The variational iteration method for solving linear and nonlinear systems of PDEs

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Abstract

In this work, the variational iteration method (VIM) is used for analytic treatment of the linear and nonlinear systems of partial differential equations. The method reduces the calculation size and overcomes the difficulty of handling nonlinear terms. Numerical examples are examined to highlight the significant features of the VIM method. The method shows improvements over existing numerical techniques.

Keywords: Variational iteration method; Systems of PDEs

1. Introduction

In 1999, the variational iteration method (VIM) was proposed by He in [1–10]. This method is now widely used by many researchers to study linear and nonlinear partial differential equations. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications, linear or nonlinear, homogeneous or inhomogeneous, equations and systems of equations as well. It was shown by many authors [11–22] that this method is more powerful than existing techniques such as the Adomian method [23,24], perturbation method, etc. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The existing numerical techniques suffer from the restrictive assumptions that are used to handle nonlinear terms. The VIM has no specific requirements, such as linearization, small parameters, Adomian polynomials, etc. for nonlinear operators. Another important advantage is that the VIM method is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications.

A substantial amount of research work has been invested in the study of linear and nonlinear systems of partial differential equations (PDEs). Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of the chemical reaction–diffusion model. To
achieve our goal in handling systems of nonlinear partial differential equations, we write a system in an operator form as

\[
\begin{align*}
L_t u + R_1(u, v, w) + N_1(u, v, w) &= g_1, \\
L_t v + R_2(u, v, w) + N_2(u, v, w) &= g_2, \\
L_t w + R_3(u, v, w) + N_2(u, v, w) &= g_3,
\end{align*}
\]

with initial data

\[
\begin{align*}
u(x, 0) &= f_1(x), \\
v(x, 0) &= f_2(x), \\
w(x, 0) &= f_3(x),
\end{align*}
\]

where \( L_t \) is considered a first-order partial differential operator, \( R_j, 1 \leq j \leq 3, \) and \( N_j, 1 \leq j \leq 3, \) are linear and nonlinear operators respectively, and \( g_1, g_2 \) and \( g_3 \) are source terms. In what follows we give the main steps of He’s variational iteration method in handling scientific and engineering problems.

2. He’s variational iteration method

In [1–10], He proposed the variational iteration method where correction functionals for equations of the system (1) can be written as

\[
\begin{align*}
u_{n+1}(x, t) &= u_n(x, t) + \int_0^t \lambda_1 (Lu_n(\xi) + R_1(\bar{u}_n, \bar{v}_n, \bar{w}_n) + N_1(\bar{u}_n, \bar{v}_n, \bar{w}_n) - g_1(\xi)) \, d\xi, \\
v_{n+1}(x, t) &= v_n(x, t) + \int_0^t \lambda_2 (Lv_n(\xi) + R_2(\bar{u}_n, \bar{v}_n, \bar{w}_n) + N_2(\bar{u}_n, \bar{v}_n, \bar{w}_n) - g_2(\xi)) \, d\xi, \\
w_{n+1}(x, t) &= w_n(x, t) + \int_0^t \lambda_3 (Lw_n(\xi) + R_3(\bar{u}_n, \bar{v}_n, \bar{w}_n) + N_3(\bar{u}_n, \bar{v}_n, \bar{w}_n) - g_3(\xi)) \, d\xi,
\end{align*}
\]

where \( \lambda_j, 1 \leq j \leq 3, \) are general Lagrange multipliers, which can be identified optimally via the variational theory, and \( \bar{u}_n, \bar{v}_n, \) and \( \bar{w}_n \) are restricted variations which means \( \delta \bar{u}_n = 0, \delta \bar{v}_n = 0 \) and \( \delta \bar{w}_n = 0. \) It is required first to determine the Lagrange multipliers \( \lambda_j \) that will be identified optimally via integration by parts. The successive approximations \( u_{n+1}(x, t), v_{n+1}(x, t), w_{n+1}(x, t), n \geq 0, \) of the solutions \( u(x, t), v(x, t) \) and \( w(x, t) \) will follow immediately upon using the Lagrange multipliers obtained and by using selected functions \( u_0, v_0, \) and \( w_0. \) The initial values are usually used for the selected zeroth approximations. With the Lagrange multipliers \( \lambda_j \) determined, then several approximations \( u_j(x, t), v_j(x, t), w_j(x, t), j \geq 0, \) can be determined. Consequently, the solutions are given by

\[
\begin{align*}
u(x, t) &= \lim_{n \to \infty} u_n(x, t), \\
v(x, t) &= \lim_{n \to \infty} v_n(x, t), \\
w(x, t) &= \lim_{n \to \infty} w_n(x, t).
\end{align*}
\]

To give a clear overview of the analysis introduced above, four illustrative systems of partial differential equations, two linear and two nonlinear, have been selected to demonstrate the efficiency of the method.

