Discrete Mathematics 67 (1987) 27–42
North-Holland

PACKING TREES IN COMPLETE GRAPHS

Arthur M. HOBBS
Department of Mathematics, Texas A & M University, College Station, TX 77843, U.S.A.

Brian A. BOURGEOIS
Applied Mathematical Sciences Department, University of Houston-Downtown, Houston, TX 77002, U.S.A.

Jothi KASIRAJ
Department of Mathematics, Texas A & M University, College Station, TX 77843, U.S.A.

Received 8 June 1981
Revised 22 October 1986

Graphs $G_1, G_2, \ldots, G_n$ are *packed in* a graph H if the edges of H are colored with the colors $0, 1, 2, \ldots, n$ so that each edge of H has exactly one color, and the subgraph of H induced by the edges colored i, together with additional vertices of H not incident with edges colored i, if necessary, is isomorphic to G_i for each $i \in \{1, 2, \ldots, n\}$. We prove several results about packing trees in complete graphs, including that three trees of different orders can be packed in K_n, and that if $T_2, T_3, \ldots, T_n$ are trees, at most one of which has diameter more than three, and if T_i has order i for each i, then $T_2, \ldots, T_n$ will pack in K_n. Finally we make the new conjecture that trees $T_2, \ldots, T_n$, with T_i having order i, pack into $K_{n-1,\lceil \frac{1}{2}n \rceil}$, and provide some support for the conjecture.

Our terminology comes mainly from [1]. We use $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ for the greatest integer and least integer functions, respectively. In a packing of $G_1, \ldots, G_n$ in graph H, we say that the packing is done *using colors* $0, 1, \ldots, n$. Throughout the rest of this paper, except where otherwise stated, a graph with subscript i has order i. Gyárfás and Lehel [5] showed that, if $T_2, \ldots, T_n$ are trees and if all but two of these trees are stars, then there is a packing of these trees in K_n. They conjectured:

Conjecture A. *If $T_1, \ldots, T_n$ are trees, then there is a packing of these trees in K_n.*

An interesting consequence of this was pointed out by Graham during the first author's visit to Bell Laboratories in 1981:

Consequence B. *If $s_1, \ldots, s_{n-1}$ are sequences of positive integers such that s_i sums to $2i$ for each i, then there is an $n-1 \times n$ matrix M such that some reordering* *of s_i appears in row i together with enough zeros to fill in the row and each column of M sums to $n-1$.*

Consequence B follows from Conjecture A by counting the numbers of edges of each color at each vertex and recording the results in matrix M with one row for each color and one column for each vertex.

Straight [8] proved Conjecture A for $n \leq 7$, and showed that if $T_1, T_2, \ldots, T_n$ are trees and if each of these trees is a path, a star, or a caterpillar with maximum degree 3 in which exactly $\lfloor \frac{1}{3}(i-3) \rfloor$ vertices have degree 3 and in which vertices of degree two are adjacent to end vertices, then they can be packed in K_n. More recently, Fishburn [4] demonstrated the truth of Consequence B. As the difficulty of Conjecture A has become more evident, extensions have been made to packings of small numbers of graphs (e.g., two) with orders and sizes near n in K_n (see, for example, [3] and [7]).

A vertex v of a tree T is *penultimate* in T if it has degree greater than one and is adjacent to at most one vertex of degree greater than one. A *matching* is a graph M such that each component of M has exactly one edge. In this paper, if v is a penultimate vertex of tree T, we use T/v to stand for a component of maximum order of the graph obtained from T by erasing v and all of its incident edges.

We need three lemmas; the first two are easily proved by examining the possible diameters of the tree involved, and so their proofs are omitted. In the figures referred to in the proofs of Lemma 3 and Theorem 1, the closed curve always represents a complete graph with the named graphs packed in it.

Lemma 1. *If, for every choice of a vertex v of degree one in a tree T and for every choice w of a penultimate vertex in T, the distance between v and w is at most two, then T has diameter at most three.*

Lemma 2. *If, for every choice of vertices v and w of degree one in a tree T of order n, either $T-\{v, w\}$ is a star, or v and w are at distance two in T, then T has maximum degree at least $n-2$.*

Lemma 3. *Suppose T and U are trees of orders $n_1 < n_2 = n$, respectively, suppose $n \geq 5$, suppose T and U are not stars, and suppose M is a matching of size $\leq \lfloor \frac{1}{2}n \rfloor$. Then T, U, and M can be packed into K_n.*

Proof. By adding edges to T and M, we may suppose $|M| = \lfloor n/2 \rfloor$ and $n_1 = n-1$. Let M' be M less one component. The packings for n equal to 5 or 6 are easily done by construction. Thus we may suppose $n \geq 7$. Suppose T (U) has vertices x_{1j} (x_{2j}), $j \in \{1, 2\}$, such that x_{ij} has degree one and is adjacent to a vertex a_{ij} with $a_{i1} \neq a_{i2}$, and such that $T-\{x_{11}, x_{12}\}$ and $U-\{x_{21}, x_{22}\}$ are not stars. By induction we may pack $T-\{x_{11}, x_{12}\}$, $U-\{x_{21}, x_{22}\}$, and M' in K_{n-2}. The packing of T, U, and M in K_n is completed as indicated in Fig. 1.

