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Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors

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Abstract

The asymptotic behavior of classical solutions of the bipolar hydrodynamical model for semiconductors is considered in the present paper. This system takes the form of Euler–Poisson with electric field and frictional damping added to the momentum equation. The global existence of classical solutions is proven, and the nonlinear diffusive phenomena is observed in large time in the sense that both densities of electron and hole tend to the same unique nonlinear diffusive wave.

1. Introduction

The mathematical modelling of semiconductors takes an important place in the fields of applied and computational mathematics. It ranges from kinetic transport equations for charge carriers (electrons and holes) to fluid dynamical models. The kinetic modelling consist of classical models in the description of the motion of particle ensembles based on Newton's second law applied to ballistic transport and scattering events of the charge currencies, semiclassical models with additional consideration of crystal lattice effects, and quantum mechanical models on other shorter scales. The typical semiclassical model is the Boltzmann equation. Fluid dynamical models can be derived from kinetic models. For instance, the (unipolar)

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full hydrodynamical model for semiconductors was derived from the Boltzmann equation (for one carrier, electron or hole for instance) by moment method based on the shifted Maxwellian ansatz for the equilibrium phase space distribution in order to describe high field phenomena such as hot electron propagation.

Similarly, applying the moment method to the bipolar semiconductor Boltzmann equations for electron and hole, we derive the bipolar full hydrodynamical models. Mathematically, the full bipolar hydrodynamical model comprises a quasi-linear hyperbolic–parabolic system of balance laws for the densities, velocities and the temperatures (of electron and hole), which are coupled through the Poisson's equation for electric potential. Such mathematical structure makes it complicated to understand the qualitative behavior of solutions.

We are interested in the time-asymptotic behavior of solutions of one-dimensional isentropic bipolar hydrodynamical model in the present paper. It takes the form of compressible Euler–Poisson system with frictional damping:

$$n_t + J_x = 0, \tag{1.1}$$

$$J_t + \left(\frac{J^2}{n} + p(n)\right)_x = nE - \frac{J}{\mu_n \tau}, \tag{1.2}$$

$$m_t + I_x = 0, \tag{1.3}$$

$$I_t + \left(\frac{I^2}{m} + q(p)\right)_x = -mE - \frac{I}{\mu_m \tau}, \tag{1.4}$$

$$\lambda^2 E_x = n - m, \tag{1.5}$$

where $n>0$, $m>0$, J, I and E denote the particle densities, current densities, and the electric field, respectively. $p = p(n)$ and $q = q(m)$ are the pressure–density functions which satisfy

$$(\rho^2 p'(\rho))' > 0, \quad (\rho^2 q'(\rho))' > 0, \quad \rho > 0. \tag{1.6}$$

And $\tau_n > 0$, $\tau_m > 0$ are the momentum relaxation times, λ is the re-scaled Debye number. The device domain is chosen to be the whole real line.

Recently, many efforts were made for unipolar hydrodynamical equation of semiconductors (one carrier type), on the whole position space or spatial bounded domain, both on the mathematical modelling [1,25] and on the rigorous mathematical analysis, such as well-posedness of steady-state solutions [4,6,7] and their stability [21,19], global existence of classical and/or entropy weak solutions [3,20,24,29,32], large time behavior of solutions [10,15,19,22] and zero relaxation limit problems [2,9,16], etc. The reader can refer to [18] for a review. The influence of Poisson coupling and damping (relaxation) dissipation on the qualitative behavior of solutions was well studied. In particular, it has been shown that the stationary wave of the system is the time-asymptotical state of both classical and entropy weak solutions.

However, the study of bipolar hydrodynamical equations for semiconductors is far from being well. Only recently, Natalini [26], Hsiao–Zhang [17] and Zhang [33] obtained the global entropy weak solutions in the framework of compensated compactness on the whole real line or spatial bounded domain respectively, and Hattori–Zhu [11] proved the stability of steady-state solutions for a recombined bipolar hydrodynamical model.

It is well known that the frictional damping will cause the nonlinear diffusive phenomena of hyperbolic waves. Namely, the hyperbolic waves behave like diffusive wave in large time. Since the works by Hsiao-Liu [12–14], such interesting phenomena for compressible Euler equation with damping was well investigated, such as, optimal time-decay rates by Nishara–Yang–Wang [27,28,30,31], time-decay under boundary effects by Marcati-Mei [23], and time-decay to strong (diffusion) wave by Zhao [34]. Furthermore, in real micro-electron devices the Debye number represents the relation between a characteristic non-neutral length and a characteristic length in the fluid determined by the space scale, and is also very small compared with other physical relevant scalings. The re-scaled Debye limit and relaxation limit lead to diffusive approximate model of the bipolar hydrodynamical model [8]. Therefore, we expect that there exist other asymptotic states, different from the steady-state solutions, to which the solutions of bipolar hydrodynamical equations converge time-asymptotically. To have some intuition, let us first apply re-scaling analysis as follows. The combined Debye–relaxation re-scaling

$$t \to \tau t, \quad \lambda = \tau^{1+\alpha}, \quad -1 < \alpha,$$

$$n^\tau = n\left(\frac{t}{\tau}, x\right), \quad J^\tau = \frac{1}{\tau} J\left(\frac{t}{\tau}, x\right),$$

$$m^\tau = m\left(\frac{t}{\tau}, x\right), \quad I^\tau = \frac{1}{\tau} I\left(\frac{t}{\tau}, x\right),$$

gives

$$n_t^\tau + J_x^\tau = 0, \tag{1.7}$$

$$\tau^2 J_t^\tau + \left(\tau^2 \frac{(J^\tau)^2}{n^\tau} + p(n^\tau)\right)_x = n^\tau E^\tau - \frac{J^\tau}{\mu_n}, \tag{1.8}$$

$$m_t^\tau + I_x^\tau = 0, \tag{1.9}$$

$$\tau^2 I_t^\tau + \left(\tau^2 \frac{(I^\tau)^2}{m^\tau} + q(p^\tau)\right)_x = -m^\tau E^\tau - \frac{I^\tau}{\mu_m}, \tag{1.10}$$

$$\tau^{1+\alpha} E_x^\tau = n^\tau - m^\tau. \tag{1.11}$$

Let $\tau \to 0+$, the variable $[n^\tau - m^\tau]$ tends to zero formally. Combine (1.7) and (1.9) by $[(1.7)/\mu_n + (1.9)/\mu_m]$, in which eliminate I^τ/τ_n and J^τ/τ_m by (1.8) and (1.10), and pass formal limit ($\tau \to 0+$). Let w be the limiting density of n^τ and m^τ. Then we get the following porous media equation for w

$$w_t = P_{xx}, \quad P(w) = \frac{\mu_n \mu_m}{\mu_n + \mu_m}(p(w) + q(w)). \tag{1.12}$$

As shown in [5], there is a unique self-similar solution, the nonlinear diffusive wave, $w(x,t) = W(\xi)$, $(\xi = \frac{x}{\sqrt{1+t}})$ of (1.12) with boundary conditions:

$$W(\pm\infty) = \rho_\pm. \tag{1.13}$$

We first consider the initial value problems (labelled as IVP) for (1.1)–(1.5) with initial data given by

$$(n, J, m, I)(x, 0) = (n_0, J_0, m_0, I_0)(x), \quad x \in \mathbb{R}, \tag{1.14}$$

where

$$(n_0, J_0, m_0, I_0)(\pm\infty) = (\rho_\pm, J_\pm, \rho_\pm, I_\pm), \tag{1.15}$$

and the electric field at $x = -\infty$, $E_-(t)$, is set to be zero for simplicity. We will show in the present paper that the classical solutions of IVP (1.1)–(1.5) and(1.14) exist globally in time and the nonlinear diffusive phenomena happens in large time in the sense that both densities tend to the same unique nonlinear diffusive wave determined by (1.12) and (1.13) at algebraic decay rate, the velocities tend to zero (the thermal equilibrium state) at algebraic decay rate.

To begin with, we assume in the present paper that the pressure–density function satisfy

$$p(\rho) = q(\rho) = \frac{1}{\gamma}\rho^\gamma, \quad \gamma \geqslant 1, \tag{1.16}$$

and set μ_n, μ_m, τ and λ to be one for simplicity. And hence, $P(\rho)$ is replaced by $p(\rho)$ in (1.12). Moreover, we assume that the initial densities n_0 and m_0 are perturbations of the nonlinear diffusive wave. As in [13,14] such perturbations cause a shift of the nonlinear diffusive wave in the following sense:

$$\int_{-\infty}^{\infty} [(n_0(x) - W(x + x_0, t = 0)]\, dx = \mu_n \tau (J_+ - J_-), \tag{1.17}$$

$$\int_{-\infty}^{\infty} [(m_0(x) - W(x + y_0, t = 0)] \, dx = \mu_m \tau (I_+ - I_-), \tag{1.18}$$

with x_0, $y_0 \in \mathbb{R}$ two constants. And the solutions (n, I, m, J) are expressed by (ϕ, η, ψ, ξ) through equations

$$\phi(x,t) = \int_{-\infty}^{x} [n(x,t) - W(x + x_0, t)] \, dx, \tag{1.19}$$

$$\psi(x,t) = \int_{-\infty}^{x} [m(x,t) - W(x + y_0, t)] \, dx, \tag{1.20}$$

$$\eta(x,t) = J(x,t) + p(W(x,t))_x, \quad \zeta(x,t) = I(x,t) + p(W(x,t))_x. \tag{1.21}$$

or alternatively

$$n(x,t) = \phi_x(x,t) + W(x + x_0, t), \quad m(x,t) = \psi_x(x,t) + W(x + y_0, t), \tag{1.22}$$

$$J(x,t) = \eta(x,t) - p(W(x + x_0, t))_x, \quad I(x,t) = \zeta(x,t) - p(W(x + y_0), t)_x, \tag{1.23}$$

where (ϕ, η, ψ, ξ) is the solution of IVP (3.7)–(3.12). The particle states (I_-, I_+) and (J_-, J_+) can be set to be in the thermal equilibrium at infinity. In fact, if not, due to the damping dissipation of momentum equations (at infinity), we can define the approximate functions $(I_e, J_e)(x,t)$ of momentum as ([13,14])

$$J_e(x,t) = J_- e^{-\frac{1}{\mu_n \tau} t} + (J_+ - J_-) e^{-\frac{1}{\mu_n \tau} t} \int_{-\infty}^{x} \tilde{n}(x) \, dx, \tag{1.24}$$

$$I_e(x,t) = I_- e^{-\frac{1}{\mu_m \tau} t} + (I_+ - I_-) e^{-\frac{1}{\mu_m \tau} t} \int_{-\infty}^{x} \tilde{m}(x) \, dx, \tag{1.25}$$

which carry out the initial momentum at infinity. Here $\tilde{n}(x) \geqslant 0$ and $\tilde{n}(x) \geqslant 0$ are C^∞ functions with compact support, and satisfy

$$\int_{-\infty}^{\infty} \tilde{n}(x) \, dx = 1, \quad \int_{-\infty}^{\infty} \tilde{m}(x) \, dx = 1. \tag{1.26}$$