3. The homogeneous linear system

We first consider the homogeneous linear system

\[
\begin{align*}
u_t - v_x + (u + v) &= 0, \\
v_t - u_x + (u + v) &= 0, \\
I.C \, u(x, 0) &= \sinh x, \quad v(x, 0) &= \cosh x.
\end{align*}
\]
The correction functionals for (5) read

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(\xi) \left( \frac{\partial u_n(x, \xi)}{\partial \xi} - \frac{\partial v_n(x, \xi)}{\partial x} + \bar{u}_n(x, \xi) + \bar{v}_n(x, \xi) \right) d\xi, \]

\[ v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(\xi) \left( \frac{\partial v_n(x, \xi)}{\partial \xi} - \frac{\partial u_n(x, \xi)}{\partial x} + \bar{u}_n(x, \xi) + \bar{v}_n(x, \xi) \right) d\xi, \quad n \geq 0. \tag{6} \]

This yields the stationary conditions

\[ 1 + \lambda_1 = 0, \quad \lambda_1'(\xi) = t = 0, \]

\[ 1 + \lambda_2 = 0, \quad \lambda_2'(\xi) = t = 0. \tag{7} \]

As a result we find

\[ \lambda_1 = \lambda_2 = -1. \tag{8} \]

Substituting these values of the Lagrange multipliers into the functionals (6) gives the iteration formulas

\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left( \frac{\partial u_n(x, \xi)}{\partial \xi} - \frac{\partial v_n(x, \xi)}{\partial x} + \bar{u}_n(x, \xi) + \bar{v}_n(x, \xi) \right) d\xi, \]

\[ v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left( \frac{\partial v_n(x, \xi)}{\partial \xi} - \frac{\partial u_n(x, \xi)}{\partial x} + \bar{u}_n(x, \xi) + \bar{v}_n(x, \xi) \right) d\xi, \quad n \geq 0. \tag{9} \]

We can select \( u_0(x, t) = \sinh x, \) \( v_0(x, t) = \cosh x \) by using the given initial values. Accordingly, we obtain the following successive approximations:

\[ u_0(x, t) = \sinh x, \]

\[ v_0(x, t) = \cosh x, \]

\[ u_1(x, t) = \sinh x - t \cosh x, \]

\[ v_1(x, t) = \cosh x - t \sinh x, \]

\[ u_2(x, t) = \sinh x - t \cosh x - \frac{t^2}{2!} \sinh x, \]

\[ v_2(x, t) = \cosh x - t \sinh x - \frac{t^2}{2!} \cosh x, \]

\[ u_3(x, t) = \sinh x - t \cosh x - \frac{t^2}{2!} \sinh x + \frac{t^3}{3!} \cosh x, \]

\[ v_3(x, t) = \cosh x - t \sinh x - \frac{t^2}{2!} \cosh x + \frac{t^3}{3!} \sinh x, \]

\[ \vdots, \]

\[ u_n(x, t) = \sinh x \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots \right) - \cosh x \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} \cdots \right), \]

\[ v_n(x, t) = \cosh x \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots \right) - \sinh x \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} \cdots \right). \tag{10} \]

Recall that

\[ u(x, t) = \lim_{n \to \infty} u_n(x, t), \]

\[ v(x, t) = \lim_{n \to \infty} v_n(x, t). \tag{11} \]
Consequently, the exact solutions

\[ u(x, t) = \sinh(x - t), \]
\[ v(x, t) = \cosh(x - t), \]

follow immediately upon using the Taylor expansion for \( \sinh t \) and \( \cosh t \).

