Tracking in Matrix Systems

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ABSTRACT

This paper provides conditions, and a sense, in which the matrix of a system of linear differential or linear difference equations can be assumed to be similar to a diagonal matrix.

INTRODUCTION

The primary motivation of this paper came from the classical paper of Simon and Ando [3] which introduced nearly decomposable matrices. They studied the behavior of a solution x_a to an equation $x_a(k+1) = x_a(k)A$ where A was a nearly decomposable matrix, showing that for a short time x_a behaves approximately as if A were completely decomposable and afterward approximately like a reduced system. In the analysis, the authors assumed that A was similar to a diagonal matrix, remarking that "this assumption is not too restrictive as a description of reality" [3, p. 115]. Of course, the analysis of a matrix system is much simpler when the matrix is similar to a diagonal matrix.

Concerning the above matrix system, the following question can be asked. Given $\epsilon > 0$, when there is a matrix $\hat{A}$ arbitrarily close to A, similar to a diagonal matrix, and such that $|x_a(k) - x_{\hat{a}}(k)| < \epsilon$ for all $k \geqslant 0$. When this occurs, the behavior of $x_{\hat{a}}$ is within ϵ of that of x_a, and so, at least for many behavior problems, the analysis can be done on

$$x_{\hat{a}}(k+1) = x_{\hat{a}}(k)\hat{A},$$

a much simpler matrix system.

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The content of this paper provides some answers to the above question. We present the work for the differential matrix system. The corresponding theorems for difference matrix systems, obtained by similar proof techniques, can be found by replacing e^{At} with A^k in the theorems.

RESULTS

In the context of the system $x' = Ax$, an eigenvalue λ of A is called stable if and only if $\operatorname{Re}\lambda < 0$, while for the system $x(k+1) = Ax(k)$, λ is stable if and only if $|\lambda| < 1$. The corresponding system matrix A is stable if and only if all of its eigenvalues are stable.

Given the system $x' = Ax$, we now present a sequence of results which show when there is a system $y' = By$ for which B is similar to a diagonal matrix and arbitrarily close to A, and $|x(t) - y(t)|_2$ is arbitrarily small for all $t \geqslant 0$. Thus, y tracks x, and hence we call these results tracking theorems.

The first such result follows.

LEMMA 1. *Let ϵ be a positive number, and let A be an $n \times n$ stable matrix. Then there is a neighborhood N of A such that if $B \in N$ then*

$$|e^{At} - e^{Bt}|_2 < \epsilon \qquad \textit{for all} \quad t \geqslant 0.$$

Proof. Let ϵ_1 be any positive number. Let λ be an eigenvalue of A for which $\operatorname{Re}\lambda$ is largest. Since $e^{\operatorname{Re}\lambda} < 1$, we can find a norm $\|\cdot\|$ such that $\|e^A\| < 1$ [5, p. 174]. Let γ be a positive number such that $\|e^A\| < \gamma < 1$. Define

$$K' = \{C : C \text{ is an } n \times n \text{ matrix and } \|e^C\| < \gamma\}.$$

Thus, K' is an open set about A. Let $K \subseteq K'$ be an open set about A such that $\overline{K}$, the closure of K, is compact. Define

$$M = \max_{\substack{C \in \overline{K} \\ 0 \leqslant t \leqslant 1}} \|e^{Ct}\|.$$

Let T be a positive number such that for $t \geqslant T$, $M\gamma^{t-1} \leqslant \epsilon_1/2$. Now, for any $t \geqslant T$, write $t = r + s$, where $r \in (0,1)$ and s is an integer. Then, for any

$C \in K$,

$$\begin{aligned}\|e^{Ct}\| &= \|e^{C(r+s)}\| \\ &= \|e^{Cr}e^{Cs}\| \\ &\leqslant \|e^{Cr}\|\,\|e^{Cs}\| \\ &\leqslant M\|e^{C}\|^{s} \\ &\leqslant M\gamma^{s} \\ &\leqslant \epsilon_1/2.\end{aligned}$$

Thus, for $t \geqslant T$ and any $C \in K$,

$$\begin{aligned}\|e^{At}-e^{Ct}\| &\leqslant \|e^{At}\|+\|e^{Ct}\| \\ &\leqslant 2(\epsilon_1/2) \\ &\leqslant \epsilon_1.\end{aligned}$$