In analogy the approximate functions $(n_e, m_e)(x,t)$ of densities are given as

$$n_e(x,t) = \frac{J_+ - J_-}{\mu_n \tau} \tilde{n}(x) e^{-\frac{1}{\mu_n \tau} t}, \tag{1.27}$$

$$m_e(x,t) = \frac{I_+ - I_-}{\mu_m \tau} \tilde{m}(x) e^{-\frac{1}{\mu_m \tau} t}. \tag{1.28}$$

In addition, the perturbations (ϕ, ψ) are chosen to be

$$\phi = \int_{-\infty}^{x} [n(x,t) - W(x+x_0,t) - n_e(x,t)]\,dx, \quad \eta = J + p(W)_x - J_e, \tag{1.29}$$

$$\psi = \int_{-\infty}^{x} [m(x,t) - W(x+y_0,t) - m_e)(x,t)]\,dx, \quad \zeta = I + p(W)_x - I_e. \tag{1.30}$$

For convenience, we only consider the case that

$$I_\pm = J_\pm = 0 \tag{1.31}$$

in the present paper. This implies, due to (1.24)–(1.28), that

$$n_e = I_e = m_e = J_e \equiv 0. \tag{1.32}$$

And by (1.12), (1.19), (1.21), (1.17), and (1.18), it holds from (1.31) that

$$\phi(x = \pm\infty, t) = \psi(x = \pm\infty, t) = 0, \quad t > 0. \tag{1.33}$$

Set

$$\phi_0 = \int_{-\infty}^{x} (n_0(y) - W(y+x_0,0))\,dy, \quad \phi_1 = J_0 + p(W(x+x_0,0))_x, \tag{1.34}$$

$$\psi_0 = \int_{-\infty}^{x} (m_0(y) - W(y+y_0,0))\,dy, \quad \psi_1 = I_0 + p(W(x+y_0,0))_x, \tag{1.35}$$

with x_0 and y_0 determined by (1.17) and (1.18). The main result in the present paper is given in the following Theorem:

Theorem 1.1. *Assume that* (1.6) *and* (1.16) *hold. Suppose that* $(\phi_0, \phi_1, \psi_0, \psi_1) \in H^3 \times H^2 \times H^3 \times H^2$ *with* $x_0 = y_0$, *and* $\delta_0 = |\rho_+ - \rho_-| \ll 1$. *Then, there is a* $\eta_0 > 0$ *such that if* $\|(\phi_0, \psi_0)\|_3 + \|(\phi_1, \psi_1)\|_2 \leqslant \eta_0$, *the global classical solutions* (n, J, m, I, E) *of IVP* (1.1)–(1.5) *and* (1.14) *exist and satisfy*

$$\sum_{k=0}^{2} (1+t)^{k+1} \|\partial_x^k (n - W, m - W)(.,t)\|^2 [-2mm] \\ + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k (J + p(W)_x, I + p(W)_x)(.,t)\|^2 \leqslant C(\Phi_0 + \delta_0), \tag{1.36}$$

$$\|(n - m, E)(.,t)\|_2 \leqslant C(\Phi_0 + \delta_0) e^{-\alpha t}, \tag{1.37}$$

with $\alpha > 0$ *a constant and* $\Phi_0 = \|(\phi_0, \psi_0)\|_3^2 + \|(\phi_1, \psi_1)\|_2^2$.

Furthermore, if it holds in addition that $(\phi_0, \psi_0) \in L^1$, *the following optimal* L_p $(2 \leqslant p \leqslant \infty)$ *decay rates can be obtained*

$$||\partial_x^k (n - W, m - W)(., t)||_{L^p} \leqslant C\delta_0 (1 + t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{k+1}{2}}, \tag{1.38}$$

$$||\partial_x^k (J + p(W)_x, I + p(W)_x)(., t)||_{L^p} \leqslant C\delta_0 (1 + t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{k+2}{2}}, \tag{1.39}$$

for any $k \leqslant 2$ *if* $p = 2$ *and* $k \leqslant 1$ *if* $p \in (2, \infty]$.

This theorem is proved by energy method. First, we prove the global existence of classical solutions by establishing energy estimates to the damped wave equations for the perturbation ϕ and ψ respectively and combining them together so as to get the estimates for electric field. Then, we investigate their large time behavior. By applying energy method to a new damped Klein–Gordon equation for electric field, we can obtain its exponential time decay rate to zero. Due to the coupling effect, more regularity of the perturbations (ϕ, ψ) of densities are required in order to obtain the desired time-decay rates of solutions. In general, to obtain H^n exponential decay of electric field the initial perturbation of densities is required to belong to H^{n+1}. After we obtain the exponential time decay rate of electric field, we can decouple the two damped wave equations and apply the methods by Nishihara [27,28] to obtain algebraic decay rate of other quantities. Furthermore, more careful a-priori estimates should be done in order to control the convection terms.

Remark 1.2. Undoubtedly one can obtain the global existence of classical solutions for general pressure–density functions instead of the case (1.16), for instance for the case that initial densities are the perturbations of a positive constant (i.e., $W \equiv$ constant) so as to avoid the complicated calculations. Here, however we only consider case (1.16) in order to simplify the discussion in the large time behavior of solutions.

This paper is arranged as follows. In Section 2, we state some known results on the diffusive wave solution W of (1.12) and (1.13). The Theorem 1.1 is proved in Section 3 where we first reformulate the original problem to a new one in Section 3.1 and establish the uniformly a-priori estimates in Section 3.2, then in Section 3.3 we prove the algebraic convergence of two densities to the nonlinear diffusive wave and the exponential decay of their difference and the electric field to zero. Also, (1.38), (1.39) are proven therein.

Notation. Throughout this paper C always denotes generic positive constant. $L^2(\mathbb{R})$ is the space of square integrable real valued function defined on $\mathbb{R}$ with the norm $||\cdot||$, and $H^k(\mathbb{R})$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $||\cdot||_k$, especially, $||\cdot||_0 = ||\cdot||$.

2. Nonlinear diffusive waves

We list some known results concerning the self-similar solution of the nonlinear parabolic equation (1.12) in this section.

Assume that the pressure–density functions satisfy (1.16) and μ_n, μ_m, τ and λ are set to be one. Then the nonlinear parabolic equation (1.12) reads:

$$w_t = p(w)_{xx}, \qquad p'(w) > 0, \tag{2.1}$$

which possesses a unique self-similar solution $w(x,t)$ (see [5])

$$w(x,t) \triangleq W(\zeta), \quad \zeta = \frac{x}{\sqrt{t+1}}, \tag{2.2}$$

satisfying

$$\begin{aligned} &W''(\zeta) + \frac{p''(W(\zeta))W'(\zeta) - \frac{1}{2}\zeta}{p'(W(\zeta))} W'(\zeta) = 0, \\ &W(\pm\infty) = \rho_\pm, \quad (\rho_+, \rho_- > 0). \end{aligned} \tag{2.3}$$

This solution is increasing if $\rho_- < \rho_+$ and decreasing if $\rho_- > \rho_+$, and satisfies

$$\sum_{k=1}^{6} \left| \frac{d^k}{d\zeta^k} \Phi(\zeta) \right| + |W(\zeta) - \rho_+|_{\zeta>0} + |W(\zeta) - \rho_-|_{\zeta<0} \leqslant C\delta_0 e^{-c\zeta^2}, \tag{2.4}$$

$$|W_t(x,t)| \leqslant C\delta_0 (1+t)^{-1}, \quad |W_x(x,t)| \leqslant C\delta_0(1+t)^{-\frac{1}{2}}, \tag{2.5}$$

where and throughout $\delta_0 = |\rho_+ - \rho_-|$. Moreover, we can obtain the L^2-estimates and L^p-estimates of the derivatives of W as ([12,13,27,28]):

Lemma 2.1. *Let W be the self-similar solution of* (1.12) *and* (1.13). *Then it holds that*

$$\int_{-\infty}^{\infty} |W_x(x,t)|^2 \, dx \leqslant C\delta_0^2 (1+t)^{-\frac{1}{2}}, \tag{2.6}$$

$$\int_{-\infty}^{\infty} (|W_t(x,t)|^2 + |W_{xx}(x,t)|^2) \, dx \leqslant C\delta_0^2 (1+t)^{-\frac{3}{2}}, \tag{2.7}$$

$$\int_{-\infty}^{\infty} (|W_{xt}(x,t)|^2 + |W_{xxx}(x,t)|^2) \, dx \leqslant C\delta_0^2 (1+t)^{-\frac{5}{2}}, \tag{2.8}$$

$$\int_{-\infty}^{\infty} |W_{tt}(x,t)|^2 \, dx \leqslant C\delta_0^2 (1+t)^{-\frac{7}{2}}, \tag{2.9}$$

$$\int_{-\infty}^{\infty} |W_{xtt}(x,t)|^2 \, dx \leqslant C\delta_0^2 (1+t)^{-\frac{9}{2}}, \tag{2.10}$$

$$\int_{-\infty}^{\infty} |W_{xttt}(x,t)|^2 \, dx \leqslant C\delta_0^2 (1+t)^{-\frac{13}{2}}, \tag{2.11}$$

for some constant $C, c > 0$ *independent of* ζ, *and*

$$\|\partial_t^l \partial_x^k W(.,t)\|_{L^p} \leqslant C\delta_0 (1+t)^{-l-\frac{k}{2}+\frac{1}{2p}}, \tag{2.12}$$

for $1 \leqslant l+k \leqslant 6$ *and* $p \in [2, \infty]$.

3. Global existence and the asymptotic of solutions

By the standard methods, we can prove the local existence of classical solutions of the IVP (1.1)–(1.5) and (1.14). To extend such solutions globally in time, we need to reformulate the original problems and establish uniform a-priori estimates then.