4. The inhomogeneous linear system

We next consider the inhomogeneous linear system

\[ u_t - v_x - (u - v) = -2, \]
\[ v_t + u_x - (u - v) = -2, \]
\[ \text{I.C } u(x, 0) = 1 + e^x, \quad v(x, 0) = -1 + e^x. \]

The correction functionals for (13) read

\[
\begin{align*}
  u_{n+1}(x, t) &= u_n(x, t) + \int_0^t \lambda_1(\xi) \left( \frac{\partial u_n(x, \xi)}{\partial \xi} - \frac{\partial v_n(x, \xi)}{\partial x} - \tilde{u}_n(x, \xi) + \tilde{v}_n(x, \xi) + 2 \right) \, d\xi, \\
  v_{n+1}(x, t) &= v_n(x, t) + \int_0^t \lambda_2(\xi) \left( \frac{\partial v_n(x, \xi)}{\partial \xi} + \frac{\partial u_n(x, \xi)}{\partial x} - \tilde{u}_n(x, \xi) + \tilde{v}_n(x, \xi) + 2 \right) \, d\xi.
\end{align*}
\]

The stationary conditions are given by

\[
\begin{align*}
  1 + \lambda_1 &= 0, \quad \lambda_1(\xi = t) = 0, \\
  1 + \lambda_2 &= 0, \quad \lambda_2(\xi = t) = 0,
\end{align*}
\]

so that

\[ \lambda_1 = \lambda_2 = -1. \]

Substituting these values of the Lagrange multipliers into the functionals (14) gives the iteration formulas

\[
\begin{align*}
  u_{n+1}(x, t) &= u_n(x, t) - \int_0^t \left( \frac{\partial u_n(x, \xi)}{\partial \xi} - \frac{\partial v_n(x, \xi)}{\partial x} - \tilde{u}_n(x, \xi) + \tilde{v}_n(x, \xi) + 2 \right) \, d\xi, \\
  v_{n+1}(x, t) &= v_n(x, t) - \int_0^t \left( \frac{\partial v_n(x, \xi)}{\partial \xi} + \frac{\partial u_n(x, \xi)}{\partial x} - \tilde{u}_n(x, \xi) + \tilde{v}_n(x, \xi) + 2 \right) \, d\xi, \quad n \geq 0.
\end{align*}
\]

The zeroth approximations \( u_0(x, t) = 1 + e^x \), \( v_0(x, t) = -1 + e^x \) are selected by using the given initial values. Accordingly, we obtain the following successive approximations:

\[
\begin{align*}
  u_0(x, t) &= 1 + e^x, \\
  v_0(x, t) &= -1 + e^x, \\
  u_1(x, t) &= 1 + e^x + te^x, \\
  v_1(x, t) &= -1 + e^x - te^x, \\
  u_2(x, t) &= 1 + te^x + \frac{t^2}{2!} e^x, \\
  v_2(x, t) &= -1 + e^x - te^x + \frac{t^2}{2!} e^x, \\
  u_3(x, t) &= 1 + e^x + te^x + \frac{t^2}{2!} e^x + \frac{t^3}{3!} e^x, \\
  v_3(x, t) &= -1 + e^x - te^x + \frac{t^2}{2!} e^x - \frac{t^3}{3!} e^x,
\end{align*}
\]
\[ u_n(x, t) = 1 + e^x \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots \right), \]
\[ v_n(x, t) = -1 + e^x \left( 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \cdots \right). \]

Proceeding as before, the exact solutions
\[ u(x, t) = 1 + e^{x+t}, \]
\[ v(x, t) = -1 + e^{x-t}, \]
are readily obtained upon using the Taylor expansion for \( e^t \) and \( e^{-t} \).

5. The inhomogeneous nonlinear system

We now consider the inhomogeneous nonlinear system
\[ u_t + v u_x + u = 1, \]
\[ v_t - u v_x - v = 1, \]
I.C \( u(x, 0) = e^x, \quad v(x, 0) = e^{-x}. \)

The correction functionals for (20) read
\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(\xi) \left( \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{v}_n(x, \xi) \frac{\partial u_n(x, \xi)}{\partial x} + \tilde{u}_n(x, \xi) - 1 \right) \, d\xi, \]
\[ v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(\xi) \left( \frac{\partial v_n(x, \xi)}{\partial \xi} - \tilde{u}_n(x, \xi) \frac{\partial v_n(x, \xi)}{\partial x} - \tilde{v}_n(x, \xi) - 1 \right) \, d\xi. \]

The stationary conditions are given by
\[ 1 + \lambda_1 = 0, \quad \lambda_1'(\xi = t) = 0, \]
\[ 1 + \lambda_2 = 0, \quad \lambda_2'(\xi = t) = 0, \]
so that
\[ \lambda_1 = \lambda_2 = -1. \]

