is*

$$\dot{x}_1 = x_2 + \gamma_1\,x_1^2 + O(\|x\|^{10}),$$

$$\dot{x}_2 = x_1^2 + \gamma_3 x_1^3 + \gamma_4 x_1^4 + \gamma_5 x_1^5 + \gamma_6 x_1^7 + \gamma_7 x_1^8 + O(\|x\|^{10}).$$

(b) *If* $\beta_{2,0}=1$ *and* $2\alpha_{2,0}+\beta_{1,1}=0$, *then a non-degenerate normal form of order* 9 *of the dynamical system* (17) *is*

$$\dot{x}_1 = x_2 + \gamma_1' x_1^4 + \gamma_2' x_1^5 + \gamma_3' x_1^7 + \gamma_7' x_1^8 + O(\|x\|^{10}),$$
$$\dot{x}_2 = x_1^2 + \gamma_5' x_1^3 + \gamma_6' x_1^6 + O(\|x\|^{10}).$$

(c) *If* $\beta_{2,0}=0$ *and* $5\beta_{1,1}\alpha_{2,0}-4\alpha_{2,0}^2-\beta_{1,1}^2-9\beta_{3,0}\neq 0$, *then a non-degenerate normal form of order* 9 *of the dynamical system* (17) *is*

$$\dot{x}_1 = x_2 + \gamma_1 x_1^2 + \gamma_2'' x_1^3 + O(\|x\|^{10}),$$
$$\dot{x}_2 = \gamma_3'' x_1^3 + \gamma_4'' x_1^4 + \gamma_5'' x_1^5 + \gamma_6'' x_1^6 + \gamma_7'' x_1^7 + \gamma_8'' x_1^8 + \gamma_9'' x_1^9 + O(\|x\|^{10}).$$

γ_j, γ_j', γ_j'' *are parameters depending only on the coefficients of F.*

Moreover, the above normal forms are optimal or unique in the given order with respect to equivalence by near identity changes of variables in the sense that two normal forms are equivalent by a near identity change of variables if and only if all of the parameters in the normal forms are equal.

The non-degeneracy conditions are algebraic conditions on the coefficients of the system, for instance, $\gamma_1'\neq 0$, etc. The parameters γ_j, γ_j', γ_j'' can be given by explicit formulae from the coefficients of F. For example,

$$\gamma_3 = -\beta_{1,1}\alpha_{2,0}+\beta_{3,0}+\alpha_{1,1} \qquad \text{and} \qquad \gamma_5' = \alpha_{1,1}+\beta_{3,0}+2\,\alpha_{2,0}^2.$$

In case (c), if $\gamma_1\neq 0$ then it can be reduced to 1 by the linear transformation (18) with $u_0=0$, $u=1/\gamma_1$.

Remark that we have reduced all terms of degrees 6 and 9 to zero.

To simplify the computations we assume that the dynamical system is in the Takens' normal form as in Section 2.2, i.e.,

$$f_1 = \sum_{k\geqslant 2} \alpha_k x_1^k, \qquad f_2 = \sum_{k\geqslant 2} \beta_k x_1^k,$$

with $\beta_2=1$. Let $Z_0^{(2)}=(U_1, U_2)^t$, $Z_0^{(3)}=(U_3, U_4)^t$, and $Z_0^{(4)}=(U_5, U_6)^t$, we obtain for $k=4$,

$$Z_1^{(4)} = \begin{bmatrix} U_6+\alpha_2 U_1^2 \\ U_1^2 \end{bmatrix}, \qquad Z_2^{(4)} = \begin{bmatrix} -\frac{3}{2}U_1^2+\alpha_2 U_3 \\ -\alpha_3 U_1 + U_3 \end{bmatrix},$$