3.1. Reformulation of original problems

For any $T > 0$, assume that (n, J, m, I, E) is a classical solution of the IVP (1.1)–(1.5) and (1.14). Then by (1.1)–(1.5) and (1.19)–(1.21), the corresponding IVP for $(\phi, \eta, \psi, \xi, E)$, with (ϕ, η, ψ, ξ) defined by (1.19) and (1.21), are obtained

$$\phi_t + \eta = 0, \tag{3.1}$$

$$\eta_t + \left[\frac{(p(W)_x + \eta)^2}{W + \phi_x} + p(W + \phi_x) - p(W) \right]_x = (W + \phi_x)E - \eta - p(W)_{xt}, \tag{3.2}$$

$$\psi_t + \zeta = 0, \tag{3.3}$$

$$\zeta_t + \left[\frac{(p(W)_x + \zeta)^2}{W + \psi_x} + p(W + \psi_x) - p(W) \right]_x = -(W + \psi_x)E - \zeta - p(W)_{xt}, \tag{3.4}$$

$$E = \phi - \psi, \tag{3.5}$$

$$\phi(x,0) = \phi_0(x), \;\; \eta(x,0) = \phi_1(x), \;\; \psi(x,0) = \psi_0(x), \;\; \zeta(x,0) = \psi_1(x), \quad x \in R. \tag{3.6}$$

By (3.1)–(3.2) and (3.3)–(3.4), we obtain the damped "wave equation" for ϕ and ψ respectively as

$$\phi_{tt} - (p(W+\phi_x) - p(W))_x + \phi_t + (W+\phi_x)E = -p(W)_{xt} + (f_1)_x, \quad (3.7)$$

$$\psi_{tt} - (p(W+\psi_x) - p(W))_x + \psi_t - (W+\psi_x)E = -p(W)_{xt} + (f_2)_x, \quad (3.8)$$

where

$$f_1 = \frac{(p(W)_x - \phi_t)^2}{W+\phi_x}, \quad (3.9)$$

$$f_2 = \frac{(p(W)_x - \psi_t)^2}{W+\psi_x}. \quad (3.10)$$

The corresponding initial data are

$$\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = -\phi_1(x), \quad (3.11)$$

$$\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = -\psi_1(x). \quad (3.12)$$

By (3.7), (3.8) and (3.5), we have the IVP for the damped "Klein–Gordon" equation for the electric field E as

$$E_{tt} + E_t + 2WE = f_3(x,t), \quad (3.13)$$

$$E(x,0) = E_0(x) := (\phi_0 - \psi_0)(x), \quad E_t(x,0) = E_1 := -(\phi_1 - \psi_1)(x), \quad (3.14)$$

where

$$\begin{aligned} f_3 &= (p(W+\phi_x) - p(W+\psi_x))_x - (\phi_x+\psi_x)E \\ &\quad + \left(\frac{(p(W)_x - \phi_t)^2}{W+\phi_x} - \frac{(p(W)_x - \psi_t)^2}{W+\psi_x}\right)_x := (f_{31})_x + f_{32} + (f_{33})_x. \end{aligned} \quad (3.15)$$

For $T > 0$, denote the basic space for the IVP (3.1)–(3.6) as

$$X(T) = \{(\phi, \phi_t, \psi, \psi_t, E) \in H^3 \times H^2 \times H^3 \times H^2 \times H^2, \ 0 \leqslant t \leqslant T\}, \quad (3.16)$$

with norm given by

$$M(0,T) = \max_{0 \leqslant t \leqslant T} \{||(\phi,\psi)(.,t)||_3 + ||(\phi_t, \psi_t, E)(.,t)||_2\},$$

and assume that the following assumption holds:

$$N(T) = \max_{0 \leqslant t \leqslant T} \{||(\phi, \psi)(.,t)||_3 + ||(\phi_t, \psi_t)(.,t)||_2\} \ll 1. \tag{3.17}$$

It is easy to verify that under the assumption (3.17) it holds that

$$0 < \frac{1}{2}\rho_- \leqslant W + \phi_x \leqslant \frac{3}{2}\rho_+, \quad \frac{1}{2}\rho_- \leqslant W + \psi_x \leqslant \frac{3}{2}\rho_+. \tag{3.18}$$

Then, to prove global existence of classical solutions, it is sufficient to prove the following a-priori estimates for IVP (3.7)–(3.12):

Theorem 3.1. *For $T > 0$ let $(\phi, \psi, E) \in X(T)$ be the solutions of IVP* (3.7)–(3.12). *Then, it holds for $N(T) + \delta_0 \ll 1$ that*

$$\begin{aligned} &||(\phi, \psi)||_3^2 + ||(\phi_t, \psi_t, E)(.,t)||_2^2 + \int_0^T ||(\phi_t, \phi_x, \psi_t, \psi_x, E)(.,s)||_2^2 \, ds \\ &\quad \leqslant C(||(\phi_0, \psi_0)||_3^2 + ||(\phi_1, \psi_1)||_2^2 + \delta_0), \quad 0 < t < T. \end{aligned} \tag{3.19}$$

3.2. The a-priori estimates

Theorem 3.1 can be proven by the following Lemmas 3.2–3.4. First, we have the basic estimate:

Lemma 3.2. *Let $T > 0$. It holds, for $(\phi, \psi, E) \in X(T)$, that*

$$\begin{aligned} &||(\phi_t, \phi_x, \phi, \psi_t, \psi_x, \psi, E)(.,t)||^2 + \int_0^T ||(\phi_t, \phi_x, \psi_t, \psi_x, E)(.,s)||^2 \, ds \\ &\quad \leqslant C(||(\phi_0, \psi_0)||_3^2 + ||(\phi_1, \psi_1)||_2^2 + \delta_0), \end{aligned} \tag{3.20}$$

provided that $N(T) + \delta_0$ small enough.

Proof. Multiplying (3.7) with ϕ and integrating over $\mathbb{R}$, we have after integration by parts

$$\begin{aligned} &\frac{d}{dt}\left(\int_{-\infty}^{\infty} \left[\phi\phi_t + \frac{1}{2}\phi^2\right] dx\right) + \int_{-\infty}^{\infty} [p'(W + \theta_1\phi_x)\phi_x^2 - \phi_t^2] \, dx + \int_{-\infty}^{\infty} (W + \phi_x)\phi E dx \\ &\quad = \int_{-\infty}^{\infty} p'(W) W_t \phi_x \, dx - \int_{-\infty}^{\infty} f_1 \phi_x \, dx, \end{aligned} \tag{3.21}$$

where $0<\theta_1<1$. By Lemma 2.1, the first term in the right-hand side of (3.21) can be estimated as

$$\int_{-\infty}^{\infty} p'(W)W_t\phi_x\,dx \leqslant C(\kappa)\int_{-\infty}^{\infty} W_t^2\,dx + \kappa\int_{-\infty}^{\infty}\phi_x^2\,dx, \tag{3.22}$$

hereafter the constants κ, p^*, and p_* are given by

$$\kappa \leqslant \frac{1}{8}\min\{p_*,\ 1\} \tag{3.23}$$

and

$$p_* = \min_{y\in[\frac{1}{2}\rho_-,\frac{3}{2}\rho_+]} p'(y), \quad p^* = \max_{y\in[\frac{1}{2}\rho_-,\frac{3}{2}\rho_+]} p'(y). \tag{3.24}$$

The other term can be estimated after integration by parts as

$$-\int_{-\infty}^{\infty} f_1\phi_x\,dx \leqslant C(\kappa)\int_{-\infty}^{\infty}|W_x|^3 + C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_x^2+\phi_t^2]\,dx. \tag{3.25}$$

Substituting (3.22) and (3.25) into (3.21), we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\phi_t\phi+\frac{1}{2}\phi_t^2\right]dx\right) - (1+\kappa)\int_{-\infty}^{\infty}\phi_t^2\,dx + (p_*-2\kappa)\int_{-\infty}^{\infty}\phi_x^2\,dx\\ &\quad+\int_{-\infty}^{\infty}(W+\phi_x)E\phi\,dx \leqslant C\int_{-\infty}^{\infty}(W_x^4+W_t^2)\,dx,\end{aligned} \tag{3.26}$$

where we have used (3.24), and (3.17) so that $C(N(T)+\delta_0)\leqslant\kappa$.

Multiplying (3.7) with ϕ_t and integrating it by parts over $\mathbb{R}$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_t^2+\sigma(\phi_x,W)\right]dx\right) + \int_{-\infty}^{\infty}\left[\phi_t^2 - \frac{\partial\sigma(\phi_x,W)}{\partial W}W_t\right]dx\\ &\quad + + \int_{-\infty}^{\infty}(W+\phi_x)E\phi_t\,dx\\ &= -\int_{-\infty}^{\infty}p(W)_{xt}\phi_t\,dx + \int_{-\infty}^{\infty}(f_1)_x\phi_t\,dx,\end{aligned} \tag{3.27}$$

where

$$\sigma(\xi,W) = \int_0^{\xi}[p(W+\tau)-p(W)]\,d\tau \tag{3.28}$$

satisfying (cf. [14])

$$p_*\xi^2 \leqslant \sigma(\xi, W) \leqslant p^*\xi^2. \tag{3.29}$$

By Lemma 2.1 it holds

$$\left|\int_{-\infty}^{\infty} p(W)_{xt}\phi_t\,dx\right| \leqslant C\int_{-\infty}^{\infty}[W_t^2 W_x^2 + W_{xt}^2]\,dx + \kappa\int_{-\infty}^{\infty}\phi_t^2\,dx. \tag{3.30}$$

Due to the fact

$$\begin{aligned}(f_1)_x = & -\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xx} - \frac{2(p(W)_x-\phi_t)}{W+\phi_x}\phi_{xt}\\ & -\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}W_x - \frac{2(p(W)_x-\phi_t)}{W+\phi_x}p(W)_{xx}\end{aligned} \tag{3.31}$$

$$\begin{aligned}= & -\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xx} - \frac{2(p(W)_x-\phi_t)}{W+\phi_x}\phi_{xt}\\ & + O(1)[(W_x+\phi_t)W_t + W_x^3 + \phi_t^2 W_x],\end{aligned} \tag{3.32}$$

we obtain, after integration by parts, that

$$\begin{aligned}&\int_{-\infty}^{\infty}(f_1)_x\phi_t\,dx\\ &\leqslant \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_x^2\,dx\right) + \int_{-\infty}^{\infty}\left(\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)_x\phi_x\phi_t\,dx\\ &\quad -\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)_t\phi_x^2\,dx + \int_{-\infty}^{\infty}\left(\frac{p(W)_x-\phi_t}{W+\phi_x}\right)_x\phi_t^2\,dx\\ &\quad + C\int_{-\infty}^{\infty}(|W_tW_x| + |\phi_tW_t| + |W_x^3| + \phi_t^2|W_x|)|\phi_t|\,dx\end{aligned} \tag{3.33}$$

$$\begin{aligned}&\leqslant \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_x^2\,dx\right)\\ &\quad + C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_t^2+\phi_x^2]\,dx + C\delta_0,\end{aligned} \tag{3.34}$$

provided that $N(T)+\delta_0 \ll 1$. Where we have used

$$|\phi_{tt}| \leqslant C|\phi_{xx} + \phi_{xt} + \phi_x + \phi_t + \phi + \psi + W_{xt} + W_x + W_t|, \tag{3.35}$$

which leads to

$$||\phi_{tt}||^2 \leqslant C(N(T) + \delta_0). \tag{3.36}$$

By (3.30), (3.34), we estimate (3.27) as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[2\sigma(\phi_x, W) - \frac{(p(W)_x - \phi_t)^2}{(W+\phi_x)^2}\phi_x^2 + \phi_t^2\right]dx\right)\\
&\quad + \int_{-\infty}^{\infty}\left[(1-\kappa)\phi_t^2 - \frac{\partial\sigma(\phi_x, W)}{\partial W}W_t\right]dx + \int_{-\infty}^{\infty}(W+\phi_x)E\phi_t\,dx\\
&\leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_x^2 + \phi_t^2]\,dx + C\delta_0.
\end{aligned} \tag{3.37}$$