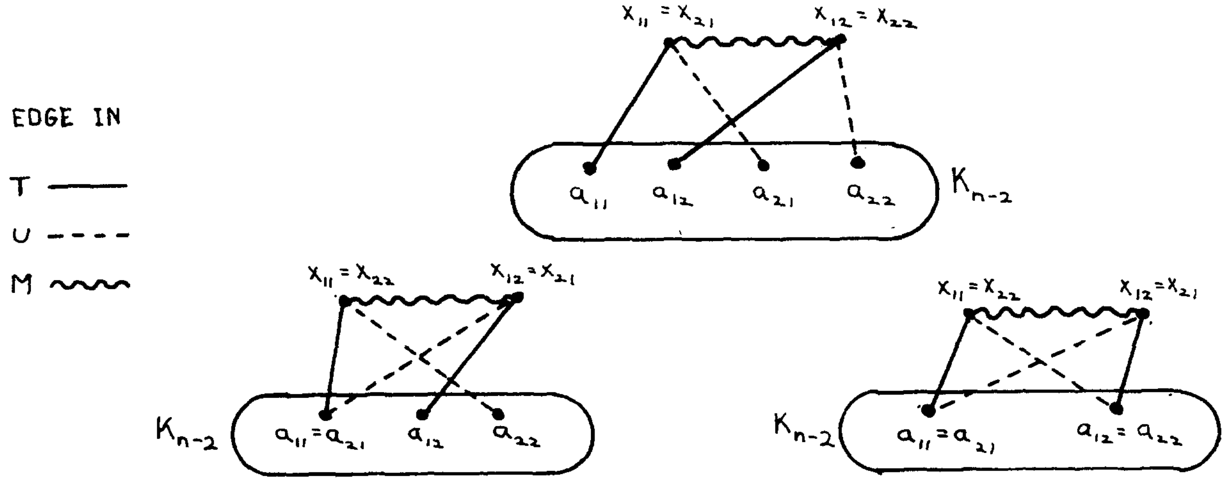

Fig. 1.

Suppose U has maximum degree $n-2$. Then we may, by induction, pack T and M' into K_{n-1}. Let y be a vertex of K_{n-1} which does not meet M'. Since T is not a star, there is an edge yz of K_{n-1} which is not in T in the packing. Let w be a vertex not in K_{n-1} joined to every vertex in K_{n-1}, color yw the color of M, and color yz and all uncolored edges from w to vertices in K_{n-1} with the color of U, thus packing T, U, and M into K_n.

Suppose the maximum degree of U is at most $n-3$ and the maximum degree of T is $n-3$. Applying Lemma 2, let x and y be vertices of degree one of U whose adjacent vertices a and b, respectively, are distinct, and such that $U-\{x, y\}$ is not a star. Pack M' and $U-\{x, y\}$ into K_{n-2}. If both a and b have degree at most $n-5$ in $U-\{x, y\}$, let r be a vertex of K_{n-2} such that ra does not have the color of M' or of U in the packing in K_{n-2}. (r exists because at most $n-5$ edges of U are at a and M' uses up only one more edge at a; thus at most $n-4$ edges are used at a in the packing of K_{n-2} and one more neighbor of a is available.) The required extension to a packing of T, U, and M is indicated in the two cases shown in Fig. 2.

Otherwise, one of a or b (say a) has degree $n-4$ in $U-\{x, y\}$. Then the cases of the packing of M' and $U-\{x, y\}$ into K_{n-2} are indicated in Fig. 3(a) and their extensions to a packing of M, T, and U into K_n are shown in Fig. 3(b). The

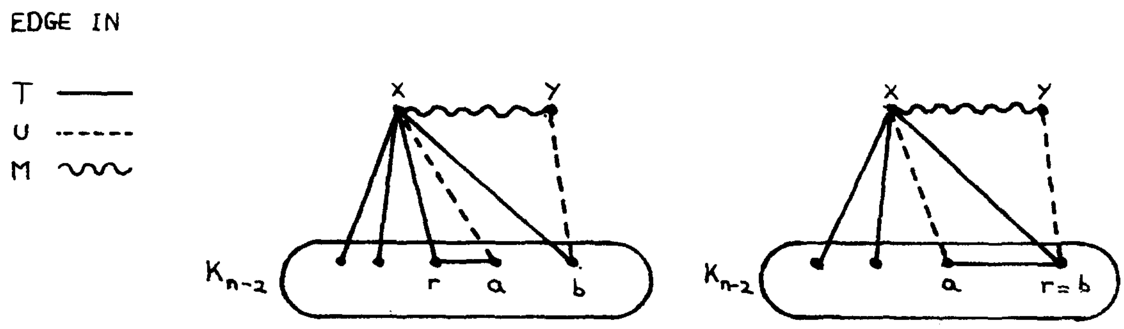

Fig. 2.

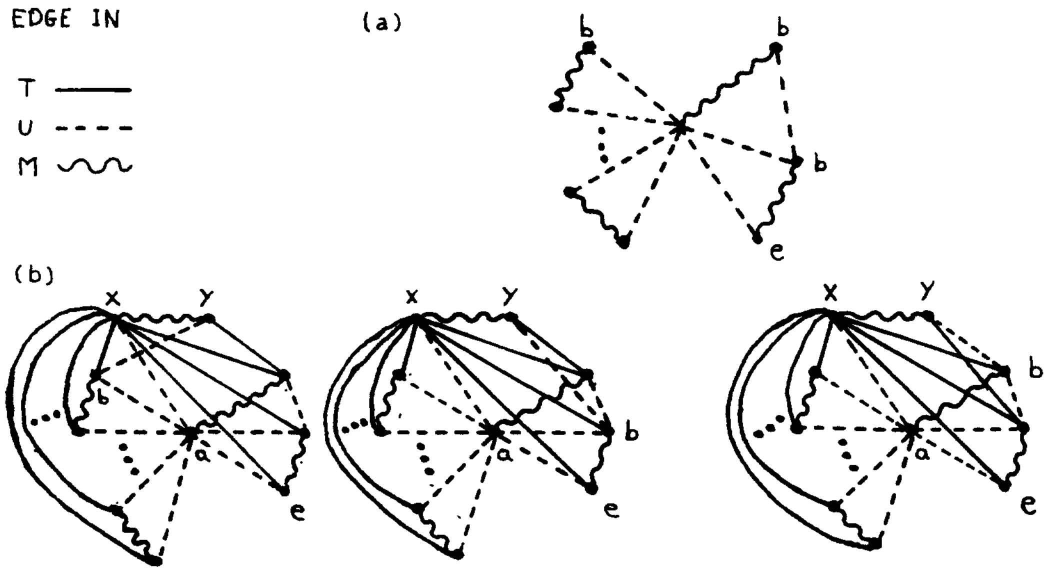

Fig. 3.

possible locations of vertex b are labelled b in Fig. 3(a). In Fig. 3, vertex e is omitted if n is odd. □

In the theorems, we use color i for the color of the edges corresponding to tree T_i in the packing.