$$Z_3^{(4)} = \begin{bmatrix} \frac{2}{3}\alpha_3 U_1 + \frac{1}{3}\alpha_2 U_4 - \frac{2}{3}U_3 \\ -\frac{1}{3}U_4+\frac{1}{3}\alpha_3^2+U_1\beta_3 \end{bmatrix}, \qquad Z_4^{(4)} = \begin{bmatrix} -\frac{1}{4}U_1\beta_3+\frac{5}{6}\alpha_3^2-\frac{7}{12}U_4 \\ \frac{1}{4}\beta_3\alpha_3 \end{bmatrix},$$

$$Z_5^{(4)} = \begin{bmatrix} -\frac{2}{3}\alpha_2 U_1 - \frac{1}{3}\alpha_2^2\alpha_3+\alpha_4-\frac{3}{4}\beta_3\alpha_3 \\ -\frac{5}{3}U_1+\frac{2}{3}\alpha_2\alpha_3+\beta_4 \end{bmatrix}.$$

If $\alpha_2 \neq 0$ then one can choose

$$U_1 = \frac{12\alpha_4 - 4\alpha_2^2\alpha_3 - 9\beta_3\alpha_3}{8\alpha_2},$$

and the homogeneous part of degree 4 in the new system is

$$\begin{bmatrix} 0 \\ \dfrac{12\alpha_2^2\alpha_3 - 20\alpha_4 + 15\beta_3\alpha_3 + 8\beta_4\alpha_2}{8\alpha_2} x_1^4 \end{bmatrix}.$$

The algorithm can be used for higher order normal forms. We obtain by computations in Maple the normal forms of the orders given in the above proposition.

We give a comparison of the normal forms derived via Takens' method and Ushiki's method. In [22] Ushiki obtained a normal form of order 4. The way he used is hardly applicable for computations of higher order normal forms, so there are some empty spaces in the following table. The normal form of Cushman and Sanders [14] and that of Elphick *et al.* [15] contain two non-zero parameters in each order, the same as the Takens' normal form. The same is true for the normal form defined in [9] (see also [10]). Since the goal for obtaining normal forms of dynamical systems is to eliminate as many monomials from each order as possible, we will list in the following table the number of monomials of each degree that is still present in the normal form. For example, 0 means that all terms of a given degree are eliminated (see also [13, p. 60]).

Degree:	2nd	3rd	4th	5th	6th	7th	8th	9th
Takens	2	2	2	2	2	2	2	2
Ushiki	2	1	1					
Case (a)	2	1	1	1	0	1	1	0
Case (b)	1	1	1	1	1	1	1	0
Case (c)	1	2	1	1	1	1	1	1

EXAMPLE 3.2. We now consider the case where the matrix of the linear part is $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As proved in the preceding section, the homogeneous part of even degree can be reduced to 0. We suppose that this is done to simplify the notation.

PROPOSITION 3.2. *Consider a dynamical system of the form* (17). *Assume that* $c_0 = -1$, $c_1 = 0$. *Suppose that the homogeneous parts of degree* 2 *in* f_1 *and* f_2 *are zero.*

(a) *If* $\gamma_1 = \frac{1}{3}(\beta_{2,1} + \alpha_{1,2} + 3\,\beta_{0,3} + 3\,\alpha_{3,0}) \neq 0$, *then a normal form of order* 9 *is*

$$\dot{x}_1 = x_2 + \gamma_1\, x_1^3 + \gamma_2 x_1^5 + O(\|x\|^{10}),$$
$$\dot{x}_2 = -x_1 + \gamma_3\, x_1^3 + O(\|x\|^{10}),$$

(b) *If* $\gamma_1 = 0$ *and* $\gamma_3' = \frac{1}{3}(-3\,\alpha_{0,3} + 3\,\beta_{3,0} + \beta_{1,2} - \alpha_{2,1}) \neq 0$, *then a non-degenerate normal form of order* 9 *is*

$$\dot{x}_1 = x_2 + \gamma_1' x_1^5 + \gamma_2' x_1^7 + \gamma_4' x_1^9 + O(\|x\|^{10}),$$
$$\dot{x}_2 = -x_1 + \gamma_3'\, x_1^3 + O(\|x\|^{10}),$$

where γ_j *and* γ_j' *are parameters depending only on the coefficients of F.*

Moreover, the above normal forms are optimal or unique in the given order in the same sense as in Proposition 3.1.