We now consider the interval $[0,T]$. Suppose that in each open set about A there is a C such that $\|e^{At}-e^{Ct}\| > \epsilon_1$ for some t in $[0,T]$. Then there is a sequence $C_1, C_2, \dots$ of matrices in K and a sequence $t_1, t_2, \dots$ of numbers in $[0,T]$ such that $\lim_{r\to\infty} C_i = A$ while

$$\|e^{At_i}-e^{C_i t_i}\| > \epsilon_1.$$

Choose any subsequence t_{i_k} of these numbers so that $\lim_{k\to\infty} t_{i_k} = t_0$. Then

$$\lim_{k\to\infty}\left\|\exp(At_{i_k})-\exp(C_{i_k}t_{i_k})\right\| = \|e^{At_0}-e^{At_0}\| = 0.$$

Thus, we have a contradiction, and so there is an open set Θ about A such that if $B \in \Theta$ then

$$\|e^{At}-e^{Bt}\| < \epsilon_1 \qquad \text{for all} \quad t \geqslant 0.$$

Finally, it is known [5, p. 170] that there are positive numbers α and β such that $\alpha|X|_2 \leqslant \|X\| \leqslant \beta|X|_2$ for all $n \times n$ matrices X. Thus, choosing $\epsilon_1 = \epsilon\alpha$ yields the result. ■

For nonstable eigenvalues we have the following

THEOREM 1. *Let ϵ be a positive number, and let A be an $n \times n$ matrix with nonstable eigenvalues having linear elementary divisors. Then there exists a neighborhood N of A such that if $B \in N$ and B has nonstable eigenvalues and corresponding eigenspaces exactly those of A, then $|e^{At} - e^{Bt}|_2 < \epsilon$ for all $t \geq 0$.*

Proof. Let P be a nonsingular matrix such that

$$A = PJP^{-1}, \qquad \text{where} \quad J = \begin{bmatrix} D & 0 \\ 0 & K \end{bmatrix},$$

D a diagonal matrix with main diagonal consisting of the unstable eigenvalues of A. Using Lemma 1, let $\overline{N}$ be a neighborhood of K such that if $R \in \overline{N}$ then

$$|e^{Kt} - e^{Rt}|_2 \leq \left(|P|_2 |P^{-1}|_2\right)^{-1} \epsilon \qquad \text{for all} \quad t \geq 0.$$

Suppose D is an $r \times r$ matrix. Let M_k denote the set of all $k \times k$ matrices. Define $\Pi : M_n \to M_{n-r}$ by $\Pi(X) = Y$, where Y is obtained from $P^{-1}XP$ by deleting the first r rows and the first r columns. Then $\Pi(A) = K$. Since Π is continuous, $N = \Pi^{-1}(\overline{N})$ is neighborhood of A.

Let $B \in N$ be such that the nonstable eigenvalues and corresponding eigenspaces are exactly those of A. Then

$$P^{-1}BP = C = \begin{bmatrix} D' & 0 \\ 0 & \overline{R} \end{bmatrix},$$

where D' is an $r \times r$ diagonal matrix with unstable eigenvalues. Now $\Pi(B) = \overline{R} \in \overline{N}$. Thus

$$|e^{Kt} - e^{\overline{R}t}|_2 \leq \left(|P|_2 |P^{-1}|_2\right)^{-1} \epsilon \qquad \text{for all} \quad t \geq 0.$$

Hence

$$\begin{aligned} |e^{At} - e^{Bt}|_2 &\leq |P|_2 |e^{Jt} - e^{Ct}|_2 |P^{-1}|_2 \\ &\leq |P|_2 |P^{-1}|_2 |e^{Kt} - e^{\overline{R}t}|_2 \\ &\leq |P|_2 |P^{-1}|_2 \left(|P|_2 |P^{-1}|_2\right)^{-1} \epsilon \\ &\leq \epsilon \qquad \text{for all} \quad t \geq 0. \end{aligned}$$

■

Assuming the hypothesis of Theorem 1, to describe the behavior of the system

$$x' = Ax,$$

$$x(0) = x_0$$

we have that there is a matrix B, similar to a diagonal matrix and arbitrarily close to A, such that the solution to

$$y' = By,$$

$$y(0) = x_0$$

satisfies

$$\begin{aligned} |x(t) - y(t)|_2 &= |e^{At}x_0 - e^{Bt}x_0|_2 \\ &\leqslant |e^{At} - e^{Bt}|_2 |x_0|_2 \\ &\leqslant \epsilon |x_0|_2 \qquad \text{for all} \quad t \geqslant 0. \end{aligned}$$

So y describes x to within $\epsilon|x_0|_2$. This, of course, always occurs if A is a stable matrix.