Theorem 1. *Trees T_{n_2}, T_{n_1}, and T_n of orders $n_2 < n_1 < n$, respectively, can be packed into K_n.*

Proof. By Straight's result, we may suppose $n > 7$. By adding edges and new vertices to the smaller trees, we can suppose $n_i = n - i$.

If one of T_n, T_{n-1}, or T_{n-2} is a star, the packing exists by the Gyárfás–Lehel result. If T_n has a penultimate vertex of degree exceeding two, let v be such a vertex adjacent to a vertex a of degree exceeding one. Pack T_n/v, T_{n-1}, and T_{n-2} into K_{n-1}. Add a vertex x not in K_{n-1} and color xa and all edges from x to K_{n-1} not meeting T_n/v with color n. Thus we may suppose T_n has diameter at least 4; by Lemma 1 it follows that T_n has a penultimate vertex and a vertex of degree 1 whose distance apart is at least three. Further, every penultimate vertex of T_n has degree two.

If T_n has two penultimate vertices u and v adjacent to vertices a and b, respectively, of $T_n/\{u, v\}$ with $a \neq b$, let r and s be the vertices of degree 1 of T_n adjacent to u and v, respectively, and let w and x be two vertices of degree one in T_{n-1} whose adjacent vertices c and d are distinct. We note that $T_n/\{u, v\}$ has $n-4$ vertices, $T_{n-1} - \{w, x\}$ has $n-3$ vertices, and T_{n-2} has $n-2$ vertices. Pack these three trees into K_{n-2}. Then we may construct a packing of T_n, T_{n-1}, and T_{n-2} into K_n as indicated in Fig. 4. In Fig. 4, if $a = c$ and d is one of the vertices of K_{n-2} not included in $T_n/\{u, v\}$, we can freely choose d to be s.

Finally, suppose all penultimate vertices of T_n are adjacent to the same vertex z

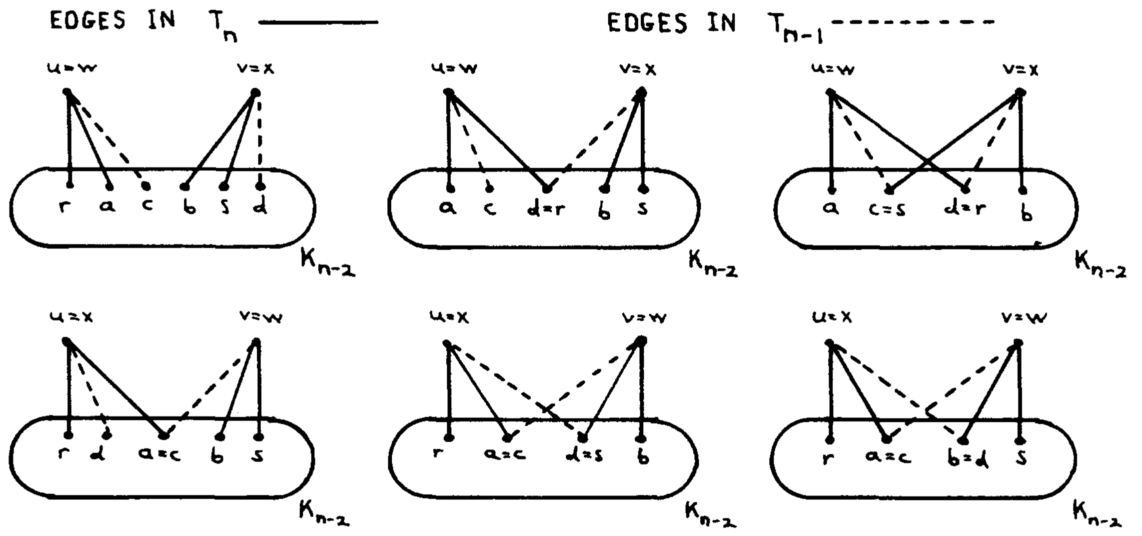

Fig. 4.

of T_n. Then $T_n - z$ is a matching together with, perhaps, some isolated vertices. Since $n > 6$, we may use Lemma 3. Pack $T_n - z$ less its isolated vertices, T_{n-1}, and T_{n-2} into K_{n-1} in accordance with that lemma, coloring the edges in $T_n - z$ with color n. We can complete the packing by adding a vertex x not in K_{n-1} and coloring with color n one edge from x to each of the edges of the matching $T_n - z$ and the edges from x to the vertices not in the packing of $T_n - z$. □

Suppose graph G has edges colored with colors from $\{2, 3, \ldots, k\}$. A *part of color i at vertex v* of G is any one vertex v of G and all of the edges colored i which are incident with that vertex. The *degree* of a part is the number of edges of that part. If v is a vertex of G, S is a set of parts at v, and $vs_1, \ldots, vs_l$ are all of the edges in the parts in S, and if w is a vertex not in G but joined to every vertex of G, we *lift the parts in S to (a new vertex) w* by erasing the colors from $vs_1, \ldots, vs_l$ and coloring the edges $ws_1, \ldots, ws_l$ so that ws_i receives color j if and only if vs_i was colored j.

Our next theorem, on partitions of integers, is the key to Theorems 4 and 5.