Substituting these values of the Lagrange multipliers into the functionals (21) gives the iteration formulas
\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left( \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{v}_n(x, \xi) \frac{\partial u_n(x, \xi)}{\partial x} + \tilde{u}_n(x, \xi) - 1 \right) \, d\xi, \]
\[ v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left( \frac{\partial v_n(x, \xi)}{\partial \xi} - \tilde{u}_n(x, \xi) \frac{\partial v_n(x, \xi)}{\partial x} - \tilde{v}_n(x, \xi) - 1 \right) \, d\xi, \quad n \geq 0. \]

The zeroth approximations \( u_0(x, t) = e^x \), and \( v_0(x, t) = e^{-x} \) are selected by using the given initial conditions. The following successive approximations:
\[ u_0(x, t) = e^x, \]
\[ v_0(x, t) = e^{-x}, \]
\[ u_1(x, t) = e^x - te^x, \]
\[ v_1(x, t) = e^{-x} + te^{-x}, \]
The stationary conditions are thus given by

\( u_2(x, t) = e^x - te^x + \frac{t^2}{2!} e^x, \)
\( v_2(x, t) = e^{-x} + te^{-x} + \frac{t^2}{2!} e^{-x}, \)
\( u_3(x, t) = e^x - te^x + \frac{t^2}{2!} e^x - \frac{t^3}{3!} e^x, \)
\( v_3(x, t) = e^{-x} + te^{-x} + \frac{t^2}{2!} e^{-x} + \frac{t^3}{3!} e^{-x}, \)
\vdots
\( u_n(x, t) = e^x \left( 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots \right), \)
\( v_n(x, t) = e^{-x} \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right), \)

follow immediately. This gives the exact solutions

\( u(x, t) = e^{x-\xi}, \)
\( v(x, t) = e^{-x+\xi}, \)

obtained upon using the Taylor expansion for \( e^{-\xi} \) and \( e^\xi \). It is obvious that we did not use any transformation formulas or linearization assumptions for handling the nonlinear terms. The approach is followed in a parallel manner to the analysis conducted for the previous two linear systems.

In what follows, a system of three nonlinear partial differential equations in three unknown functions \( u(x, y, t), v(x, y, t) \) and \( w(x, y, t) \) will be studied. It is worth noting that handling this system by traditional methods is quite complicated.

6. The homogeneous nonlinear system

We finally examine the homogeneous nonlinear system [17]

\[ \begin{align*}
    u_t + u_x v_x + u_y v_y + u &= 0, \\
    v_t + v_x w_x - v_y w_y - v &= 0, \\
    w_t + w_x u_x + w_y u_y - w &= 0, \\
    \text{I.C} u(x, y, 0) = e^{x+y}, \quad v(x, y, 0) = e^{x-y}, \quad w(x, y, 0) = e^{-x+y}.
\end{align*} \]

The correction functionals for (27) read

\[ \begin{align*}
    u_{n+1}(x, y, t) &= u_n(x, y, t) + \int_0^t \lambda_1(\xi) \left( \frac{\partial u_n}{\partial \xi} + \frac{\partial u_n}{\partial x} \frac{\partial v_n}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial v_n}{\partial y} + \tilde{u}_n \right) d\xi, \\
    v_{n+1}(x, y, t) &= v_n(x, y, t) + \int_0^t \lambda_2(\xi) \left( \frac{\partial v_n}{\partial \xi} + \frac{\partial v_n}{\partial x} \frac{\partial w_n}{\partial x} - \frac{\partial v_n}{\partial y} \frac{\partial w_n}{\partial y} - \tilde{v}_n \right) d\xi, \\
    w_{n+1}(x, y, t) &= w_n(x, y, t) + \int_0^t \lambda_3(\xi) \left( \frac{\partial w_n}{\partial \xi} + \frac{\partial w_n}{\partial x} \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} \frac{\partial u_n}{\partial y} - \tilde{w}_n \right) d\xi.
\end{align*} \]

The stationary conditions are thus given by

\[ \begin{align*}
    1 + \lambda_1 = 0, \quad \lambda_1(\xi = t) = 0, \\
    1 + \lambda_2 = 0, \quad \lambda_2(\xi = t) = 0, \\
    1 + \lambda_3 = 0, \quad \lambda_3(\xi = t) = 0.
\end{align*} \]
so that
\[ \lambda_1 = \lambda_2 = \lambda_3 = -1. \]  