In case (b) we obtain some non-degeneracy conditions such as $\gamma_1' \neq 0$. The parameters γ_j, γ_j' can be given by explicit formulae from the coefficients of F. For example, $\gamma_1' = \frac{1}{5}\alpha_{1,4} + \frac{1}{5}\alpha_{3,2} + \alpha_{5,0} + \frac{1}{5}\beta_{4,1} + \frac{1}{5}\beta_{2,3} + \beta_{0,5} - \frac{1}{5}\alpha_{1,2}\,\alpha_{2,1} - \frac{1}{5}\alpha_{2,1}\,\beta_{0,3} - \frac{1}{5}\alpha_{1,2}\,\beta_{1,2} - \frac{3}{5}\,\beta_{0,3}\,\alpha_{0,3} - \frac{1}{5}\alpha_{1,2}\,\alpha_{0,3} - \frac{2}{5}\alpha_{3,0}\,\alpha_{2,1} - \frac{2}{5}\alpha_{3,0}\,\beta_{1,2} - \frac{1}{5}\beta_{1,2}\,\beta_{0,3} - \frac{3}{5}\beta_{0,3}\,\beta_{3,0} - \frac{1}{5}\alpha_{1,2}\,\beta_{3,0}$.

To obtain the normal forms of the proposition, we apply the algorithm given in the above paragraph. For $k = 3$, with $Z_0^{(3)} = (U_1, U_2)^t$, we have

$$Z_1^{(3)} = \begin{bmatrix} U_2 + \alpha_{0,3} \\ -U_1 + \beta_{0,3} \end{bmatrix}, \qquad Z_2^{(3)} = \begin{bmatrix} U_1 + \frac{1}{2}\beta_{0,3} + \frac{1}{2}\alpha_{1,2} \\ U_2 - \frac{1}{2}\alpha_{0,3} + \frac{1}{2}\beta_{1,2} \end{bmatrix},$$

$$Z_3^{(3)} = \begin{bmatrix} U_2 + \frac{1}{3}\alpha_{2,1} + \frac{1}{2}\alpha_{0,3} + \frac{1}{6}\beta_{1,2} \\ -U_1 + \frac{1}{3}\beta_{2,1} + \frac{1}{2}\beta_{0,3} - \frac{1}{6}\alpha_{1,2} \end{bmatrix},$$

$$Z_4^{(3)} = \begin{bmatrix} \alpha_{3,0} + \frac{1}{3}\beta_{2,1} + \beta_{0,3} + \frac{1}{3}\alpha_{1,2} \\ \beta_{3,0} - \frac{1}{3}\alpha_{2,1} - \alpha_{0,3} + \frac{1}{3}\beta_{1,2} \end{bmatrix}.$$

One then has in case (a) that $(\gamma_1, \gamma_3)^t = Z_4^{(3)}$. In this case, $\mathscr{C}^3$ is of dimension 2 spanned by

$$\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}.$$

For $k = 4$ we find that $\mathscr{C}^4 = \{0\}$. And we continue with $k = 5$. If $\gamma_1 \neq 0$, Theorem 2.2 is applicable for $\ell = 3$ and $k = 5$ as stated in the proposition.

One can continue the algorithm with $k=6, 7, \ldots$. The computations are similar to those above. We obtain by computations in Maple the normal forms of the orders given in the proposition.