Using $[(3.26) + 2(1+\kappa) \times (3.37)]$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[(1+\kappa)\phi_t^2 + \frac{1}{2}\phi^2 + \phi\phi_t\right]dx\right)\\
&\quad + \frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[2(1+\kappa)\sigma(\phi_x, W) - \frac{2(1+\kappa)(p(W)_x - \phi_t)^2}{(W+\phi_x)^2}\phi_x^2\right]dx\right)\\
&\quad + \int_{-\infty}^{\infty}\left[(1+\kappa)(1-2\kappa)\phi_t^2 + (p_* - 2\kappa)\phi_x^2 - 2(1+\kappa)\frac{\partial\sigma(\phi_x, W)}{\partial W}W_t\right]dx\\
&\quad + \int_{-\infty}^{\infty}(W+\phi_x)E(\phi + 2(1+\kappa)\phi_t)\,dx\\
&\leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_x^2 + \phi_t^2]\,dx + C\delta_0,
\end{aligned} \tag{3.38}$$

where we have used (3.29), (2.1).

Multiplying (3.8) by $[\psi + 2(1+\kappa)\psi_t]$, applying the similar procedures as above, noticing the negative symbol of the electric term of (3.8) and

$$f_2 = O(1)(W_x + \psi_t),$$

$$\begin{aligned}
(f_2)_x = &-\frac{(p(W)_x - \psi_t)^2}{(W+\psi_x)^2}\psi_{xx} - \frac{2(p(W)_x - \psi_t)}{W+\psi_x}\psi_{xt}\\
&+ O(1)[(W_x + \psi_t)W_t + W_x^3 + \psi_t^2 W_x],
\end{aligned} \tag{3.39}$$

we obtain

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[(1+\kappa)\psi_t^2 + \frac{1}{2}\psi^2 + \psi\psi_t\right]dx\right)$$

$$
\begin{aligned}
&+\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[2(1+\kappa)\sigma(\psi_x, W)-\frac{2(1+\kappa)(p(W)_x-\psi_t)^2}{(W+\psi_x)^2}\psi_x^2\right]dx\right)\\
&+\int_{-\infty}^{\infty}\left[(1+\kappa)(1-2\kappa)\psi_t^2+(p_*-2\kappa)\psi_x^2-2(1+\kappa)\frac{\partial\sigma(\psi_x, W)}{\partial W}W_t\right]dx\\
&-\int_{-\infty}^{\infty}(W+\psi_x)E(\psi+2(1+\kappa)\psi_t)\,dx\\
&\leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\psi_x^2+\psi_t^2]\,dx+C\delta_0, \qquad (3.40)
\end{aligned}
$$

provided that $N(T)+\delta_0 \ll 1$.

Taking $[(3.38)+(3.40)]$, integrating it over $[0, T]$, using (3.23) and the following facts

$$
\left|\frac{\partial\sigma(\xi, W)}{\partial W}W_t\right| \leqslant \kappa\xi^2 + C(\kappa)W_t^2, \quad \xi=\phi_x,\ \psi_x, \qquad (3.41)
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}[(W+\phi_x)E(\phi+2(1+\kappa)\phi_t)-(W+\psi_x)E(\psi+2(1+\kappa)\psi_t)]\,dx\\
&\geqslant \frac{d}{dt}\left(\int_{-\infty}^{\infty}(1+\kappa)WE^2\,dx\right)+\int_{-\infty}^{\infty}(W-(1+\kappa)W_t)E^2\,dx\\
&\quad - CN(T)\int_{-\infty}^{\infty}[E^2+\phi_x^2+\phi_t^2+\psi_x^2+\psi_t^2]\,dx, \qquad (3.42)
\end{aligned}
$$

and

$$
\|E_0\|_2^2+\|E_1\|_1^2 \leqslant C(\|(\phi,\psi)\|_3^2+\|(\phi_1,\psi_1)\|_2^2) \leqslant C\Phi_0^2, \qquad (3.43)
$$

we obtain (3.20) provided that $(N(T)+\delta_0) \ll 1$. □

Now, we turn to higher-order estimates:

Lemma 3.3. *Let $T>0$. It holds, for $(\phi,\psi,E)\in X(T)$, that*

$$
\begin{aligned}
&\|(\phi_{xx},\phi_{xt},\phi_{tt},\phi_x,\psi_{xx},\psi_{xt},\psi_{tt},\psi_x,E_x)(.,t)\|^2\\
&\quad+\int_0^T\|(\phi_{xt},\phi_{xx},\psi_{xt},\psi_{xx},E_x)(.,s)\|^2\,ds\\
&\leqslant C(\|(\phi_0,\psi_0)\|_3^2+\|(\phi_1,\psi_1)\|_2^2+\delta_0), \qquad (3.44)
\end{aligned}
$$

provided that $N(T)+\delta_0$ small enough.

Proof. Differentiate (3.7) and (3.8) with respect to x, we obtain

$$\phi_{xtt} - (p'(W+\phi_x)\phi_{xx})_x + \phi_{xt} + (WE)_x = -p(W)_{xxt} + (f_1)_{xx} + g_1, \tag{3.45}$$

$$\psi_{xtt} - (p'(W+\psi_x)\psi_{xx})_x + \psi_{xt} - (WE)_x = -p(W)_{xxt} + (f_2)_{xx} + g_2, \tag{3.46}$$

where

$$g_1 = ((p'(W+\phi_x) - p'(W))W_x)_x - (\phi_x E)_x, \tag{3.47}$$

$$g_2 = ((p'(W+\psi_x) - p'(W))W_x)_x + (\psi_x E)_x. \tag{3.48}$$

Multiply (3.45) with ϕ_x and integrate it by parts over $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\phi_x\phi_{xt} + \frac{1}{2}\phi_x^2\right]dx\right) &+ \int_{-\infty}^{\infty}[p'(W+\phi_x)\phi_{xx}^2 - \phi_{xt}^2]\,dx \\ &+ \int_{-\infty}^{\infty}(WE)_x\phi_x\,dx \\ &= \int_{-\infty}^{\infty} p(W)_{xt}\phi_{xx}\,dx - \int_{-\infty}^{\infty}(f_1)_x\phi_{xx}\,dx + \int_{-\infty}^{\infty} g_1\phi_x\,dx.\end{aligned} \tag{3.49}$$

By Lemma 2.1, the first term on the right-hand side of (3.49) can be estimated as

$$\int_{-\infty}^{\infty} p(W)_{xt}\phi_{xx}\,dx \leqslant C(\kappa)\int_{-\infty}^{\infty}[W_{xt}^2 + W_x^2W_t^2]\,dx + \frac{1}{3}\kappa\int_{-\infty}^{\infty}\phi_{xx}^2\,dx. \tag{3.50}$$

By Cauchy's inequality and the fact that

$$g_1 = O(1)(\phi_x(W_{xx} + W_x^2) + W_x\phi_{xx}) - \phi_x E_x - \phi_{xx}E, \tag{3.51}$$

the last term can be estimated as

$$\int_{-\infty}^{\infty} g_1\phi_{xx}\,dx \leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_x^2 + \phi_{xx}^2]\,dx. \tag{3.52}$$

and the other term can be estimated, after integration by parts and with the help of (3.32), as

$$-\int_{-\infty}^{\infty}(f_1)_x\phi_{xx}\,dx \leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xx}^2 + \phi_{xt}^2 + \phi_t^2]\,dx + C\delta_0. \tag{3.53}$$

Substituting (3.50), (3.52), (3.53) into (3.49), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\phi_x\phi_{xt}+\frac{1}{2}\phi_x^2\right]dx\right)\\
&\quad+\int_{-\infty}^{\infty}[(p'(W+\phi_x)-\kappa)\phi_{xx}^2-(1+\kappa)\phi_{xt}^2]\,dx+\int_{-\infty}^{\infty}(WE)_x\phi_x\,dx\\
&\leqslant C\delta_0+C\int_{-\infty}^{\infty}[\phi_x^2+\phi_t^2]\,dx, \qquad (3.54)
\end{aligned}$$

provided that (3.17) holds so that $C(N(T)+\delta_0)\leqslant\frac{1}{3}\kappa$.