Theorem 2. *If n is a positive integer, if $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$ is a partition of n with $k > \frac{1}{2}n$, and if m is any integer in $\{1, 2, \ldots, n\}$, then there exist $i_1, \ldots, i_f$ such that $m = \sum_{j=1}^{f} p_{i_j}$.*

Proof. Suppose $p_1, \ldots, p_t$ are the members of the partition greater than 2, and suppose the partition includes exactly s twos and r ones. Then

$$\frac{\sum_{i=1}^{t} p_i + r + 2s}{r + s + t} = \frac{n}{k} < \frac{n}{(\frac{1}{2}n)} = 2. \tag{A}$$

Hence $r > \sum_{i=1}^{t} (p_i - 2) \geqslant 0$. In particular, if $t > 0$, then $r \geqslant p_1 - 1$. In this case we can use ones to obtain any number in $\{1, \ldots, p_1 - 1\}$; then use p_1 for p_1; then use p_1 and ones to obtain the integers between p_1 and $p_1 + p_2$. We continue until we obtain $p_1 + p_2 - \cdots + p_t$, and finally we use this sum with ones and twos as described in the next case to construct the other integers up to n.

If $t = 0$, we use the ones to construct the integers from 1 through r. Then use $r - 1$ ones and 1 two for $r + 1$, then r ones and 1 two for $r + 2$, then $r - 1$ ones and 2 twos for $r + 3$, etc., until all integers in $\{1, 2, \ldots, n\}$ have been constructed. □

If the integer n is even, this theorem can be extended to partitions with exactly $\frac{1}{2}n$ parts as follows:

Theorem 3. *Let n be even. Let $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{\frac{1}{2}n}$ be a partition of n, and let m be any member of $\{1, 2, \ldots, n\}$. Then either*

(1) *$p_1 = p_2 = \cdots = p_{\frac{1}{2}n} = 2$, or*

(2) *$p_1 = \frac{1}{2}n + 1$ and $p_2 = p_3 = \cdots = p_{\frac{1}{2}n} = 1$, or*

(3) *there exist $i_1, \ldots, i_f$ such that $\sum_{j=1}^{f} p_{i_j} = m$.*

Proof. Suppose $p_1, \ldots, p_t$ exceed 2, suppose that there are s twos, and suppose there are r ones in the partition. Then, by (A) in the proof of Theorem 2, $r = \sum_{i=1}^{t} (p_i - 2)$. Suppose neither conclusion (1) nor conclusion (2) holds. Then $t \geqslant 1$ and $r \geqslant 1$. If $t = 1$, then there are $p_1 - 2$ ones and $\frac{1}{2}n - (p_1 - 2) - 1 = \frac{1}{2}n + 1 - p_1$ twos. Since $p_1 \neq \frac{1}{2}n + 1$, we can use ones to produce any positive integer up to $p_1 - 2$, then use $p_1 - 3$ ones and a two to produce $p_1 - 1$, then use p_1, and finally p_1 and a suitable combination of ones and twos to produce all larger integers up to n. If $t \geqslant 2$, then $r = p_1 + \sum_{i=2}^{t} (p_i - 2) - 2 \geqslant p_1 - 1$. Then we construct m as in Theorem 2. □

Theorem 2 is used in the next two theorems, on packing trees of diameter three:

Theorem 4. *Suppose $T_2, T_3, \ldots, T_n$ are trees, suppose $T_2, T_3, \ldots, T_{n-1}$ can be packed in K_{n-1}, and suppose T_n has diameter at most three. Then $T_2, \ldots, T_n$ can be packed in K_n.*

Proof. If T_n is a star, we can add a vertex and edges colored n from it to every vertex of K_{n-1}, thus completing the desired packing. Suppose T_n has diameter 3. In $T_2, \ldots, T_{n-1}$, there are $\sum_{i=2}^{n-1} i = \frac{1}{2}((n-1)^2 + n - 3)$ parts, so the average number of parts at each vertex of K_{n-1} in a packing of $T_2, \ldots, T_{n-1}$ is $\frac{1}{2}(n-1) + (1 - 2/(n-1))$. Thus there is a vertex v which meets at least $\frac{1}{2}(n-1)$ parts. Since these parts partition the $n-2$ edges at v, by Theorem 2 for any integer m from 1 through $n-2$ there is a selection of the parts at v whose set of edges has m members. Let y be a penultimate vertex of T_n. We can select a set S

of parts at v whose set of edges has cardinality $(\deg y)-1$. We lift the parts in S to a new vertex w, and then color with n the edges at v that were in S and the edges incident with w not involved in the lifting. This completes the desired packing. □

Corollary. *Suppose $T_2, \ldots, T_n$ are trees such that T_i has order i and diameter at most 3. Then $T_2, \ldots, T_n$ can be packed into K_n.*

We can strengthen this corollary by allowing one tree to have diameter in excess of three.

Theorem 5. *Let $T_2, T_3, \ldots, T_n$ be trees, T_i having order i, such that at most one of the trees has diameter more than 3. Then $T_2, \ldots, T_n$ can be packed in K_n.*

Proof. The theorem is true if $n \leqslant 7$ as shown by Straight [8]. Continuing by induction, if T_n has diameter at most 3, then the trees pack by Theorem 4. Thus we may suppose T_n has diameter at least 4. Let w be a vertex of degree one in T_n and let its neighbor be x. Pack $T_2, \ldots, T_{n-2}$, $T_n - w$ in K_{n-1}, using color n for $T_n - w$. If T_{n-1} has diameter 2, we can complete the packing in K_n by adding vertex w and edges from it to all vertices of K_{n-1}, coloring wx with color n and coloring all of the other edges with color $n-1$. Thus we may suppose T_{n-1} has diameter 3. Let y and z be the penultimate vertices of T_{n-1}, y having the smaller degree of the two if their degrees differ. Then y is joined to at most $\frac{1}{2}(n-3)$ vertices of degree one in T_{n-1}. Among $T_2, \ldots, T_{n-2}$ there are $\sum_{i=2}^{n-2} i = \frac{1}{2}(n^2-3n)$ vertices, so that there is an average of $\frac{1}{2}(n-2) - 1/(n-1)$ parts colored $2, \ldots, n-2$ at the vertices of K_{n-1}.