We give in the following table, for comparison, the numbers of non-zero parameters in the normal forms, as for the preceding example. In [22] Ushiki obtained a normal form of order 7 in this case. For a normal form of general order, Baider gave partial results in this semisimple case in [3]. Complete reduction is obtained by Chua and Kokubu, by Ushiki's method in [13] for case (a) and by Gaeta in [17] for a more general case. Note also that in this case Gaeta's normal form is the best possible. But these methods compute only a general normal form of dynamical systems with the given linear part. They are not concerned with the computations of a formal diffeomorphism that realizes the normalization. One can compute by our method at the same time a normal form and a normalization diffeomorphism.

degree	2nd	3rd	4th	5th	6th	7th	8th	9th
Poincaré	0	2	0	2	0	2	0	2
Takens	0	2	0	2	0	2	0	
Ushiki	0	2	0	1	0	0		
Gaeta	0	2	0	1	0	0	0	0
Case (a)	0	2	0	1	0	0	0	0
Case (b)	0	1	0	1	0	1	0	1

4. DYNAMICAL SYSTEMS OF DIMENSION 3

Consider dynamical systems of dimension 3 of the form

$$\frac{dx}{dt} = F(x) = \begin{pmatrix} x_2 + f_1(x) \\ x_2 + f_2(x) \\ \eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + f_3(x) \end{pmatrix} \tag{19}$$

where the matrix of the linear part is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix}$$

with $\eta_j \in K$. Write for $k \geqslant 2$, $q=(q_1, q_2, q_3)$,

$$F^k(x)=\begin{pmatrix} \sum_{|q|=k} \alpha_q x^q \\ \sum_{|q|=k} \beta_q x^q \\ \sum_{|q|=k} \gamma_q x^q \end{pmatrix}, \qquad G^k(y)=\begin{pmatrix} \sum_{|q|=k} \alpha'_q y^q \\ \sum_{|q|=k} \beta'_q y^q \\ \sum_{|q|=k} \gamma'_q x^q \end{pmatrix}, \qquad \varphi^k(y)=\begin{pmatrix} \sum_{|q|=k} \alpha_q y^q \\ \sum_{|q|=k} b_q y^q \\ \sum_{|q|=k} c_q y^q \end{pmatrix},$$

where $\alpha_q, \beta_q, \gamma_q \in K$ and where $a_q, b_q, c_q, \alpha'_q, \beta'_q, \gamma'_q \in K$ are to be determined. Let

$$\mathcal{M}_q=\begin{pmatrix} a_q \\ b_q \\ c_q \end{pmatrix}, \qquad F_q=\begin{pmatrix} \alpha_q \\ \beta_q \\ \gamma_q \end{pmatrix}, \qquad G_q=\begin{pmatrix} \alpha'_q \\ \beta'_q \\ \gamma'_q \end{pmatrix}.$$

Then the homological equations (7) in dimension 3 can be written as:

$$\begin{aligned}(q_1+1)\, \mathcal{M}_{q+e_1-e_2} + (q_2+1)\, \mathcal{M}_{q+e_2-e_3} + \eta_1(q_3+1)\, \mathcal{M}_{q-e_1+e_3} \\ + \eta_2(q_3+1)\, \mathcal{M}_{q-e_2+e_3} + (\eta_3 q_3 I - A)\, \mathcal{M}_q = F_q - G_q\end{aligned}$$

where I denotes the identity matrix of order 3. Let $q_1=k-i-j$, $q_2=i$, $q_3=j$, then

$$Z_{ij}=\mathcal{M}_{(k-i-j,\, i,\, j)}, \qquad \Lambda_{i,j}=F_q, \qquad \text{and} \qquad \Lambda'_{i,j}=G_q.$$

We then have

$$\begin{aligned}(i+1)\, Z_{i+1,\, j-1} + (k-i-j+1)\, Z_{i-1,\, j} + \eta_1(j+1)\, Z_{i,\, j+1} \\ + \eta_2(j+1)\, Z_{i-1,\, j+1} + (\eta_3\, jI - A)\, Z_{i,\, j} = \Lambda_{i,\, j} - \Lambda'_{i,\, j}.\end{aligned}$$