Multiply (3.45) with ϕ_{xt} and integrate it by parts over $\mathbb{R}$. We get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}[\phi_{xt}^2+p'(W+\phi_x)\phi_{xx}^2]dx\right)\\
&\quad+\int_{-\infty}^{\infty}\left[\phi_{xt}^2-\frac{1}{2}p'(W+\phi_x)_t\phi_{xx}^2\right]dx+\int_{-\infty}^{\infty}(WE)_x\phi_{xt}\,dx\\
&=-\int_{-\infty}^{\infty}p(W)_{xxt}\phi_{xt}\,dx+\int_{-\infty}^{\infty}(f_1)_{xx}\phi_{xt}\,dx+\int_{-\infty}^{\infty}g_1\phi_{xt}\,dx, \qquad (3.55)
\end{aligned}$$

The right-hand side terms of (3.55) can be estimated as follows. By Lemma 2.1 it holds

$$\left|\int_{-\infty}^{\infty}p(W)_{xxt}\phi_{xt}\,dx\right|\leqslant C(\kappa)\int_{-\infty}^{\infty}W_{tt}^2\,dx+\frac{1}{6}\kappa\int_{-\infty}^{\infty}\phi_{xt}^2\,dx. \qquad (3.56)$$

It follows from (3.51) that

$$\int_{-\infty}^{\infty}g_1\phi_{xt}\,dx\leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xx}^2+\phi_{xt}^2+\phi_x^2]\,dx. \qquad (3.57)$$

Due to (1.12) and (3.31) it holds that

$$\begin{aligned}
(f_1)_{xx}=&-\left(\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xx}\right)_x-\frac{2(p(W)_x-\phi_t)}{W+\phi_x}\phi_{xxt}\\
&+O(1)(W_x+\phi_t)(W_{xx}(W_x+\phi_t)+W_{xt})\\
&+O(1)(\phi_{xt}+W_t)[(\phi_{xx}+W_x)(\phi_t+W_x)+W_t+\phi_{xt}]\\
&+O(1)W_x[(\phi_{xx}+W_x)(\phi_t^2+W_x^2)+(W_t+\phi_{xt})(W_x+\phi_t)], \qquad (3.58)
\end{aligned}$$

which implies, after integration by parts, that

$$\int_{-\infty}^{\infty}(f_1)_{xx}\phi_{xt}\,dx\leqslant\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xx}^2\,dx\right)$$
$$+C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xx}^2+\phi_{xt}^2]\,dx+C\delta_0. \tag{3.59}$$

Substituting (3.56), (3.57), (3.59) into (3.55), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\phi_{xt}^2+\left(p'(W+\phi_x)-\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)\phi_{xx}^2\right]dx\right)$$
$$+\int_{-\infty}^{\infty}\left[(1-\kappa)\phi_{xt}^2-\frac{1}{2}\kappa\phi_{xx}^2\right]dx+\int_{-\infty}^{\infty}(WE)_x\phi_{xt}\,dx$$
$$\leqslant C\int_{-\infty}^{\infty}\phi_x^2\,dx+C\delta_0, \tag{3.60}$$

where we have used (3.17) so that $C(N(T)+\delta_0)\leqslant\frac{1}{6}\kappa$.

Having $[(3.54)+2\times(3.60)]$, we obtain

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_x^2+\phi_x\phi_{xt}+\phi_{xt}^2\right]dx\right)$$
$$+\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[p'(W+\phi_x)-\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right]\phi_{xx}^2\,dx\right)$$
$$+\int_{-\infty}^{\infty}[(p'(W+\phi_x)-2\kappa)\phi_{xx}^2+(1-3\kappa)\phi_{xt}^2]\,dx$$
$$+\int_{-\infty}^{\infty}(WE)_x(\phi_x+2\phi_{xt})\,dx\leqslant C\delta_0+C\int_{-\infty}^{\infty}[\phi_x^2+\phi_t^2]\,dx. \tag{3.61}$$

Multiplying (3.46) by $[\psi_x+2\psi_{xt}]$, applying the similar procedures as above, noticing (3.39) and the facts

$$(f_2)_{xx}=-\left(\frac{(p(W)_x-\psi_t)^2}{(W+\psi_x)^2}\psi_{xx}\right)_x-\frac{2(p(W)_x-\psi_t)}{W+\psi_x}\psi_{xxt}$$
$$+O(1)(W_x+\phi_t)(W_{xx}(W_x+\phi_t)+W_{xt})$$
$$+O(1)(\psi_{xt}+W_t)[(\psi_{xx}+W_x)(\psi_t+W_x)+W_t+\psi_{xt}]$$
$$+O(1)W_x[(\psi_{xx}+W_x)(\psi_t^2+W_x^2)+(W_t+\psi_{xt})(W_x+\psi_t)], \tag{3.62}$$

$$g_2=O(1)(\psi_x(W_{xx}+W_x^2)+W_x\psi_{xx})+\psi_xE_x+\psi_{xx}E, \tag{3.63}$$

we obtain

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\psi_x^2+\psi_x\psi_{xt}+\psi_{xt}^2\right]dx\right)$$
$$+\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[p'(W+\psi_x)-\frac{(p(W)_x-\psi_t)^2}{(W+\psi_x)^2}\right]\phi_{xx}^2\,dx\right)$$
$$+\int_{-\infty}^{\infty}[(p''(W+\psi_x)-2\kappa)\psi_{xx}^2+(1-3\kappa)\psi_{xt}^2]\,dx$$
$$-\int_{-\infty}^{\infty}(WE)_x(\psi_x+2\psi_{xt})\,dx\leqslant C\delta_0+C\int_{-\infty}^{\infty}[\psi_x^2+\psi_t^2]\,dx. \tag{3.64}$$

Combining $[(3.61)+(3.64)]$ and integrating it over $[0,T]$, noticing that

$$\int_{-\infty}^{\infty}[(WE)_x(\phi_x+2\phi_{xt})-(WE)_x(\psi_x+2\psi_{xt})]\,dx$$
$$\geqslant\frac{d}{dt}\left(\int_{-\infty}^{\infty}[WE_{xx}^2+2W_xEE_x]\,dx\right)$$
$$+\int_{-\infty}^{\infty}\left[(W-W_t)E_{xx}^2-\frac{1}{2}(W-W_t)_{xx}E_x^2\right]dx$$
$$-C\int_{-\infty}^{\infty}[\psi_x^2+\psi_t^2+\phi_x^2+\phi_t^2]\,dx, \tag{3.65}$$

and using Lemma 3.2, we have

$$\int_{-\infty}^{\infty}[\phi_{xt}^2+\phi_{xx}^2+\phi_x^2+\psi_{xt}^2+\psi_{xx}^2+\psi_x^2+E_x^2]\,dx$$
$$+\int_0^T\int_{-\infty}^{\infty}[\phi_{xt}^2+\phi_{xx}^2+\psi_{xt}^2+\psi_{xx}^2+E_x^2]\,dx$$
$$\leqslant C(||(\phi_0,\psi_0)||_3^2+||(\phi_1,\psi_1)||_2^2+\delta_0), \tag{3.66}$$

provided that $N(T)+\delta_0\ll 1$. Where we have used (3.43).

Multiply (3.7) with ϕ_{tt} and (3.8) with ϕ_{tt}, and integrate them over $\mathbb{R}$ and $\mathbb{R}\times[0,T]$ respectively. We get by (3.66) and (3.20) that

$$\int_{-\infty}^{\infty}[\phi_{xt}^2+\phi_{xx}^2]\,dx+\int_0^T\int_{-\infty}^{\infty}[\phi_{xt}^2+\phi_{xx}^2]\,dx\,ds$$
$$\leqslant C(||(\phi_0,\psi_0)||_3^2+||(\phi_1,\psi_1)||_2^2+\delta_0), \tag{3.67}$$

The combination of (3.66) and (3.67) leads to (3.44). □

To enclose the a-priori assumption (3.17), we need the following estimate:

Lemma 3.4. *Let $T>0$. It holds, for $(\phi,\psi,E)\in X(T)$, that*

$$\begin{aligned}
&||(\phi_{xxx},\phi_{xxt},\phi_{xtt},\phi_{ttt},\psi_{xxx},\psi_{xxt},\psi_{ttt},\psi_{xtt},E_{xx})(.,t)||^2\\
&\quad+\int_0^T ||(\phi_{xxt},\phi_{xxx},\psi_{xxt},\psi_{xxx},E_{xx})(.,s)||^2\,ds\\
&\leqslant C(||(\phi_0,\psi_0)||_3^2+||(\phi_1,\psi_1)||_2^2+\delta_0),
\end{aligned}\tag{3.68}$$

provided that $N(T)+\delta_0$ small enough.

Proof. Differentiate (3.45) and (3.46) with respect to x,[1] we obtain

$$\phi_{xxtt}-(p'(W+\phi_x)\phi_{xxx})_x+\phi_{xxt}+(WE)_{xx}=-p(W)_{xxxt}+(f_1)_{xxx}+g_3, \tag{3.69}$$

$$\psi_{xxtt}-(p'(W+\psi_x)\psi_{xxx})_x+\psi_{xxt}-(WE)_{xx}=-p(W)_{xxxt}+(f_2)_{xxx}+g_4, \tag{3.70}$$

where

$$g_3=(p'(W+\phi_x)_x\phi_{xx})_x+((p'(W+\phi_x)-p'(W))W_x)_{xx}-(\phi_xE_x)_{xx}, \tag{3.71}$$

$$g_4=(p'(W+\psi_x)_x\psi_{xx})_x+((p'(W+\psi_x)-p'(W))W_x)_x+(\psi_xE_x)_{xx}. \tag{3.72}$$

Multiply (3.69) by $[\phi_{xx}+2\phi_{xxt}]$ and integrate it by parts over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_{xx}^2+\phi_{xx}\phi_{xxt}+\phi_{xxt}^2+p'(W+\phi_x)\phi_{xxx}^2\right]dx\right)\\
&\quad+\int_{-\infty}^{\infty}[(p'(W+\phi_x)-p'(W+\phi_x)_t)\phi_{xxx}^2+\phi_{xxt}^2]\,dx\\
&\quad+\int_{-\infty}^{\infty}(WE)_{xx}(\phi_{xx}+2\phi_{xxt})\,dx\\
&=-\int_{-\infty}^{\infty}p(W)_{xxxt}(\phi_{xx}+2\phi_{xxt})\,dx-\int_{-\infty}^{\infty}(f_1)_{xx}\phi_{xxx}\,dx\\
&\quad+2\int_{-\infty}^{\infty}(f_1)_{xxx}\phi_{xxt}\,dx+\int_{-\infty}^{\infty}g_3(\phi_{xx}+2\phi_{xxt})\,dx.
\end{aligned}\tag{3.73}$$

[1] For the proof, we first assume that the solutions (ϕ,ψ) have high-order regularity because the a-priori estimates (3.20) and (3.44) will be still valid for these solutions by applying the Friedrich's mollifier under the same assumptions (3.17). We omit the detail here.