To give another interesting special case of Theorem 1, we need two lemmas. The first lemma, with other versions in [4], uses the notation f for the n-component column vector all of whose entries are $1/\sqrt{n}$.

LEMMA 2. *Let A be an $n \times n$ stochastic matrix. Then there is an $n \times n$ orthogonal matrix Q with first column f, a $1 \times (n-1)$ matrix h, and an $(n-1) \times (n-1)$ matrix H such that*

$$A = Q\begin{bmatrix} 1 & h \\ 0 & H \end{bmatrix} Q^t.$$

Proof. Choose $q_2, \dots, q_n$ n-component column vectors so that $Q = [f\ q_2\ \cdots\ q_n]$ is orthogonal. Then there is a $1 \times (n-1)$ matrix h and an

$(n-1)\times(n-1)$ matrix H such that

$$AQ = Q\begin{bmatrix} 1 & h \\ 0 & H \end{bmatrix}.$$

Thus, $A = Q\begin{bmatrix} 1 & h \\ 0 & H \end{bmatrix}Q^t$. ■

LEMMA 3. *Let* A *be an* $n \times n$ *stochastic matrix. Given any number* $\epsilon > 0$, *there is an* $n \times n$ *stochastic matrix* B, *with distinct eigenvalues, such that* $|A - B|_2 < \epsilon$.

Proof. Let C be any $n \times n$ positive stochastic matrix such that $|A - C|_2 < \epsilon/2$. Now, using Lemma 2, factor

$$C = Q\begin{bmatrix} 1 & h \\ 0 & H \end{bmatrix}Q^t.$$

Choose an $(n-1)\times(n-1)$ matrix G so that G has distinct eigenvalues and so that

$$\left|\begin{bmatrix} 1 & h \\ 0 & H \end{bmatrix} - \begin{bmatrix} 1 & h \\ 0 & G \end{bmatrix}\right|_2 < \frac{\epsilon}{2},$$

and

$$B = Q\begin{bmatrix} 1 & h \\ 0 & G \end{bmatrix}Q^t$$

is positive. Let e be the n-component column vector of 1's, and ϵ_1 the n-component $(0,1)$ column vector with a 1 only in the first position. Then

$$\begin{aligned} Be &= Q\begin{bmatrix} 1 & h \\ 0 & G \end{bmatrix}Q^t e \\ &= Q\begin{bmatrix} 1 & h \\ 0 & G \end{bmatrix}\left(\frac{n}{\sqrt{n}}e_1\right) \\ &= Q\left(\frac{n}{\sqrt{n}}e_1\right) \\ &= e, \end{aligned}$$

and it follows that B is stochastic. Finally, since

$$\begin{aligned}|A-B|_2 &= |A-C+C-B|_2\\ &\leqslant |A-C|_2+|C-B|_2\\ &\leqslant \frac{\epsilon}{2}+\left\|\begin{bmatrix}1 & h\\ 0 & H\end{bmatrix}-\begin{bmatrix}1 & h\\ 0 & G\end{bmatrix}\right\|_2\\ &\leqslant \frac{\epsilon}{2}+\frac{\epsilon}{2}\\ &\leqslant \epsilon,\end{aligned}$$

the result follows. ■

An interesting corollary to Theorem 1, which shows that the Simon-Ando assumption of P being similar to a diagonal matrix is acceptable, follows.

COROLLARY 1. *Let A be an $n\times n$ regular stochastic matrix and ϵ a positive number. Then there is an $n\times n$, regular stochastic matrix B, with distinct eigenvalues, such that $|A^k-B^k|_2<\epsilon$ for $k=1,2,\dots$.*

Proof. Using the difference equation version of Theorem 1, there is a neighborhood N about A such that for any $B\in N$ having 1 as an eigenvalue and having the same eigenspace for 1 as does A,

$$|A^k-B^k|_2<\epsilon \qquad \text{for all} \quad k\geqslant 1.$$

Now, from Lemma 3, such a B can be chosen so that it is stochastic and has distinct eigenvalues. The result follows. ■

A similar result can be given for primitive nonnegative matrices.