Suppose there are at least $\lfloor \frac{1}{2}n \rfloor$ parts with colors in $\{2, \ldots, n-2\}$ at a vertex $v \neq x$. If edge vx is colored n, then at most $n-3$ edges of color $\leqslant n-2$ are partitioned by $\lfloor \frac{1}{2}n \rfloor$ or more parts; by Theorem 2, parts containing $(\deg y)-1$ edges can be found at v which may be lifted to w to accommodate T_{n-1}. If vx is not colored n, and recalling that $T_n - w$ meets every vertex of K_{n-1}, there are at most $n-4$ edges at v not colored n and not in the part P at v which includes vx. Further there are at least $\lfloor \frac{1}{2}(n-2) \rfloor > \frac{1}{2}(n-4)$ parts other than P in the parts at v and not colored n. Thus parts containing $(\deg y)-1$ edges can be found at v which can be lifted to w to allow T_{n-1} to be packed.

Suppose x meets r parts with colors in $\{2, \ldots, n-2\}$, and suppose s other vertices meet exactly $\lfloor \frac{1}{2}(n-2) \rfloor$ such parts. Then there are at most $r + s\lfloor \frac{1}{2}(n-2) \rfloor + (n-1-s-1)\lfloor \frac{1}{2}(n-4) \rfloor$ parts colored $2, \ldots, n-2$. Hence $r + s + (n-2)\lfloor \frac{1}{2}(n-4) \rfloor \geqslant \frac{1}{2}(n^2-3n)$. If n is even, $s \geqslant \frac{1}{2}(3n-8) - r$. Since $r \leqslant n-3$, we have $s \geqslant \frac{1}{2}(n-2)$. Similarly, if n is odd, $s \geqslant 2n-5-r \geqslant n-2$. Since $s \leqslant n-2$ also, we get $s = n-2$ and $r = n-3$ in this case.

Suppose n is even, and let vertex v meet $\frac{1}{2}(n-2)$ parts with colors in $\{2, \ldots, n-2\}$. If vx is colored n, then there are least $\frac{1}{2}(n-2)$ parts colored differently from n at at most $n-3$ edges in these parts, so the edge set of a subset of the parts at v has cardinality $(\deg y)-1$ and thus T_{n-1} can be packed. If vx is not colored n, let P be the part at v which contains vx. Then the number of edges at v not colored n and not in P is $\leq n-4$, since v meets an edge of $T_n - w$. If P contains more than one edge, or if more than one edge at v is colored n, then we have $\frac{1}{2}(n-4)$ parts partitioning a number less than or equal to $n-5$, so the edge set of a subset of the parts at v has cardinality $(\deg y)-1$ and T_{n-1} can be packed. Thus vx is the only edge in P.

It follows that x is adjacent to at least $\frac{1}{2}(n-2)$ vertices v at which the tree T_i, $i \leq n-2$, containing edge vx has degree one at v. Since only T_2 can also have degree one at x, x meets at least $2(\frac{1}{2}(n-2)-1)+1=n-3$ edges from these trees. Since x also meets an edge of T_n, T_2 must contain one of these edges. Thus one of the vertices, say v', meets $\frac{1}{2}(n-2)$ parts with colors in $\{2, \ldots, n-2\}$, and $v'x$ is colored 2. Allowing the use of $v'x$ at v', we can find a subset of the parts at v' whose edge set has cardinality $(\deg y)-1$ and thus pack T_{n-1}. Color wx with color n. If the part containing edge $v'x$ is lifted, then there is an edge from w which receives neither color n nor color $n-1$. This edge can be given color 2, thus completing the packing of $T_2, \ldots, T_n$ in K_n.

Suppose n is odd, so $s=n-2$ and $r=n-3$. Let vx be an edge colored n and incident with x. Since $r=n-3$, $T_n - w$ has degree 1 at x, so its degree at v is more than 1. Hence the $\frac{1}{2}(n-3)$ parts at v partition a number of less than or equal to $n-4$. Thus we can find a subset of the parts at v whose edge set has cardinality $(\deg y)-1$, so T_{n-1} can be packed by placing y at v. □

A connected bipartite graph has a unique decomposition of its vertices into two sets so that each edge joins a vertex of one set with a vertex of the other set. We call these two sets the *sides* of the bipartite graph. If one of the sides has more vertices than the other, we say it is the *large* side and the other side is the *small* side. If the sides are of the same order, let the *larger* one be the one with a larger number of vertices of degree one, if such exists, and otherwise let the *larger* one be either side.

A path in a tree T is *maximal* if both of its end vertices are of degree one in T and the path has length at least one. The center of a star is the only penultimate vertex of the star. If a tree has diameter at least three, the second and next-to-the-last vertices of any maximal length path in the tree are penultimate vertices of the tree.

In attempting to determine the largest number of trees of different orders that could be packed into a complete graph with n vertices, we have found it useful to pack trees into complete bipartite graphs. Efforts in this direction led to the following conjecture:

Conjecture C. *If trees $T_2, T_3, \ldots, T_n$ are chosen arbitrarily with T_i having i vertices for each i, then these trees can be packed into $K_{n-1,\lceil \frac{1}{2}n \rceil}$.*

If n is even, the number of edges available in the complete bipartite graph of the conjecture is exactly the number needed to pack the trees. When n is odd, $\frac{1}{2}(n-1)$ vertices on the small side would not provide enough edges in the complete bipartite graph to pack the trees. Also, $n-1$ vertices are needed on the large side in order to allow T_n to be a star. Direct construction of the packings has verified this conjecture if $n \leqslant 6$.

Suppose $T_2, \ldots, T_n$ are stars, and let T_1 be K_1. If n is even, numbering the vertices on the small side of $K_{n-1,\lceil\frac{1}{2}n\rceil}$ with $1, 2, 3, \ldots, \frac{1}{2}n$, we can pack the stars by packing T_i and T_{n+1-i} with their centers on vertex i, using any $i-1$ of the edges from vertex i for T_i and the remaining $n-i$ edges for T_{n+1-i}. A similar packing works for n odd, although in this case some edges will not be used for packing the stars. Thus the conjecture is true if all of the trees are stars.