The terms on the right-hand side of (3.73) are estimated as follows. By (2.1) and integration by parts, we obtain

$$-\int_{-\infty}^{\infty} p(W)_{xxxt}(\phi_{xx}+2\phi_{xxt})\,dx \leqslant C\int_{-\infty}^{\infty}[|W_{tt}|+|W_{xtt}|]\,dx+\delta_0\int_{-\infty}^{\infty}[\phi_{xxx}^2+\phi_{xxt}^2]\,dx. \tag{3.74}$$

In view of (3.71), it follows

$$g_3 = C(N(T)+\delta_0)(\phi_{xxx}+\phi_{xx}+\phi_x+\psi_{xx}). \tag{3.75}$$

Therefore, we have

$$\int_{-\infty}^{\infty} g_3(\phi_{xx}+2\phi_{xxt})\,dx \leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xxx}^2+\phi_{xxt}^2+\phi_{xx}^2+\phi_x^2+\psi_{xx}^2]\,dx. \tag{3.76}$$

By (3.58), we have

$$-\int_{-\infty}^{\infty}(f_1)_{xx}\phi_{xxx}\,dx \leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xxx}^2+\phi_{xxt}^2+\phi_{xx}^2+\phi_{xt}^2]\,dx+C\delta_0. \tag{3.77}$$

Since

$$\begin{aligned}(f_1)_{xxx} = &-\left(\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xxx}\right)_x - \frac{2(p(W)_x-\phi_t)}{W+\phi_x}\phi_{xxxt}\\
&+O(1)\phi_{xxt}[W_t+\phi_{xt}+(W_x+\phi_t)(W_x+\phi_{xx})]\\
&+O(1)\phi_{xxx}[(W_x^2+\phi_t^2)(W_x+\phi_{xx})+(W_x+\phi_t)(W_t+\phi_{xt})]\\
&+O(1)(W_{xxt}+W_{xxx}(W_x+\phi_t))(W_x+\phi_t)\\
&+O(1)W_{xt}[W_t+\phi_{xt}+(W_x+\phi_{xx})(W_x+\phi_t)]\\
&+O(1)W_{xx}[(W_t+\phi_{xt})(W_x+\phi_t)+(W_x+\phi_{xx})(W_x^2+\phi_t^2)]\\
&+O(1)(W_t+\phi_{xt})[(W_t+\phi_{xt})(W_x+\phi_{xx})+(W_x+\phi_t)(W_x^2+\phi_{xx}^2)]\\
&+O(1)(W_x+\phi_{xx})(W_x^2+\phi_t^2)(W_x^2+\phi_{xx}^2)\end{aligned} \tag{3.78}$$

$$= -\left(\frac{(p(W)_x - \phi_t)^2}{(W+\phi_x)^2}\phi_{xxx}\right)_x - \frac{2(p(W)_x - \phi_t)}{W+\phi_x}\phi_{xxxt}$$
$$+ O(1)(W_{xxt} + W_{xxx} + W_{xt} + W_{xx} + W_t + W_x^2)$$
$$+ O(N(T)+\delta_0)(\phi_{xxx} + \phi_{xxt} + \phi_{xt} + \phi_{xx}), \tag{3.79}$$

the other term on the right-hand side of (3.73) is bounded by

$$\begin{aligned}
&2\int_{-\infty}^{\infty}(f_1)_{xxx}\phi_{xxt}\,dx \\
&\leqslant \frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xxx}^2\,dx\right) \\
&\quad - \int_{-\infty}^{\infty}\left(\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)_t\phi_{xxx}^2 dx + \int_{-\infty}^{\infty}\left(\frac{2(p(W)_x-\phi_t)}{W+\phi_x}\right)_x\phi_{xxt}^2\,dx \\
&\quad + C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xxx}^2+\phi_{xxt}^2+\phi_{xt}^2+\phi_{xx}^2]\,dx + C\delta_0.
\end{aligned} \tag{3.80}$$

Substituting (3.74), (3.76), (3.77), (3.80) into (3.73), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_{xx}^2+\phi_{xx}\phi_{xxt}+\phi_{xxt}^2\right]dx\right) \\
&\quad + \frac{d}{dt}\left(\int_{-\infty}^{\infty}\left(p'(W+\phi_x)-\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)\phi_{xxx}^2\,dx\right) \\
&\quad + \int_{-\infty}^{\infty}[p'(W+\phi_x)\phi_{xxx}^2+\phi_{xxt}^2]\,dx + \int_{-\infty}^{\infty}(WE)_{xx}(\phi_{xx}+2\phi_{xxt})\,dx \\
&\leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xxx}^2+\phi_{xxt}^2+\phi_{xt}^2+\phi_{xx}^2+\phi_x^2+\psi_{xx}^2]\,dx + C\delta_0.
\end{aligned} \tag{3.81}$$

Multiply (3.70) by $[\psi_{xx} + 2\psi_{xxt}]$ and integrate it by parts over $\mathbb{R}$. Noticing (3.62),

$$(f_2)_{xxx} = -\frac{(p(W)_x - \psi_t)^2}{(W+\psi_x)^2}\psi_{xxxx} - \frac{2(p(W)_x-\psi_t)}{W+\psi_x}\psi_{xxxt}$$
$$+ O(1)(W_{xxt} + W_{xxx} + W_{xt} + W_{xx} + W_t + W_x^2)$$
$$+ O(N(T)+\delta_0)(\psi_{xxx} + \psi_{xxt} + \psi_{xt} + \psi_{xx}), \tag{3.82}$$

and

$$|g_4| \leqslant C(N(T)+\delta_0)|\psi_{xxx} + \psi_{xx} + \psi_x + \phi_{xx}| \tag{3.83}$$

we obtain after a tedious calculation that

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\psi_{xx}^2+\psi_{xx}\psi_{xxt}+\psi_{xxt}^2\right]dx\right)\\
&\quad+\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left(p'(W+\psi_x)-\frac{(p(W)_x-\psi_t)^2}{(W+\psi_x)^2}\right)\psi_{xxx}^2\,dx\right)\\
&\quad+\int_{-\infty}^{\infty}[p'(W+\psi_x)\psi_{xxx}^2+\psi_{xxt}^2]\,dx-\int_{-\infty}^{\infty}(WE)_{xx}(\psi_{xx}+2\psi_{xxt})\,dx\\
&\leqslant C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\psi_{xxx}^2+\psi_{xxt}^2+\psi_{xt}^2+\psi_{xx}^2+\psi_x^2+\phi_{xx}^2]\,dx+C\delta_0. \qquad (3.84)
\end{aligned}
$$

Combining [(3.81) + (3.84)] and integrating it over $[0,T]$, noticing that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}[(WE)_{xx}(\phi_{xx}+2\phi_{xxt})-(WE)_{xx}(\psi_{xx}+2\psi_{xxt})]\,dx\\
&\quad=\int_{-\infty}^{\infty}(WE_{xx}+W_{xx}E+2W_xE_x)(E_{xx}+2E_{xxt})\,dx\\
&\quad\geqslant\frac{d}{dt}\left(\int_{-\infty}^{\infty}[WE_{xx}^2+2W_{xx}EE_{xx}]\,dx\right)+\int_{-\infty}^{\infty}(W-W_t)E_{xx}^2\,dx\\
&\qquad-C(N(T)+\delta_0)\int_{-\infty}^{\infty}[\phi_{xxt}^2+\psi_{xxt}^2+\phi_{xx}^2+\psi_{xx}^2]\,dx, \qquad (3.85)
\end{aligned}
$$

and using Lemmas 3.2–3.3, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}[\phi_{xxt}^2+\phi_{xxx}^2+\phi_{xx}^2+\psi_{xxt}^2+\psi_{xxx}^2+\psi_{xx}^2+E_{xx}^2]\,dx\\
&\quad+\int_0^T\int_{-\infty}^{\infty}[\phi_{xxt}^2+\phi_{xxx}^2+\psi_{xxt}^2+\psi_{xxx}^2+E_{xx}^2]\,dx\\
&\leqslant C(\|(\phi_0,\psi_0)\|_3^2+\|(\phi_1,\psi_1)\|_2^2+\delta_0), \qquad (3.86)
\end{aligned}
$$

provided that $N(T)+\delta_0\ll 1$.

Differentiate (3.7) and (3.8) with respect to t, Multiply them with ϕ_{ttt} and ψ_{ttt} respectively and integrate them over $\mathbb{R}$ and $\mathbb{R}\times[0,T]$ respectively. We obtain by (3.86), (3.20) and (3.44) that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}[\phi_{ttt}^2+\psi_{ttt}^2]\,dx+\int_0^T\int_{-\infty}^{\infty}[\phi_{ttt}^2+\psi_{ttt}^2]\,dx\,ds\\
&\leqslant C(\|(\phi_0,\psi_0)\|_3^2+\|(\phi_1,\psi_1)\|_2^2+\delta_0), \qquad (3.87)
\end{aligned}
$$

which together with (3.86) yields (3.68). □

The combination of Lemmas 3.2–3.4 gives Theorem 3.1.

By Theorem 3.1 and a standard continuity argument, we can extend the local classical solutions of IVP (3.7)–(3.14) globally in time:

Theorem 3.5. *Under the assumptions of Theorem* 1.1, *the classical solutions* (ϕ, ψ, E) *of IVP* (3.7)–(3.14) *exist globally in time if* $\Phi_0 + \delta_0 \ll 1$, *and satisfy that*

$$||(\phi, \psi)(., t)||_3^2 + ||(\phi_t, \psi_t, E)(., t)||_2^2 + \int_0^t ||(\phi_t, \phi_x, \psi_t, \psi_x, E)(., s)||_2^2 \, ds \leqslant C(\Phi_0 + \delta_0), \quad t > 0, \tag{3.88}$$

$$||(\phi_x, \psi_x, \phi_t, \psi_t, E)(., t)||_2^2 \to 0, \quad t \to \infty. \tag{3.89}$$

3.3. The time-decay rate of solutions

In this section, we prove the time-decay rate of classical solutions (ϕ, ψ, E) of IVP (3.7)–(3.14) obtained by Theorem 3.5. We first prove the exponential decay of electric field to zero state, and then, obtain the algebraic convergence of (ϕ, ψ) to zero state. Due to the results in Section 3.2, we know that the global classical solutions (ϕ, ψ, E) satisfy

$$||(\phi, \psi)||_3^2 + ||(\phi_t, \psi_t, E)||_2^2 \leqslant C(\Phi_0 + \delta_0), \tag{3.90}$$

which leads to, in terms of Sobolev embedding theorem, that

$$||(\phi, \phi_x, \phi_{xx}, \phi_t, \phi_{xt}, \psi, \psi_x, \psi_{xx}, \psi_t, \psi_{xt}, E, E_x)||_{L^\infty} \leqslant C(\Phi_0 + \delta_0). \tag{3.91}$$

By (3.91), (3.7), (3.9), (3.8), and (3.10), we know that

$$||(\phi_{tt}, \psi_{tt})||_{L^\infty} \leqslant C(\Phi_0 + \delta_0). \tag{3.92}$$

Lemma 3.6. *Let* (ϕ, ψ, E) *be the global classical solutions of IVP* (3.7)–(3.14) *with initial data satisfying* $||(\phi_0, \psi_0)||_3 + ||(\phi_1, \psi_1)||_2 \ll 1$. *Then it holds for electric field E that*

$$||(E, E_t, E_x, E_{xt}, E_{xx}, E_{tt})||^2 \leqslant C(\Phi_0 + \delta_0) \exp\{-\beta_0 t\} \tag{3.93}$$

for $t > 0$.