Let $Z_{0,\, j}(0 \leqslant j \leqslant k)$ be given arbitrarily. One can determine $Z_{i+1,\, j-1}$ for $0 \leqslant i \leqslant k-1$, $1 \leqslant j \leqslant k$, and $i+j \leqslant k$ such that $\Lambda'_{i,\, j}=0$. In fact,

$$\begin{aligned}(i+1)\, Z_{i+1,\, j-1} = \Lambda_{i,\, j} - (k-i-j+1)\, Z_{i-1,\, j} - \eta_1(j+1)\, Z_{i,\, j+1} \\ - \eta_2(j+1)\, Z_{i-1,\, j+1} - (\eta_3\, jI - A)\, Z_{i,\, j}.\end{aligned}$$

The remaining equations are (with $j=0$)

$$\Lambda'_{i,\, 0} = \Lambda_{i,\, 0} - (k-i+1)\, Z_{i-1,\, 0} - \eta_1 Z_{i,\, 1} - \eta_2 Z_{i-1,\, 1} + AZ_{i,\, 0}.$$

Since we have determined all $Z_{i,j}$ for $i \geqslant 1$ as functions of $Z_{0,j}$, these equations are linear equations for $Z_{0,j}$ which may be solved in some cases to make some of the $\Lambda'_{i,0}$ zero.

EXAMPLE 4.1. Consider now a dynamical system of the form (19) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{20}$$

as its linear part.

With notations as above, for $k = 2$ one can compute $Z_{1,1}$, $Z_{1,0}$, $Z_{2,0}$ to get

$$\Lambda'_{0,0} = \begin{bmatrix} Z_{0,0_2} + \alpha_{2,0,0} \\ Z_{0,0_3} + \beta_{2,0,0} \\ -Z_{0,0_2} + \gamma_{2,0,0} \end{bmatrix},$$

$$\Lambda'_{1,0} = \begin{bmatrix} -2\,Z_{0,0_1} + Z_{0,1_1} + Z_{0,1_3} + \beta_{1,0,1} + \alpha_{1,1,0} \\ -2\,Z_{0,0_2} + \gamma_{1,0,1} + \beta_{1,1,0} \\ -2\,Z_{0,0_3} - \beta_{1,0,1} + \gamma_{1,1,0} \end{bmatrix},$$

$$\Lambda'_{2,0} = \begin{bmatrix} -\frac{3}{2}Z_{0,1_2} - \alpha_{1,0,1} + \frac{3}{2}Z_{0,2_2} + \alpha_{0,0,2} + \frac{1}{2}\gamma_{0,0,2} + \frac{1}{2}\beta_{0,1,1} + \alpha_{0,2,0} \\ -\frac{3}{2}Z_{0,1_3} - \beta_{1,0,1} + \frac{3}{2}Z_{0,2_3} + \frac{1}{2}\beta_{0,0,2} + \frac{1}{2}\gamma_{0,1,1} + \beta_{0,2,0} \\ \frac{3}{2}Z_{0,1_2} - \gamma_{1,0,1} - \frac{3}{2}Z_{0,2_2} + \frac{1}{2}\gamma_{0,0,2} - \frac{1}{2}\beta_{0,1,1} + \gamma_{0,2,0} \end{bmatrix}$$

where we have denoted by $Z_{0,1_i}$ the ith element of $Z_{0,1}$. After resolution for $Z_{0,0}$ by making $\Lambda'_{1,0} = 0$ and resolution of the last two elements of $Z_{0,1}$ by making the first two elements of $\Lambda'_{2,0}$ equal zero, we obtain

$$\Lambda'_{0,0} = \begin{bmatrix} \frac{1}{2}\gamma_{1,0,1} + \frac{1}{2}\beta_{1,1,0} + \alpha_{2,0,0} \\ -\frac{1}{2}\beta_{1,0,1} + \frac{1}{2}\gamma_{1,1,0} + \beta_{2,0,0} \\ -\frac{1}{2}\gamma_{1,0,1} - \frac{1}{2}\beta_{1,1,0} + \gamma_{2,0,0} \end{bmatrix}, \qquad \Lambda'_{1,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\Lambda'_{2,0} = (0, 0, \gamma_{0,0,2} - \alpha_{1,0,1} + \alpha_{0,0,2} + \alpha_{0,2,0} - \gamma_{1,0,1} + \gamma_{0,2,0})^t.$$