If $T_2, T_3, \ldots, T_n$ are all paths and stars, T_i having order i for each i, then Zaks and Liu [9] have verified the conjecture for these trees when n is even, and Hobbs [6] has verified it for them when n is odd. Zaks and Liu's work suggests the conjecture might also be provable if the complete bipartite graph were $K_{n,\frac{1}{2}(n-1)}$ with n odd. We next show that two trees of different order can be packed in the complete bipartite graph if the orders of the trees and of the sides of the complete bipartite graph satisfy the conditions of the conjecture.

Lemma 4. *If every maximal path in a tree T has even length, then all of the vertices of degree one in the tree are on the large side of T, and the large side of T is strictly larger than the small side. Further, if T has at least k penultimate vertices of degree three or more, the large side of T has at least $k+1$ more vertices than the small side.*

The proof of this lemma is routine and hence is omitted.

Theorem 6. *Let T_i and T_j be trees having orders $i<j$, and suppose $j \leqslant n$. Then T_i and T_j can be packed into $K_{n-1,\lceil\frac{1}{2}n\rceil}$ so that their smaller sides are on the small side of $K_{n-1,\lceil\frac{1}{2}n\rceil}$.*

Proof. Denote the complete bipartite graph $K_{n-1,\lceil\frac{1}{2}n\rceil}$ by K. We may suppose $i=n-1$ and $j=n$. We will prove this theorem by induction on n. Direct verification for $n \leqslant 5$ begins the induction. If T_n is a star, we can pack T_n with its center on one of the vertices of the small side of K. Then there remains a graph $K_{n-1,\lceil\frac{1}{2}n\rceil-1}$ in which T_{n-1} can clearly be packed with its small side on the smaller side of the graph K.

Suppose T_{n-1} is a star. Pack T_n in K with its small side on the small side of K. If every vertex on the small side of K meets at least two edges colored n, then T_n has at least $2\lceil\frac{1}{2}n\rceil \geqslant n$ edges, which is impossible. Thus the center of T_{n-1} can be

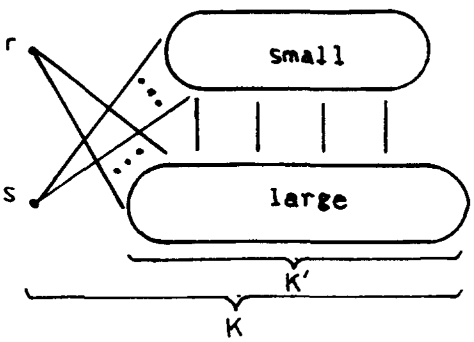

Fig. 5.

placed on a vertex of the small side of K which meets at most one edge of T_n, and $n-2$ edges at that vertex can be colored $n-1$, thus completing the packing.

The basic idea of the rest of the proof is that we remove two vertices from T_n to produce $T_n - \{x, y\}$ which is packed with T_{n-1} into a complete bipartite graph K' smaller than K. If n is odd, K' is formed by removing one vertex from each side of K. If n is even, only one vertex is removed from (the large side of) K to form K'. In terms of the packing, the necessity of K' being only one vertex smaller than K when n is even can be seen if T_{n-1} is a star and $T_n - \{x, y\}$ is a path, for then T_{n-1} uses all of the edges incident with one vertex of the small side of K' and $T_n - \{x, y\}$ requires $\frac{1}{2}(n-2)$ other vertices on that side. Algebraically, K' is $K_{n-1-1,\lceil\frac{1}{2}(n-1)\rceil}$. If n is odd, $\lceil\frac{1}{2}(n-1)\rceil = \frac{1}{2}(n-1) = \frac{1}{2}(n+1) - 1 = \lceil\frac{1}{2}n\rceil - 1$, while if n is even $\lceil\frac{1}{2}(n-1)\rceil = \frac{1}{2}n = \lceil\frac{1}{2}n\rceil$.

Suppose n is odd. Let $K' = K_{n-2,\frac{1}{2}(n-1)}$, let the vertex on the small side of K which is not in K' be r and the vertex on the large side of K not in K' be s (see Fig. 5). If T_n has an odd maximal path, let x and y be the end vertices of such a path, and let a be the vertex adjacent to x and b the vertex adjacent to y in the tree. Applying induction, pack $T_n - \{x, y\}$ and T_{n-1} in K' with their smaller sides on the small side of K'. Since a and b are at an odd distance in T_n, they are on different sides in the packing; let us suppose a is on the large side. Then x and y can be added by coloring with n the edges ar and bs.

Suppose T_n has no odd maximal paths but that it has a penultimate vertex x of degree two in T_n. Let a and y be the vertices adjacent to x, y being of degree one. Pack $T_n - \{x, y\}$ and T_{n-1} in K' with their smaller sides on the small side of K'. Observe that we have removed one vertex from each side of T_n so that the vertices on the small side of $T_n - \{x, y\}$ are on the small side of T_n. Hence a is on the large side of K'. Complete the packing in K by coloring the edges ar and rs with n.

Thus we may suppose every maximal path in T_n has even length, T_n is not a star, and no penultimate vertex of T_n has degree two. Let x and y be vertices of degree one of T_n joined to a penultimate vertex a of T_n. By Lemma 4, the larger side of T_n has at least three more vertices than the smaller side, so the vertices in the smaller side of $T_n - \{x, y\}$ are in the smaller side of T_n. Pack $T_n - \{x, y\}$ and

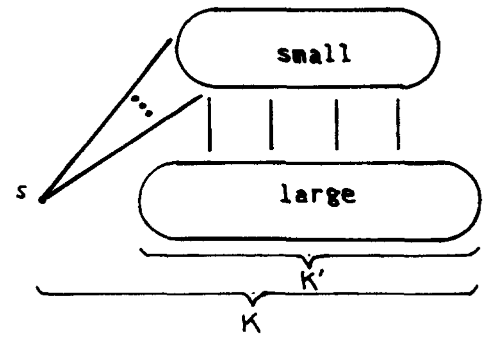

Fig. 6.