We now develop two other types of tracking results. The first such result, a relative tracking result, uses the following definition. Let A be an $n\times n$ matrix. In the context of $x'=Ax$, an eigenvalue λ of A such that $\operatorname{Re}\lambda\geqslant\operatorname{Re}\beta$ for all eigenvalues β of A is called an r-maximal eigenvalue of A. For the system $x(k+1)=Ax(k)$ an eigenvalue λ of A is r-maximal if $|\lambda|\geqslant|\beta|$ for all eigenvalues β of A.

We now develop several other types of tracking results. A result concerning relative tracking follows.

THEOREM 2. *Let ϵ be a positive number, and let A be an $n\times n$ matrix whose r-maximal eigenvalues correspond to linear elementary divisors. Then there exists a neighborhood N of A such that if $B\in N$ and B has r-maximal eigenvalues, and corresponding eigenspaces, exactly the same as those of A, then*

$$\frac{|e^{At}-e^{Bt}|_2}{|e^{At}|_2}<\epsilon \quad \textit{for all} \quad t\geqslant 0.$$

Proof. Let P be a nonsingular matrix such that

$$A=PJP^{-1}, \qquad \text{where} \quad J=\begin{bmatrix} D & 0\\ 0 & K\end{bmatrix}$$

with D a diagonal matrix having as main diagonal the r-maximal eigenvalues of A. Let λ be such an eigenvalue.

Using Lemma 1, let $\overline{N}$ be a neighborhood of $K-\lambda I$ such that if $R-\lambda I\in\overline{N}$ then

$$|e^{(K-\lambda I)t}-e^{(R-\lambda I)t}|_2\leqslant\left(|P|_2|P^{-1}|_2\right)^{-1}\epsilon \qquad \text{for all} \quad t\geqslant 0.$$

Define N as in Theorem 1. Then if $B\in N$ and B has r-maximal eigenvalues and corresponding eigenspaces to those of A, we can write

$$B=P\begin{bmatrix} D & 0\\ 0 & \overline{R}\end{bmatrix}P^{-1}.$$

Then

$$\frac{|e^{At}-e^{Bt}|_2}{|e^{At}|_2}\leqslant\frac{|P|_2|e^{Kt}-e^{\overline{R}t}|_2|P^{-1}|_2}{|e^{At}|_2}$$

$$\leqslant|P|_2|P^{-1}|_2\frac{|e^{(K-\lambda I)t}-e^{(\overline{R}-\lambda I)t}|_2}{|e^{-\lambda t}e^{At}|_2}$$

and, since 1 is an eigenvalue of $e^{-\lambda t}e^{At}$, $1 \leqslant |e^{-\lambda t}e^{At}|_2$. Thus,

$$\frac{|e^{At} - e^{Bt}|_2}{|e^{At}|_2} \leqslant |P|_2|P^{-1}|_2|e^{(K-\lambda I)t} - e^{(\bar{R}-\lambda I)t}|_2$$

$$\leqslant |P|_2|P^{-1}|_2\left(|P|_2|P^{-1}|_2\right)^{-1}\epsilon$$

$$\leqslant \epsilon.$$ ■

To convert this result into one for a solution to a system of linear differential equations requires a bit of work.

COROLLARY 2. *Let ϵ be a positive number, and let A be an $n \times n$ matrix whose r-maximal eigenvalues, denoted by $\lambda_1,\dots,\lambda_s$, correspond to linear elementary divisors. Let $p^1,\dots,p^s$ be eigenvectors corresponding to $\lambda_1,\dots,\lambda_s$ respectively, and let $P = [p^1 \ \cdots \ p^n]$ be a nonsingular matrix for which $P^{-1}AP$ is in Jordan form.*

Let x be a vector such that if $x = \alpha_1 p^1 + \cdots + \alpha_n p^n$ for scalars $\alpha_1,\dots,\alpha_n$, then at least one of the $\alpha_1,\dots,\alpha_s$ is not 0. Then there exists an $n \times n$ matrix B, which is similar to a diagonal matrix and arbitrarily close to A, such that

$$\frac{|e^{At}x - e^{Bt}x|_2}{|e^{At}x|_2} < \epsilon \qquad \textit{for all} \quad t \geqslant 0.$$

Proof. Let $\bar{A} = A - \lambda_1 I$. We first show that there is a positive number α such that $|e^{\bar{A}t}x| \geqslant \alpha$ for all $t \geqslant 0$. We argue this by contradiction.