We obtain then a new system whose homogeneous part of degree 2 contains four parameters that may be nonzero:

$$\begin{aligned} \dot{x}_1 &= x_2 + \mu_1 x_1^2 + O(\|x\|^3), \\ \dot{x}_2 &= x_3 + \mu_2 x_1^2 + O(\|x\|^3), \\ \dot{x}_3 &= -x_2 + \mu_3 x_1^2 + \mu_4 x_2^2 + O(\|x\|^3), \end{aligned} \tag{21}$$

where $(\mu_1, \mu_2, \mu_3)^t = \Lambda'_{0,0}$ and μ_4 is the nonzero element of $\Lambda'_{2,0}$.

Let

$$P = \begin{pmatrix} u & v & w \\ 0 & u-w & v \\ 0 & -v & u-w \end{pmatrix} \tag{22}$$

with $u, v, w \in K$ such that $\det P \neq 0$, then $P^{-1}AP = A$. The linear transformation $x = Py$ makes no change on the linear part of the system. If $\mu_1 \neq 0$ then, by taking $u = 1/\mu_1$ and making the linear transformation (22) on the system (21), μ_1 is reduced to 1. We suppose that this is done to simplify the notation. The above algorithm applied to the new system leads to the normal form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 + O(\|x\|^3), \\ \dot{x}_2 &= x_3 + \mu_2 x_1^2 + O(\|x\|^3), \\ \dot{x}_3 &= -x_2 + \mu_3 x_1^2 + \mu'_4 x_2^2 + O(\|x\|^3), \end{aligned}$$

where $\mu'_4 = v^2\mu_4 + v^2 + v^2\mu_3 - 2w + \mu_4 w^2 + w^2 + \mu_4 - 2\mu_4 w - 2w\mu_3 + w^2\mu_3$. It is clear that if one looks for a normal form in $\mathbf{C}$ or in an algebraic extension of K then one can generically reduce μ'_4 to zero by choosing a solution for v or w. This eliminates one more parameter in degree 2. One can now give the following example.

EXAMPLE 4.2. The two dynamical systems

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2, & \dot{x}_1 &= x_2 + x_1^2, \\ \dot{x}_2 &= x_3 + x_1^2, & \dot{x}_2 &= x_3 + x_1^2, \\ \dot{x}_3 &= -x_2 + x_1^2 - 4x_2^2 & \dot{x}_3 &= -x_2 + x_1^2 \end{aligned}$$

are not equivalent in $\mathbf{R}$ but are equivalent in $\mathbf{C}$ up to order 2 (with respect to equivalence by diffeomorphisms).

We take $v = w = 0$ in the following, since we are looking for rational normal forms. We study further reductions of systems with the given linear part by near identity transformations.

If there are undetermined elements in $Z_{0,j}$ for the computations of a normal form of a certain order, then it may be chosen for the reduction of higher order normal forms. The above algorithm is implemented in Maple V. We obtain a normal form of the fourth order that we state in the following proposition.

PROPOSITION 4.1. *Consider a dynamical system of dimension* 3 *of the form* (19) *with the matrix* (20) *as its linear part. Let notations be as above. If* $\mu_1 \neq 0$, *then a non-degenerate normal form of order* 4 *is*

$$\dot{x}_1 = x_2 + x_1^2 + \mu_5 x_1^3 + O(\|x\|^5),$$

$$\dot{x}_2 = x_3 + \mu_2 x_1^2 + \mu_6 x_1^3 + O(\|x\|^5),$$

$$\dot{x}_3 = -x_2 + \mu_3 x_1^2 + \mu_4 x_2^2 + \mu_7 x_1^3 + \mu_8 x_1^4 + O(\|x\|^5).$$

Moreover, the above normal forms are optimal or unique in the given order in the same sense as in Proposition 3.1.