T_{n-1} in K' so that the smaller sides of the trees are on the small side of K'; note that a is on the small side of K'. Since $T_n - \{x, y\}$ has $n-2$ vertices, at least one vertex c on the large side of K' does not meet $T_n - \{x, y\}$. Then lift the part in K' colored n at a to r in K and add the color n to edges rs and rc. The result is a packing of T_{n-1} and T_n in K.

Suppose n is even. Delete a vertex s from the large side of K to produce $K' = K_{n-2,\frac{1}{2}n}$ (see Fig. 6). Suppose T_n has an odd maximal path; let its ends be x and y, with x adjacent to a and y adjacent to b. Pack $T_n - \{x, y\}$ and T_{n-1} in K' and suppose a is on the larger side of K'. Since $T_n - \{x, y\}$ has $n-2$ vertices, its smaller side has at most $\frac{1}{2}n - 1$ vertices, and so there is a vertex c on the small side of K' which meets no edge colored n. If ca is not colored $n-1$, color it n and color bs with n, thus completing the packing of T_{n-1} and T_n in K. If ba is not colored $n-1$, lift the part colored n at a in K' to s and add the color n to edges ba and sc in K, thus completing the packing. Thus we may suppose ba has color $n-1$.

$T_n - \{x, y\}$ is not a star since it has an odd maximal path, so $T_n - \{x, y\}$ has at most $n-4$ vertices on the large side of K'. Let $s_1, s_2, \ldots, s_k$ be the vertices on the large side of K' which do not meet an edge colored n. If some s_ib is not colored $n-1$, lift the part colored n at a in K' to s and add the color n to the edges sc and s_ib in K, completing the packing in K. Thus we may suppose all of $s_1b, \ldots, s_kb$ have color $n-1$.

We note that s_ic cannot be colored $n-1$ for any i, for then a cycle (s_i, c, a, b, s_i) would be present in T_{n-1}. Suppose there is a vertex s_i such that, for every vertex f other than b, if s_i is joined to f by an edge colored $n-1$, then f is not joined to a by an edge colored n. Lift the part colored n in K' at a to s and the part colored $n-1$ in K' at s_i to s, and color sc and s_ib with color n to complete the packing in K. Thus we may suppose each of $s_1, \ldots, s_k$ meets an edge colored $n-1$ which is incident with a vertex joined to a by an edge colored n.

But if $i \neq j$, s_i and s_j cannot be joined to the same vertex f by edges colored $n-1$, for then the cycle (s_i, f, s_j, b, s_i) would be present in T_{n-1}. Thus a is joined to at least k vertices by edges colored n. But then there are at least $k+1$ vertices of $T_n - \{x, y\}$ on the small side of K', so there are at most $n-k-3$ vertices of T on the large side. Hence $k \geqslant k+1$, which is impossible. Thus T_{n-1} and T_n can be packed in K if T_n has an odd maximal path.

Suppose every maximal path of T_n has even length, and suppose T_n has a penultimate vertex x of degree two. Let x be adjacent to a and y, with y having degree one. Since y is on the large side of T_n, so is a. Hence a is on the large side of $T_n - \{x, y\}$. Since $T_n - \{x, y\}$ has no odd maximal paths, it has more vertices in its larger side than its smaller side, by Lemma 4. Thus it has at most $\frac{1}{2}n - 2$ vertices on its smaller side. Pack $T_n - \{x, y\}$ and T_{n-1} in K' so that the smaller sides of the trees are on the small side of K' and let $c_1, \ldots, c_p$ be the vertices of the smaller side of K' which do not meet an edge colored n; then $p \geqslant 2$. If any edge ac_i is not colored $n-1$, color ac_i and c_is with color n, thus completing the packing of T_{n-1} and T_n in K. Since $T_n - \{x, y\}$ has a vertex on the small side of K', there is at least one vertex s' on the large side which meets no edge colored n. If an edge $s'c_i$ is not colored $n-1$, then we can color sc_i and $s'c_i$ with color n to complete the packing. But otherwise, T_{n-1} includes the cycle (s', c_1, a, c_2, s'). Thus we may suppose T_n has no odd maximal paths and no penultimate vertices of degree two.

Let a be a smallest degree penultimate vertex of T_n, and suppose a is joined to exactly the vertices $x_1, \ldots, x_d$ of degree one in T_n. Let the other neighbor of a in T_n be b. Let $T' = T_n - \{x_1, \ldots, x_d\}$. Since T_n has no odd maximal paths, $T' - a$ has none, so its larger side includes all of its vertices of degree one. Further, $T' - a$ that at least one penultimate vertex of degree greater than two, so by Lemma 4 $T' - a$ has at least two more vertices on its larger side than on its smaller side. It follows that T' has a strictly larger side, and a is on its smaller side. Pack T' and T_{n-1} in K' so that the smaller sides of the trees are on the small side of K'.

Case 1. If b is the center of T_n and if b has degree two in T_n, let a' be its other neighbor in T_n. Since n is even and $d < \frac{1}{2}(n-3)$, $d \leqslant \frac{1}{2}(n-4)$. Let the vertices of the large side of K' not meeting an edge colored n form a set A. Then A has $n-2-(n-d-2) = d$ elements. The vertex a is joined to at least two vertices of A by edges colored $n-1$, for otherwise we could color $d-1$ of the edges from a to A with color n and color sa with color n, thus completing the packing of T_{n-1} and T_n in K.

Suppose a vertex y other than a or a' on the small side of K' exists such that yb is not colored $n-1$ and at most one edge from y to A has color $n-1$. Then erase the color from edge ba, and color edges yb, ys, and $d-1$ uncolored edges from y to A with color n. The result is a packing of T_{n-1} and T_n in K. Thus each of the $\frac{1}{2}(n-4)$ vertices other than a and a' on the small side of K' meets at least two edges of T_{n-1}, and a meets two edges of T_{n-1}, so T_{n-1} has at least $n-2$ edges joined to $A \cup \{b\}$. Since this is the number of edges in T_{n-1}, the vertices met by these edges are the $n-1$ vertices of T_{n-1}. Hence $\frac{1}{2}(n-4)+1+d+1 = n-1$, or $d = \frac{1}{2}(n-2)$. Since $d \leqslant \frac{1}{2}(n-4)$, we have a contradiction.