Suppose there is a sequence $t_1, t_2,\dots$ for nonnegative numbers such that $|e^{\bar{A}t_i}x|$ tends to 0. Since $e^{\bar{A}t}$ is bounded, the sequence has a limit point E and so $|Ex| = 0$. Note that there are exactly s eigenvalues of E having modulus 1, and these correspond to the eigenvectors $p^1,\dots,p^s$. But this means that $|Ex| \neq 0$, a contradiction.

Now, choose $\bar{B}$ as in Theorem 1 so that

$$|e^{\bar{A}t}x - e^{\bar{B}t}x|_2 \leqslant \alpha\epsilon \qquad \text{for all} \quad t \geqslant 0.$$

Set $B = \bar{B} + \lambda_1 I$. Then

$$\frac{|e^{At}x - e^{Bt}x|_2}{|e^{At}x|_2} = \frac{|e^{\bar{A}t}x - e^{\bar{B}t}x|_2}{|e^{\bar{A}t}x|_2} \leqslant \frac{\alpha\epsilon}{\alpha} = \epsilon,$$

which yields the corollary. ∎

The second result, a projected tracking result, necessitates a lemma.

LEMMA 4. *Let K be an $m \times m$ triangular matrix with*

(i) *main diagonal $\lambda_1, \dots, \lambda_m$, where*

$$\operatorname{Re}\lambda_i \geqslant \operatorname{Re}\lambda_j \qquad \textit{for all} \quad i \leqslant j,$$

(ii) *superdiagonal composed of 0's and 1's, and*
(iii) *all other entries* 0.

Then

$$|e^{Kt}|_1 \leqslant e^{(\operatorname{Re}\lambda_1)t}(t^m + \cdots + 1) \qquad \textit{for all} \quad t \geqslant 0.$$

Proof. Let $\beta_1, \dots, \beta_{r+1}$ be numbers such that $\operatorname{Re}\beta_1 \geqslant \cdots \geqslant \operatorname{Re}\beta_{r+1}$. Consider the differential equations

$$z_1'(t) = \beta_1 z_1(t) + z_2(t),$$

$$z_2'(t) = \beta_2 z_2(t) + z_3(t),$$

$$\vdots$$

$$z_r'(t) = \beta_r z_r(t) + z_{r+1}(t),$$

$$z_{r+1}'(t) = \beta_{r+1} z_{r+1}(t)$$

with $z_1(0) = \alpha_1, \dots, z_{r+1}(0) = \alpha_{r+1}$ and $|\alpha_i| \leqslant 1$ for all i.

Solving these equations yields that

$$z_{r+1}(t) = e^{\beta_{r+1}t}\alpha_{r_1}$$

and hence

$$|z_{r+1}(t)|_1 \leqslant e^{\operatorname{Re}\beta_r t}.$$

Continuing,

$$z_r(t) = e^{\beta_r t}\left[\int_0^t e^{-\beta_r t} z_{r+1}(t)\,dt + \alpha_r\right]$$

and

$$|z_r(t)| \leqslant e^{(\operatorname{Re}\beta_r)t}\left[\int_0^t e^{-(\operatorname{Re}\beta_r)t} e^{(\operatorname{Re}\beta_r)t}\,dt + |\alpha_r|\right]$$
$$\leqslant e^{(\operatorname{Re}\beta_{r-1})t}(t+1).$$

By further continuing this technique we have

$$|z_1(t)| \leqslant e^{(\operatorname{Re}\beta_1)t}(t^r + \cdots + 1).$$

This leads to

$$|z_i(t)| \leqslant e^{(\operatorname{Re}\beta_1)t}(t^r + \cdots + 1),$$

for all i. Applying this result to

$$y'(t) = Ky(t),$$

where $|y_i(0)| \leqslant 1$ for all i, yields that

$$|y_i(t)| \leqslant e^{(\operatorname{Re}\lambda_1)t}(t^m + \cdots + 1).$$

Now, let e_i denote the m-component $(0,1)$ vector with a 1 only in the ith position. Then

$$y'(t) = Ky(t),$$

$$y(0) = e_i$$

has solution $e^{Kt}e_i$ for all i. Thus

$$|e^{Kt}|_1 = \max_i |e^{Kt}e_i|_1 \leqslant e^{(\mathrm{Re}\,\lambda_1)t}(t^m + \cdots + 1). \qquad \blacksquare$$

We now develop a projected tracking result.