Proof. *Step* 1: Multiply (3.13) by $[E+2E_t]$ and integrate it by parts over $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[E_t^2+\frac{1}{2}E^2+EE_t+2WE^2\right]dx\right)\\ +\int_{-\infty}^{\infty}[E_t^2+2(W-W_t)E^2]\,dx\\ =\int_{-\infty}^{\infty}f_3(E+2E_t)\,dx=\int_{-\infty}^{\infty}[(f_{31})_x+f_{32}+(f_{33})_x](E+2E_t)\,dx.\end{aligned} \tag{3.94}$$

The right-hand side terms in (3.94) can be estimated as follows. Since

$$(f_{31})_x=p'(W+\phi_x)E_{xx}+(p'(W+\phi_x)-p'(W+\psi_x))(W+\psi_x)_x, \tag{3.95}$$

$$=p'(W+\phi_x)E_{xx}+O(1)(W+\psi_x)_xE_x, \tag{3.96}$$

we obtain, by integration by parts, that

$$\begin{aligned}&\int_{-\infty}^{\infty}(f_{31})_x(E+2E_t)\,dx\\ &\leqslant-\frac{d}{dt}\left(\int_{-\infty}^{\infty}p'(W+\phi_x)E_x^2\,dx\right)-\int_{-\infty}^{\infty}p'(W+\phi_x)E_x^2\,dx\\ &\quad+C(N(T)+\delta_0)\int_{-\infty}^{\infty}[E_x^2+E_t^2]\,dx,\end{aligned} \tag{3.97}$$

$$\left|\int_{-\infty}^{\infty}f_{32}(E+2E_t)\,dx\right|\leqslant C(\Phi_0+\delta_0)\int_{-\infty}^{\infty}[E^2+E_t^2]\,dx. \tag{3.98}$$

Due to the facts

$$f_{33}=O(1)(\phi_t+\psi_t+W_x)(E_x+E_t), \tag{3.99}$$

and

$$\begin{aligned}(f_{33})_x=&-\frac{2(p(W)_x-\phi_t)}{W+\phi_x}E_{xt}\\ &+\left(\frac{2(p(W)_x-\phi_t)}{W+\phi_x}-\frac{2(p(W)_x-\psi_t)}{W+\psi_x}\right)(p(W)_x-\psi_t)_x\\ &-\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}E_{xx}\\ &-\left(\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}-\frac{(p(W)_x-\psi_t)^2}{(W+\psi_x)^2}\right)(W+\psi_x)_x,\end{aligned} \tag{3.100}$$

$$= -\frac{2(p(W)_x - \phi_t)}{W + \phi_x} E_{xt} - \frac{(p(W)_x - \phi_t)^2}{(W + \phi_x)^2} E_{xx} + O(1)(\phi_t + \psi_t + W_x)(E_x + E_t), \tag{3.101}$$

we have, after integration by parts, that

$$\int_{-\infty}^{\infty} (f_{33})_x (E + 2E_t)\, dx \leqslant \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty} \frac{(p(W)_x - \phi_t)^2}{(W + \phi_x)^2} E_x^2\, dx\right) + C(\Phi_0 + \delta_0)\int_{-\infty}^{\infty} [E_x^2 + E_t^2]\, dx, \tag{3.102}$$

where we have used (2.1), and the estimate

$$|\psi_{tt}| \leqslant C|\psi_{xx} + \psi_{xt} + \psi_x + \psi_t + \psi + \phi + W_{xx} + W_{xt}|, \tag{3.103}$$

which comes from (3.7).

Substituting (3.97), (3.98), (3.102) into (3.94), we obtain for $\delta_1 > 0$

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty} \left[E_t^2 + \frac{1}{2}E^2 + EE_t + 2WE^2\right] dx\right) + \frac{d}{dt}\left(\int_{-\infty}^{\infty} \left(p'(W + \phi_x) - \frac{(p(W)_x - \phi_t)^2}{(W + \phi_x)^2}\right) E_x^2\, dx\right) + \delta_1 \int_{-\infty}^{\infty} [E_t^2 + E^2 + E_x^2]\, dx \leqslant 0, \tag{3.104}$$

provided that $N(T) + \delta_0 \ll 1$. Applying Gronwall's lemma to above differential inequality , we obtain from (3.104)

$$||(E, E_x, E_t)||^2 \leqslant C(\Phi_0 + \delta_0)\exp\{-\beta_1 t\}. \tag{3.105}$$

Step 2: Differentiating (3.13) with respect to x, we obtain

$$E_{xtt} + E_{xt} + 2WE_x + 2W_xE = (f_3)_x. \tag{3.106}$$

Multiplying (3.106) by $[E_x + 2E_{xt}]$ and integrating it by parts over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[E_{xt}^2+\frac{1}{2}E_x^2+E_xE_{xt}+2WE_x^2\right]dx\right)\\
&\quad+\int_{-\infty}^{\infty}[E_{xt}^2+2(W-W_t)E_x^2]\,dx\\
&\leqslant 2\int_{-\infty}^{\infty}W_xE(E_x+2E_{xt})\,dx\\
&\quad+\int_{-\infty}^{\infty}[(f_{31})_{xx}+(f_{32})_x+(f_{33})_{xx}](E_x+2E_{xt})\,dx. \qquad (3.107)
\end{aligned}$$

The right-hand side terms in (3.107) can be estimated as follows. It follows from (2.1) and integration by parts that

$$2\int_{-\infty}^{\infty}W_xE(E_x+2E_{xt})\,dx\leqslant C\delta_0\int_{-\infty}^{\infty}[E_{xt}^2+E_x^2+E^2]\,dx. \qquad (3.108)$$

Due to (3.95), it holds that

$$\begin{aligned}
(f_{31})_{xx} =& p'(W+\phi_x)E_{xxx}+O(1)((W+\psi_x)_{xx}\\
&+O(1)[(W+\psi_x)_x^2)E_x+(W+\phi_x+\psi_x)_xE_{xx}], \qquad (3.109)
\end{aligned}$$

which combined with (3.96) and integration by parts yields that

$$\begin{aligned}
&\int_{-\infty}^{\infty}(f_{31})_{xx}(E_x+2E_{xt})\,dx\\
&\quad\leqslant -\frac{d}{dt}\left(\int_{-\infty}^{\infty}p'(W+\phi_x)E_{xx}^2\,dx\right)-\int_{-\infty}^{\infty}p'(W+\phi_x)E_{xx}^2\,dx\\
&\qquad+C(\Phi_0+\delta_0)\int_{-\infty}^{\infty}[E_{xx}^2+E_x^2]\,dx+\kappa\int_{-\infty}^{\infty}E_{xt}^2\,dx. \qquad (3.110)
\end{aligned}$$

A direct calculation gives

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}(f_{32})_{xx}(E_x+2E_{xt})\,dx\right|\\
&\quad=\int_{-\infty}^{\infty}|((\phi_x+\psi_x)E_x+(\phi_x+\psi_x)_xE)(E_x+2E_{xt})|\,dx\\
&\quad\leqslant C(\Phi_0+\delta_0)\int_{-\infty}^{\infty}[E^2+E_x^2+E_{xt}^2]\,dx. \qquad (3.111)
\end{aligned}$$

Noticing (3.101) and

$$(f_{33})_{xx} = -\frac{2(p(W)_x - \phi_t)}{W+\phi_x}E_{xxt} - \frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}E_{xxx}, \\ + O(1)(\psi_{xxx}+\psi_{xxt})(E_x+E_t) \\ + O(1)(\Phi_0+\delta_0)(E_{xx}+E_{xt}+E_x+E_t), \tag{3.112}$$

we obtain, by integration by parts, that

$$\int_{-\infty}^{\infty}(f_{33})_{xx}(E_x+2E_{xt})\,dx \\ \leqslant \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}E_x^2\,dx\right) + \kappa\int_{-\infty}^{\infty}E_{xt}^2\,dx \\ + C(\Phi_0+\delta_0)\int_{-\infty}^{\infty}[E_{xt}^2+E_{xx}^2+E_t^2+E_x^2]\,dx, \tag{3.113}$$

where we have used (3.103).

Substituting (3.108), (3.110), (3.111),(3.113) into (3.107), we obtain, similarly to (3.104), that

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[E_{xt}^2+\frac{1}{2}E_x^2+E_xE_{xt}+2WE_x^2\right]dx\right) \\ + \frac{d}{dt}\left(\int_{-\infty}^{\infty}\left(p'(W+\phi_x)-\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)E_{xx}^2\,dx\right) \\ + \delta_3\int_{-\infty}^{\infty}[E_{xt}^2+E_x^2+E_{xx}^2]\,dx \leqslant 0, \tag{3.114}$$

provided that $\Phi_0+\delta_0 \ll 1$. Applying Gronwall's lemma to above differential inequality, we obtain

$$||(E_x, E_{xx}, E_{xt})||^2 \leqslant C(||E_0||_2^2+||E_1||_1^2+\delta_0)\exp\{-\beta_2 t\}. \tag{3.115}$$

By (3.105), (3.115), and the estimate

$$|E_{tt}| \leqslant C|E_{xt}+E_{xx}+E_t+E_x+E|, \tag{3.116}$$

we can prove (3.93) for $\beta_0 = \min\{\beta_1,\ \beta_2\}$. □

Then, we turn to prove the time-decay rate of (ϕ,ψ), for which we are able to obtain the algebraical decay rate for large time. We will use Nishihara's idea [28,27] to prove Lemma 3.7.

Lemma 3.7. *Let* (ϕ, ψ, E) *be the global classical solutions of IVP* (3.7)–(3.14) *with initial data satisfying* $||(\phi_0, \psi_0)||_3 + ||(\phi_1, \psi_1)||_2 \ll 1$. *If it holds for* (ϕ, ψ) $(t>0)$ *that*

$$\sum_{k=0}^{3} (1+t)^k ||\partial_x^k(\phi, \psi)(., t)||^2 + \sum_{k=0}^{2} (1+t)^{k+2} ||\partial_x^k(\phi_t, \psi_t)(., t)||^2 \leqslant C, \tag{3.117}$$

then we have

$$\sum_{k=0}^{3} (1+t)^k ||\partial_x^k(\phi, \psi)(., t)||^2 + \sum_{k=1}^{3} \int_0^t (1+s)^{k-1} ||\partial_x^k(\phi, \psi)(., s)||^2 \, ds \leqslant C(\Phi_0 + \delta_0), \tag{3.118}$$

$$\sum_{k=0}^{2} (1+t)^{k+2} ||\partial_x^k(\phi_t, \psi_t)(., t)||^2 + \sum_{k=0}^{2} \int_0^t (1+s)^{k+1} ||\partial_x^k(\phi_t, \psi_t)(., s)||^2 \, ds \leqslant C(\Phi_0 + \delta_0), \tag{3.119}$$

$$(1+t)^5 ||(\phi_{ttt}, \psi_{ttt}, \phi_{xtt}, \psi_{xtt})(., t)||^2 + \int_0^t (1+s)^5 ||(\phi_{ttt}, \psi_{ttt})(., s)||^2 \, ds \leqslant C(\Phi_0 + \delta_0), \tag{3.120}$$

provided that $\Phi_0 + \delta_0 \ll 1$.