(30)

Substituting these values of the Lagrange multipliers into the functionals (28) gives the iteration formulas

\[
\begin{align*}
 u_{n+1}(x, y, t) &= u_n(x, y, t) - \int_0^t \left( \frac{\partial u_n}{\partial \xi} + \frac{\partial u_n}{\partial x} \frac{\partial v_n}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial v_n}{\partial y} + \tilde{u}_n \right) \, d\xi, \\
 v_{n+1}(x, y, t) &= v_n(x, y, t) - \int_0^t \left( \frac{\partial v_n}{\partial \xi} + \frac{\partial v_n}{\partial x} \frac{\partial w_n}{\partial x} - \frac{\partial v_n}{\partial y} \frac{\partial w_n}{\partial y} - \tilde{v}_n \right) \, d\xi, \\
 w_{n+1}(x, y, t) &= w_n(x, y, t) - \int_0^t \left( \frac{\partial w_n}{\partial \xi} + \frac{\partial w_n}{\partial x} \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} \frac{\partial u_n}{\partial y} - \tilde{w}_n \right) \, d\xi.
\end{align*}
\]  

(31)

The zeroth approximations \( u_0(x, y, 0) = e^{x+y} \), \( v_0(x, y, 0) = e^{x-y} \) and \( w(x, y, 0) = e^{-x+y} \) are selected by using the given initial conditions. Proceeding as before we obtain the following successive approximations:

\[
\begin{align*}
 u_0(x, y, t) &= e^{x+y}, \\
 v_0(x, y, t) &= e^{x-y}, \\
 w_0(x, y, t) &= e^{-x+y}, \\
 u_1(x, y, t) &= e^{x+y} - te^{x+y}, \\
 v_1(x, y, t) &= e^{x-y} + te^{x-y}, \\
 w_1(x, y, t) &= e^{-x+y} + te^{-x+y}, \\
 u_2(x, y, t) &= e^{x+y} - te^{x+y} + \frac{t^2}{2!} e^{x+y}, \\
 v_2(x, y, t) &= e^{x-y} + te^{x-y} + \frac{t^2}{2!} e^{x-y}, \\
 w_2(x, y, t) &= e^{-x+y} + te^{-x+y} + \frac{t^2}{2!} e^{-x+y}, \\
 u_3(x, y, t) &= e^{x+y} - te^{x+y} + \frac{t^2}{2!} e^{x+y} - \frac{t^3}{3!} e^{x+y}, \\
 v_3(x, y, t) &= e^{x-y} + te^{x-y} + \frac{t^2}{2!} e^{x-y} + \frac{t^3}{3!} e^{x-y}, \\
 w_3(x, y, t) &= e^{-x+y} + te^{-x+y} + \frac{t^2}{2!} e^{-x+y} + \frac{t^3}{3!} e^{-x+y}, \\
 &\vdots \\
 u_n(x, y, t) &= e^{x+y} \left( 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \cdots \right), \\
 v_n(x, y, t) &= e^{x-y} \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots \right), \\
 w_n(x, y, t) &= e^{-x+y} \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots \right),
\end{align*}
\]  

readily. This gives the exact solutions

\[
\begin{align*}
 u(x, y, t) &= e^{x+y-t}, \\
 v(x, y, t) &= e^{x-y+t}, \\
 w(x, y, t) &= e^{-x+y+t},
\end{align*}
\]  

(33)
obtained upon using the Taylor expansions. As stated before, we did not use any transformation formulas or Adomian polynomials for handling the nonlinear terms. The VIM method is applied in a direct and smooth manner.

7. Discussion

The first goal of this work is employing the powerful variational iteration method to investigate systems of linear and nonlinear systems of equations. We also aim to show the power of the VIM method by reducing the size of calculations without any need to transform nonlinear terms. The two goals are achieved.

It is obvious that the method gives rapidly convergent successive approximations through determining the Lagrange multipliers. He’s variational iteration method gives several successive approximations through using the iteration of the correction functional. The VIM uses the initial values for selecting the zeroth approximation, and boundary conditions, when given for bounded domains, can be used for justification only. For nonlinear equations that arise frequently for expressing nonlinear phenomena, He’s variational iteration method facilitates the computational work and gives the solution rapidly as compared with the Adomian method.

References