LEMMA 3. *Let A be an $n \times n$ matrix such that all r-maximal eigenvalues of A have linear elementary divisors. Then, given any positive number ϵ, there is a neighborhood N of A such that if $B \in N$ and B has r-maximal eigenvalues and corresponding eigenspaces, exactly those of A, then*

$$\left| \frac{e^{At}}{|e^{At}|_1} - \frac{e^{Bt}}{|e^{Bt}|_1} \right|_1 < \epsilon \qquad \textit{for all} \quad t \geqslant 0.$$

Proof. It is sufficient to prove this result in the norm defined by

$$|C|_P = |P^{-1}CP|_1,$$

where

$$P^{-1}AP = \begin{bmatrix} D & 0 \\ 0 & J \end{bmatrix},$$

where D is an $r \times r$ diagonal matrix containing the r-maximal eigenvalues of A on its main diagonal and J a Jordan submatrix. Let ϵ be a positive number. Let λ be an r-maximal eigenvalue of A. Set $\bar{A} = A - \lambda I$. Then $e^{\bar{A}t} = e^{-\lambda t}e^{At}$.

Define N as in Theorem 1, and such that, if $B \in N$ and the r-maximal eigenvalues and corresponding eigenspaces are exactly as those of A, and if $\bar{B} = B - \lambda I$, then $|e^{\bar{A}t} - e^{\bar{B}t}|_P < \epsilon$ for all $t \geqslant 0$. Now,

$$\left| \frac{e^{At}}{|e^{At}|_1} - \frac{e^{Bt}}{|e^{Bt}|_1} \right|_P = \left| \frac{e^{\bar{A}t}}{|e^{\bar{A}t}|_1} - \frac{e^{\bar{B}t}}{|e^{\bar{B}t}|_1} \right|_P,$$

and, by using Lemma 4, there is a positive number M such that $|e^{\bar{A}t}|_1 = |e^{\bar{B}t}|_1 = 1$ for all $t \geqslant M$. Thus,

$$\left| \frac{e^{\bar{A}t}}{|e^{At}|_1} - \frac{e^{\bar{B}t}}{|e^{\bar{B}t}|} \right|_P < \epsilon \qquad \text{for all} \quad t \geqslant M.$$

Now suppose that there is a sequence of numbers $t_1, t_2, \ldots$ in $[0, M]$ and a sequence of matrices $B_1, B_2, \ldots$ satisfying the same properties as B, so that $\lim_{i\to\infty} B_i = A$ while

$$\left| \frac{e^{At_i}}{|e^{At_i}|_1} - \frac{e^{B_i t_i}}{|e^{B_i t_i}|_1} \right|_P > \epsilon \qquad \text{for all } i.$$

Without loss of generality we can assume $\lim_{i\to\infty} t_i = t'$. Then, taking limits, we have

$$\left| \frac{e^{At'}}{|e^{At'}|_1} - \frac{e^{At'}}{|e^{At'}|_1} \right|_P > \epsilon,$$

a contradiction. Thus, there is a neighborhood N' of A such that if $B \in N'$ and B has r-maximal eigenvalues and corresponding eigenspaces exactly the same as those of A, then

$$\left| \frac{e^{At}}{|e^{At}|_1} - \frac{e^{Bt}}{|e^{Bt}|_1} \right|_P < \epsilon \qquad \text{for all} \quad t \geqslant 0. \qquad \blacksquare$$

Converting this theorem to a solution result yields the following.

COROLLARY 3. *Let A be a $n \times n$ matrix such that all r-maximal eigenvalues, denoted by $\lambda_1, \ldots, \lambda_s$, of A have linear elementary divisors. Let $p^1, \ldots, p^s$ be eigenvectors corresponding to $\lambda_1, \ldots, \lambda_s$ respectively, and let $P = [p^1 \ \cdots \ p^n]$ be a nonsingular matrix for which $P^{-1}AP$ is in Jordan form.*

Let x be a vector such that if $x = \alpha_1 p^1 + \cdots + \alpha_n p^n$ for scalars $\alpha_1, \ldots, \alpha_n$, then one of $\alpha_1, \ldots, \alpha_s$ is not 0. Then, given any positive number ϵ, there is an $n \times n$ matrix B, similar to a diagonal matrix and arbitrarily close to A, such that

$$\left| \frac{e^{At}x}{|e^{At}x|_1} - \frac{e^{Bt}x}{|e^{Bt}x|_1} \right|_1 < \epsilon \qquad \textit{for all} \quad t \geqslant 0.$$

Proof. Let $\bar{A} = A - \lambda_1 I$. By the proof of Corollary 2, there is a positive number α such that $\alpha < |e^{\bar{A}t}x|_1$ for all $t \geqslant 0$. Let $\alpha_0 = \frac{1}{2}\alpha$. We first show that there are $n \times n$ matrices $\bar{B}$, with r-maximal eigenvalues and corresponding eigenvectors as those of $\bar{A}$, similar to diagonal matrices, and sufficiently close to $\bar{A}$ that $\alpha_0 \leqslant |e^{\bar{B}t}x|_1$ for all $t \geqslant 0$. We argue this by contradiction.