The following table gives a comparison for the number of non-zero parameters remaining in the normal forms (see also [13, p. 60]).

Degree	2nd	3rd	4th	5th
Poincaré	4	6	7	9
Takens	4	6	7	9
Ushiki	4	3	4	2
Our method	4	3	1	

EXAMPLE 4.3. Consider a dynamical system of the form (19) with the nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

as its linear part.

With the above notations we obtain for $k=2$,

$$\Lambda'_{0,0}=\begin{bmatrix} Z_{0,0_2}+\alpha_{2,0,0} \\ Z_{0,0_3}+\beta_{2,0,0} \\ \gamma_{2,0,0} \end{bmatrix}, \qquad \Lambda'_{1,0}=\begin{bmatrix} -2Z_{0,0_1}+Z_{0,1_3}+\beta_{1,0,1}+\alpha_{1,1,0} \\ -2Z_{0,0_2}+\gamma_{1,0,1}+\beta_{1,1,0} \\ -2Z_{0,0_3}+\gamma_{1,1,0} \end{bmatrix},$$

$$\Lambda'_{2,0}=\begin{bmatrix} -\frac{3}{2}Z_{0,1_2}-\alpha_{1,0,1}+\frac{1}{2}\gamma_{0,0,2}+\frac{1}{2}\beta_{0,1,1}+\alpha_{0,2,0} \\ -\frac{3}{2}Z_{0,1_3}-\beta_{1,0,1}+\frac{1}{2}\gamma_{0,1,1}+\beta_{0,2,0} \\ -\gamma_{1,0,1}+\gamma_{0,2,0} \end{bmatrix}$$

where $Z_{0,1_i}$ denotes the ith element in $Z_{0,1}$. After resolution we obtain finally

$$\Lambda'_{0,0}=\begin{bmatrix} \frac{1}{2}\gamma_{1,0,1}+\frac{1}{2}\beta_{1,1,0}+\alpha_{2,0,0} \\ \frac{1}{2}\gamma_{1,1,0}+\beta_{2,0,0} \\ \gamma_{2,0,0} \end{bmatrix},$$

$$\Lambda'_{1,0}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Lambda'_{2,0}=\begin{bmatrix} 0 \\ 0 \\ -\gamma_{1,0,1}+\gamma_{0,2,0} \end{bmatrix}.$$

The new system is then

$$\begin{aligned} \dot{x}_1&=x_2+\mu_1 x_1^2+O(\|x\|^3), \\ \dot{x}_2&=x_3+\mu_2 x_1^2+O(\|x\|^3), \\ \dot{x}_3&=\mu_3 x_1^2+\mu_4 x_2^2+O(\|x\|^3), \end{aligned} \tag{24}$$

where four parameters

$$\mu_1=\tfrac{1}{2}\gamma_{1,0,1}+\tfrac{1}{2}\beta_{1,1,0}+\alpha_{2,0,0}, \qquad \mu_2=\tfrac{1}{2}\gamma_{1,1,0}+\beta_{2,0,0},$$
$$\mu_3=\gamma_{2,0,0}, \qquad \text{and} \qquad \mu_4=-\gamma_{1,0,1}+\gamma_{0,2,0}$$

may be nonzero. Let

$$P=\begin{pmatrix} u & v & w \\ 0 & u & v \\ 0 & 0 & u \end{pmatrix} \tag{25}$$

with $u, v, w \in K$, $u \neq 0$, then $P^{-1}AP = A$. The linear transformation $x = Py$ makes no change on the linear part of the system. If $\mu_3 \neq 0$ then we take $u = 1/\mu_3$ and make the linear transformation (25) on the system (24). The above algorithm applied to the new system leads to the normal form