Proof. We only need to prove (3.118)–(3.120) for ϕ because we can de-couple wave equations (3.7) and (3.8) since the effort of electric field decays exponentially by Lemma 3.6. However, more efforts have to be made in dealing with the difficulties caused by convection terms. Here we just prove (3.118) in order to show how to estimate the new term.

By (3.117) and the Sobolev inequality

$$||f||_{L^\infty} \leqslant \sqrt{||f|| ||f_x||}, \tag{3.121}$$

we can obtain the a-priori L^∞ time-decay rate of (ϕ, ψ) and their derivatives and then we can obtain the a-priori L^∞ time decay rate of convection term.

Step 1: Multiply (3.7) with $(1+t)\phi_t$ and integrate over $\mathbb{R}$ (or multiply (3.27) with $(1+t)$). Using (3.27), (3.30), (3.33), (2.1), Lemma 3.6 and Cauchy–Schwartz's inequality, we have, after integration by parts, that

$$\begin{aligned}
&\frac{d}{dt}\left((1+t)\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_t^2+\sigma(\phi_x,W)-\frac{1}{2}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_x^2\right]dx\right)\\
&\quad-\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_t^2+\sigma(\phi_x,W)-\frac{1}{2}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_x^2\right]dx+(1+t)\int_{-\infty}^{\infty}\phi_t^2\,dx\\
&\leqslant C\delta_0(1+t)^{-3/2}+C(\Phi_0+\delta_0)(1+t)e^{-\beta_0 t}\\
&\quad+\frac{1}{8}(1+t)\int_{-\infty}^{\infty}\phi_t^2\,dx+C\int_{-\infty}^{\infty}[\phi_t^2+\phi_x^2]\,dx,
\end{aligned}\tag{3.122}$$

where we have used (3.117), (3.7), Lemma 3.6, and

$$\begin{cases}
\left|\left(\dfrac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)_t\right|_{L^\infty}\leqslant C(\Phi_0+\delta_0)(1+t)^{-1},\\
\left|\left(\dfrac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)_x^2\right|_{L^\infty}\leqslant C(\Phi_0+\delta_0)(1+t)^{-2}.
\end{cases}\tag{3.123}$$

Integrating (3.122) over $[0,t]$, using Lemmas 3.2–3.3 and (3.29), we obtain

$$(1+t)||(\phi_t,\phi_x)(.,t)||^2+\int_0^t(1+s)||\phi_t(.,s)||^2\,ds\leqslant C(\Phi_0+\delta_0),\tag{3.124}$$

provided that $\Phi_0+\delta_0\ll 1$.

Step 2: Multiply (3.45) with ϕ_{xt} and integrate by parts over $\mathbb{R}$ (or rewrite (3.55)). Then we have, by Cauchy–Schwartz's inequality, (3.51), (3.56), (2.1), (3.117) and (3.123), that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\phi_{xt}^2+\left(p'(W+\phi_x)-\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\right)\phi_{xx}^2\right]dx\right)+\int_{-\infty}^{\infty}\phi_{xt}^2\,dx\\
&\leqslant C(1+t)^{-1}\int_{-\infty}^{\infty}[\phi_t^2+\phi_{xx}^2]\,dx+\frac{1}{8}\int_{-\infty}^{\infty}\phi_{xt}^2\,dx+C(\Phi_0+\delta_0)e^{-\beta_0 t}\\
&\quad+C(1+t)^{-7/2},
\end{aligned}\tag{3.125}$$

provided that $\Phi_0+\delta_0\ll 1$.

Multiply (3.7) with $-\phi_{xx}$ and integrate it by parts over $\mathbb{R}$. By (3.32), (3.93) and Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\left[\frac{1}{2}\phi_x^2+\phi_x\phi_{xt}\right]dx\right)+\int_{-\infty}^{\infty}p'(W+\phi_x)\phi_{xx}^2\,dx\\ \leqslant\frac{1}{8}\int_{-\infty}^{\infty}p'(W+\phi_x)\phi_{xx}^2\,dx+\frac{7}{8}\int_{-\infty}^{\infty}\phi_{xt}^2\,dx+C(1+t)^{-5/2}\\ +C(\Phi_0+\delta_0)e^{-\beta_0 t}+C(1+t)^{-1}\int_{-\infty}^{\infty}[\phi_x^2+\phi_{xx}^2]\,dx\\ +C(1+t)^{-1}\int_{-\infty}^{\infty}\phi_t^2\,dx.\end{aligned}\tag{3.126}$$

Multiply $(1+t)$ to $[2\times(3.125)+(3.126)]$ and integrate it over $[0,t]$. We get

$$\begin{aligned}(1+t)\int_{-\infty}^{\infty}[\phi_x^2+\phi_{xt}^2]\,dx+\int_0^t(1+s)\|(\phi_{xt},\phi_{xx})(.,s)\|^2\,dxs\\ \leqslant C(\Phi_0+\delta_0).\end{aligned}\tag{3.127}$$

Multiply (3.126) with $(1+t)^2$ and integrate it over $[0,t]$, we obtain by (3.127) that

$$(1+t)^2\int_{-\infty}^{\infty}[\phi_{xx}^2+\phi_{xt}^2]\,dx+\int_0^t(1+s)^2\|\phi_{xt}(.,s)\|^2\,dx\,ds\leqslant C(\Phi_0+\delta_0).\tag{3.128}$$

The combination of (3.128) and (3.127) leads to

$$\begin{aligned}(1+t)^2\|(\phi_{xx},\phi_{xt})\|^2\\ +\int_0^t[(1+s)^2\|\phi_{xt}(.,s)\|^2+(1+s)\|\phi_{xx}(.,s)\|^2]\,dx\,ds\leqslant C(\Phi_0+\delta_0).\end{aligned}\tag{3.129}$$

Similarly, multiply $(1+t)^2$ to $[2\phi_{xxt}\times(3.69)-\phi_{xx}\times(3.45)]$ and $(1+t)^3\phi_{xxt}$ to (3.69), and integrate them over $\mathbb{R}\times[0,t]$. By (3.78), (3.58),(3.123), (3.124) and (3.117), the estimates of convection term are given by

$$\begin{aligned}\int_{-\infty}^{\infty}(f_1)_{xxx}\phi_{xxt}\,dx\\ \leqslant\frac{d}{dt}\left(\int_{-\infty}^{\infty}\frac{(p(W)_x-\phi_t)^2}{(W+\phi_x)^2}\phi_{xxx}^2\,dx\right)+\frac{1}{8}\int_{-\infty}^{\infty}\phi_{xxt}^2\,dx\\ +C(\Phi_0+\delta_0)(1+t)^{-1}\int_{-\infty}^{\infty}\phi_{xxx}^2\,dx+C(\Phi_0+\delta_0)(1+t)^{-9/2},\end{aligned}$$

$$\times \int_{-\infty}^{\infty} (f_1)_{xx}\phi_{xxx}\, dx$$
$$\leqslant \frac{1}{8}\int_{-\infty}^{\infty} [\phi_{xxt}^2 + p'(W+\phi_x)\phi_{xxx}^2]\, dx$$
$$+ C(1+t)^{-2}\int_{-\infty}^{\infty} \phi_{xx}^2\, dx + C(\Phi_0+\delta_0)(1+t)^{-7/2},$$

for $\Phi_0 + \delta_0 \ll 1$, which yields finally the high-order decay estimates:

$$(1+t)^3\|(\phi_{xxx}, \phi_{xxt})\|^2 + \int_0^t (1+s)^3\|\phi_{xxt}(.,s)\|^2\, dx\, ds$$
$$+ \int_0^t (1+s)^2\|\phi_{xxx}(.,s)\|^2\, dx\, ds \leqslant C(\Phi_0+\delta_0). \tag{3.130}$$

The combination of (3.124), (3.129), (3.130) leads to the estimate (3.118) for ϕ.

Applying above steps in proving the time-decay rates of ϕ to ψ, and using (2.1), Lemma 3.6, Lemmas 3.2–3.3 and Cauchy–Schwartz's inequality, we obtain, after tedious calculations, the time-decay estimates for ψ and its derivatives, which together with those for ϕ yields (3.118).

The estimates (3.119) and (3.120) can be obtained similarly. In fact, taking integration by parts over $\mathbb{R} \times [0,t]$ the equations $(1+t)^i \times [(\phi_t + 2\phi_{tt}) \times (3.7)_t + (\psi_t + \psi_{tt}) \times (3.8)_t]$, $(1+t)^{i+1} \times [\phi_{tt} \times (3.7)_t + \psi_{tt} \times (3.8)_t]$ for $i = 1, 2$, $(1+t)^j \times [(\phi_{xt} + 2\phi_{xtt}) \times (3.7)_{xt} + (\psi_{xt} + \psi_{xtt}) \times (3.8)_{xt}]$ and $(1+t)^{j+1} \times [\phi_{xtt} \times (3.7)_{xt} + \psi_{xtt} \times (3.8)_{xt}]$ for $j = 1, 2, 3$, and using $E = \phi - \psi$. Combining the results, we can get (3.119). Finally, integrating by parts over $\mathbb{R} \times [0,t]$ the equation $(1+t)^i \times [\phi_{ttt} \times (3.7)_{tt} + \psi_{ttt} \times (3.8)_{tt}]$ for $i = 0, 1, \ldots, 5$, and using $E = \phi - \psi$, we can get (3.120). We omitted the details. Therefore, the proof of Lemma 3.7 is completed. □

Proof of Theorem 1.1. By Theorem 3.5 and (1.22)–(1.23), we know that the classical solutions of IVP (1.1)–(1.5) and(1.14) exist globally in time and are uniformly bounded. By Lemmas 3.7–3.6, we can prove (1.36)–(1.37) for the classical solutions of IVP (1.1)–(1.5) and (1.14). The L^p decay rate can be obtained by applying the method of approximate Green's function [28] and more complicated a-priori estimates on the convection term. We omit the details.

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