Write

$$\bar{A} = P^{-1}\begin{bmatrix} D & O \\ O & J \end{bmatrix} P,$$

where D is an $r \times r$ diagonal matrix containing the r-maximal eigenvalues of $\bar{A}$, and J a Jordan submatrix. Let $K_i = J + E_i$, where E_i is a diagonal matrix with sufficiently small entries and such that K_i has distinct eigenvalues and $\lim_{i \to \infty} E_i = 0$. Set

$$\bar{B}_i = P^{-1}\begin{bmatrix} D & O \\ O & K_i \end{bmatrix} P.$$

Now, suppose there is a sequence of nonnegative numbers $t_1, t_2, \ldots$ such that $|e^{\bar{B}_i t_i}x|_1 < \alpha_0$.

We now argue cases.

Case 1: $t_1, t_2, \ldots$ *is a bounded sequence.* For this case we can assume $\lim_{i \to \infty} t_i = \bar{t}$. Then

$$\lim_{i \to \infty} |e^{\bar{B}_i t_i}x|_1 = |e^{\bar{A}\bar{t}}x|_1 > \alpha > \alpha_0.$$

This yields a contradiction.

Case 2: $t_1, t_2, \ldots$ *is an unbounded sequence.* Here, we need only argue the case where $\lim_{i \to \infty} t_i = \infty$. In this case, by inspecting the form of $e^{\bar{B}_i t_i}$ and applying Lemma 4, we see that $e^{\bar{B}_i t_i}$ is arbitrarily close to $e^{\bar{A}t_i}$ for sufficiently large t_i. Thus $|e^{\bar{B}t_i}|_1 > \alpha_0$ for sufficiently large t_i, a contradiction.

Similarly, a $\mathscr{B}_0$ can be found such that any $n \times n$ matrix $\bar{B}$ having r-maximal eigenvalues and corresponding eigenvectors to those of $\bar{A}$, and sufficiently close to $\bar{A}$, satisfies $|e^{\bar{B}t}x|_1 \leqslant \mathscr{B}_0$ for all $t \geqslant 0$.

Now, using Theorem 1, an $n \times n$ matrix $\bar{B}$ can be chosen, satisfying the hypothesis of this corollary, and such that

$$|e^{\bar{A}t}x - e^{\bar{B}t}x|_1 < \frac{\alpha_0^2}{2\mathscr{B}_0}\epsilon.$$

Define $B = \bar{B} + \lambda_1 I$. Then

$$\left|\frac{e^{At}x}{|e^{At}x|_1} - \frac{e^{Bt}x}{|e^{Bt}x|_1}\right|_1 = \left|\frac{e^{\bar{A}t}x}{|e^{\bar{A}t}x|_1} - \frac{e^{\bar{B}t}x}{|e^{\bar{B}t}x|_1}\right|_1$$

$$= \left|\frac{e^{\bar{B}t}x|_1 e^{\bar{A}t}x - |e^{\bar{A}t}x|_1 e^{\bar{B}t}x}{|e^{\bar{A}t}x|_1 |e^{\bar{B}t}x|_1}\right|_1$$

$$= \left|\frac{|e^{\bar{B}t}x|_1 e^{\bar{A}t}x - |e^{\bar{B}t}x|_1 e^{\bar{B}t}x + |e^{\bar{B}t}x|_1 e^{\bar{B}t}x - |e^{\bar{A}t}x|_1 e^{\bar{B}t}x}{|e^{\bar{A}t}x|_1 |e^{\bar{B}t}x|_1}\right|_1$$

$$\leqslant \frac{|e^{\bar{B}t}x|_1 |e^{\bar{A}t}x - e^{\bar{B}t}x|_1 + |e^{\bar{B}t}x|_1 - |e^{\bar{A}t}x|_1 |e^{\bar{B}t}x|_1}{\alpha_0^2}$$

$$\leqslant \frac{\mathscr{B}_0\left(\dfrac{\alpha_0^2}{2\mathscr{B}_2}\epsilon\right) + \left(\dfrac{\alpha_0^2}{2\mathscr{B}_0}\epsilon\right)\mathscr{B}_0}{\alpha_0^2} = \epsilon,$$

which yields the corollary. ∎

These results cover all cases of A except the case in which an eigenvalue having maximum real part has an corresponding nonlinear elementary divisor. As shown in the example below, this case has no diagonal tracking result.

EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

If B is any matrix similar to a diagonal matrix, then e^{Bt} can be written as

$$e^{Bt} = e^{\lambda t}\Pi_1 + e^{\beta t}\Pi_2$$

where λ and β are eigenvalues of B and Π_1 and Π_2 are idempotent matrices. We look at three cases:

Case 1: Consider

$$|e^{At} - e^{Bt}|_2 = \left|\begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} - e^{\lambda t}\Pi_1 - e^{\beta t}\Pi_2\right|_2.$$

Without loss of generality we can assume that $\operatorname{Re}\lambda \geqslant \operatorname{Re}\beta$. We look at cases.

(i) Suppose $\operatorname{Re}\lambda \leqslant 1$. In this case the te^t entry in A grows faster than the corresponding entry in e^{Bt}.

(ii) Suppose $\operatorname{Re}\lambda > 1$. In this case $e^{\lambda t}$ grows faster than the entries in e^{At}. Thus, in any case e^{At} cannot be approximated by e^{Bt}.

Case 2: Consider

$$\frac{\left\| \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} - e^{\lambda t}\Pi_1 - e^{\beta t}\Pi_2 \right\|_2}{\left\| \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \right\|_2} = \frac{\left\| \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} - e^{(\lambda-1)t}\Pi_1 - e^{(\beta-1)t}\Pi_2 \right\|_2}{\left\| \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\|_2}.$$

Analyzing as before, it can be seen that e^{At} cannot be approximated by e^{Bt}.

Case 3: Consider

$$\left\| \frac{\begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}}{\left\| \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \right\|_1} - \frac{e^{\lambda t}\Pi_1 + e^{\beta t}\Pi_2}{|e^{\lambda t}\Pi_1 + e^{\beta t}\Pi_2|_1} \right\|_1.$$

Note that

$$\lim_{t\to\infty} \frac{\begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}}{\left\| \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \right\|_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now $e^{\lambda t}\Pi_1 + e^{\beta t}\Pi_2$ can be written as

$$P\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\beta t} \end{bmatrix}P^{-1}$$

for some nonsingular matrix P. Without loss of generality, suppose $\operatorname{Re}\lambda \geqslant$

$\operatorname{Re}\beta$. Then we can write

$$\alpha(t)=\frac{P\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\beta t}\end{bmatrix}P^{-1}}{\left|P\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\beta t}\end{bmatrix}P^{-1}\right|_1}$$

$$=\frac{P\begin{bmatrix} 1 & 0 \\ 0 & e^{(\beta-\lambda)t}\end{bmatrix}tP^{-1}}{\left|P\begin{bmatrix} 1 & 0 \\ 0 & e^{(\beta-\lambda)t}\end{bmatrix}P^{-1}\right|_1}.$$

Suppose $\operatorname{Re}\lambda > \operatorname{Re}\beta$. Then

$$\lim_{t\to\infty}\alpha(t)=\frac{P\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}P^{-1}}{\left|P\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}P^{-1}\right|_1}.$$

Note that $\operatorname{trace}\lim_{t\to\infty}\alpha(t)=1$; however, the main diagonal of $\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$ consists of 0's. Thus,

$$\left|\frac{\begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t}\end{bmatrix}}{\left\|\begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t}\end{bmatrix}\right\|_1}-\frac{e^{\lambda t}\Pi_1+e^{\beta t}\Pi_2}{|e^{\lambda t}\Pi_1+e^{\beta t}\Pi_2|_1}\right|_1<\epsilon$$

is false. Now suppose $\operatorname{Re}\lambda=\operatorname{Re}\beta$. Then

$$\alpha(t)=\frac{P\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta t}\end{bmatrix}P^{-1}}{\left|P\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta t}\end{bmatrix}P^{-1}\right|_2},$$

where θ is real. Then trace $\alpha(t) = 1 + \epsilon^{i\theta t}$. Now, regardless of θ, since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a main diagonal of 0's, $\alpha(t)$ cannot approximate $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for t large. ■

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