$$\dot{x}_1 = x_2 + \frac{\mu_1}{\mu_3}\, x_1^2 + O(\|x\|^3),$$

$$\dot{x}_2 = x_3 + \frac{\mu_2}{\mu_3}\, x_1^2 + O(\|x\|^3),$$

$$\dot{x}_3 = x_1^2 + \frac{-2\mu_3^2 w + \mu_4 + \mu_3^3 v^2}{\mu_3}\, x_2^2 + O(\|x\|^3).$$

Then one can choose $w = (\mu_4 + \mu_3^3 v^2)/(2\mu_3^2)$ to eliminate one more term. We obtain non-degenerate fourth order normal forms which we state in the following proposition:

PROPOSITION 4.2. *Consider a dynamical system of dimension* 3 *of the form* (19) *with the nilpotent matrix* (23) *as its linear part. Let the notation be as above.*

(a) *If* $\gamma_{2,0,0} = 1$, *then a non-degenerate normal form of order* 4 *is*

$$\dot{x}_1 = x_2 + \mu_1 x_1^2 + \mu_4 x_1^3 + O(\|x\|^5),$$
$$\dot{x}_2 = x_3 + \mu_2 x_1^2 + \mu_5 x_1^3 + \mu_7 x_1^4 + O(\|x\|^5),$$
$$\dot{x}_3 = x_1^2 + \mu_6 x_1^3 + \mu_8 x_1^4 + O(\|x\|^5).$$

(b) *If* $\gamma_{2,0,0} = 0$, $\mu_2 \neq 0$, *then a non-degenerate normal form of order* 4 *is*

$$\dot{x}_1 = x_2 + \mu_1 x_1^2 + \mu'_4 x_1^3 + \mu'_7 x_1^4 + O(\|x\|^5),$$
$$\dot{x}_2 = x_3 + x_1^2 + \mu'_8 x_1^4 + O(\|x\|^5),$$
$$\dot{x}_3 = \mu'_3 x_2^2 + \mu'_5 x_1^3 + \mu'_6 x_1 x_2^2 + \mu'_9 x_1^4 + \mu'_{10} x_1^2 x_2^2 + O(\|x\|^5)$$

where μ_j *and* μ'_j *are parameters depending on the coefficients of the system.*

Moreover, the above normal forms are optimal or unique in the given order in the same sense as in Proposition 3.1.

The non-degeneracy conditions are non-nullity conditions of some polynomial expressions in some of the μ_j.

Note that Ushiki (see [22, 13]) obtained non-degenerate normal forms of order 3 in this nilpotent case. Cushman and Sanders' normal form in this case is

$$\tilde{f}_1(p_1, p_2)\begin{pmatrix}0\\0\\0\end{pmatrix}+\tilde{f}_2(p_1, p_2)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\tilde{f}_3(p_1, p_2)\begin{pmatrix}x_2\\x_3\\0\end{pmatrix},$$

where $p_1 = x_3$, $p_2 = x_2^2 - 2x_1x_3$, and $\tilde{f}_i$ are formal power series (with leading matrix A^t). Elphick *et al.* method leads to the same normal form. This is a general normal form but it contains as many non-zero parameters as in the normal form of Takens. One should also note that p_2 is not a monomial. Note that in [9] we obtained also a normal form by using a Jordan basis in H_k. The normal form up to order 5 defined in [9] contains the same number of nonzero parameters as that of Takens.

The following table gives a comparison for the number of nonzero parameters remaining in the normal forms (see also [13, p. 60]), in the non-degenerate case (a).

Degree	2nd	3rd	4th
Takens	4	6	7
Ushiki	3	3	
Our method	3	3	2

Our method can be used to find degeneracy conditions and to compute the corresponding degenerate normal forms. We compute at the same time a formal diffeomorphism that realizes the normalization.

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