Case 2. Hence either b is the center of T_n but with degree at least three, or T_n has diameter at least six. In either case, the small side of T' has at least three vertices, so there is a set A of $d+1$ vertices on the large side of K' which do not meet an edge colored n.

If a is joined to at most two of these by edges colored $n-1$, we may color $d-1$ edges from a to A and the edge sa with color n to complete the packing in K. Thus a meets at least three edges colored $n-1$ whose other ends are in A. Since T' has $n-d$ vertices, the small side of T' has fewer than $\frac{1}{2}(n-d)$ vertices, so at least $\lfloor\frac{1}{2}d\rfloor+1$ vertices of the small side of K' meet no edges colored n. Let B be a set of exactly $\lfloor\frac{1}{2}d\rfloor+1$ such vertices. If there is $y\in B$ such that yb has no color and at most two edges from y to A have color $n-1$, then we can erase the color from edge ba and color yb, ya, and $d-1$ edges from y to A with color n to complete the packing in K. If there is $y\in B$ such that yb has color $n-1$ and at most one edge from y to A has color $n-1$, lift the part colored n in K' at b to s, erase the color from edge sa, and color with n edge sy, and d edges from y to A, thus completing the packing in K. Finally, we may suppose each vertex of $B\cup\{a\}$ meets at least three edges colored $n-1$ which have their other ends in $A\cup\{b\}$. The number of vertices in these two sets is $d+\lfloor\frac{1}{2}d\rfloor+4$, and the number of edges colored $n-1$ that have both ends in these sets is at least $3(\lfloor\frac{1}{2}d\rfloor+2)>d+\lfloor\frac{1}{2}d\rfloor+3$. Since this latter number is the maximum number of edges T_{n-1} could have joining pairs of these vertices, we have a contradiction, and the theorem is proved. □

We next study how large an integer t can be chosen with a given value of integer n such that trees $T_2,\ldots,T_t$ with T_i having order i can be packed into $K_{n-1,\lceil\frac{1}{2}n\rceil}$. If we could show $t=n$, we would have proved our conjecture. We have not done so well, but we do have:

Theorem 7. *Let n be an integer $\geqslant 6$. Then $t\simeq 2\sqrt{3n}$ trees can be packed into $K_{n-1,\lceil\frac{1}{2}n\rceil}$.*

Proof. By construction, we have packed each set of trees $T_2,\ldots,T_6$ in $K_{5,3}$, so the conjecture is true for $n=6$. In $K_{17,9}$, pack $T_2,\ldots,T_6$ in a $K_{5,3}$ subgraph and pack T_7 and T_8 by Theorem 6 in a $K_{12,6}$ vertex-disjoint from the selected $K_{5,3}$. Thus we can set $n_1=6$, $t_1=6$ and $n_2=18$, $t_2=8$ for the construction of a recurrence relation. If n_i is even and $n_i\geqslant t_i+7$, we can add t_i+6 vertices to the large side of $K_{n_i-1,\frac{1}{2}n_i}$ and $\frac{1}{2}(t_i+6)$ vertices to the small side. Then, by using the above theorem, we can pack T_{t_i+1} and T_{t_i+2} between the small side of $K_{n_i-1,\frac{1}{2}n_i}$ and the newly added t_i+6 vertices, pack T_{t_i+3} and T_{t_i+4} between the large side of $K_{n_i-1,\frac{1}{2}n_i}$ and the newly added $\frac{1}{2}(t_i+6)$ vertices, and pack T_{t_i+5} and T_{t_i+6} between the two sets of newly added vertices. Then $n_{i+1}=n_i+t_i+6$ and $t_{i+1}=t_i+6$. We have $n_i\geqslant t_i+7$, so $n_{i+1}\geqslant 2t_i+13=t_{i+1}+t_i+7>t_{i+1}+7$. Thus, for $i\geqslant 2$, $n_{i+1}=n_i+t_i+6$ and $t_{i+1}=t_i+6$. Solving this relation, we find $t_1=6$, $t_i=6i-4$ if $i\geqslant 2$,

$n_1 = 6$, $n_i = 3i^2 - i + 8$ if $i \geq 2$. Combining and solving for t_i in terms of n_i, we get $t_i = -3 + \sqrt{12n_i - 95}$, of which the stated result is a simplification. □

Let us now consider the Gyárfás–Lehel conjecture. Suppose we have t_i trees $T_2, \ldots, T_{t_i}$ packed in K_{n_i}. To form a recurrence relation, we can pack T_{t_i+1}, T_{t_i+2}, and T_{t_i+3} in K_{t_i+3}. Between K_{n_i} and K_{t_i+3} there is a complete bipartite graph K_{n_i, t_i+3} with uncolored edges. If we partition the n_i vertices of K_{n_i} into subsets of orders $t_i + 4$, $t_i + 6, \ldots, t_i + 2r$ and a remainder set of fewer than $t_i + 2r + 2$ vertices, then we can pack trees T_{t_i+4} and T_{t_i+5} in the subgraph of K_{n_i, t_i+3} whose small side is the small side of the complete bipartite graph and whose large side is the set of $t_i + 4$ vertices. We can pack T_{t_i+6} and T_{t_i+7} in the subgraph using the small side of the complete bipartite graph and the $t_i + 6$ vertex set of the partition. We continue, finally packing T_{t_i+2r} and T_{t_i+2r+1} in the subgraph using the small side of the complete bipartite graph and the $t_i + 2r$ vertex set of the partition. Then we have packed $t_{i+1} = t_i + 2r + 1$ trees of orders $1, 2, 3, \ldots, t_i + 2r + 1$ into $K_{n_{i+1}}$ with $n_{i+1} = n_i + t_i + 3$. This recursion is valid if n_i is large enough compared to t_i; specifically it is valid for $n_2 = 17$ and $t_2 = 11$ and for all n_i with $i > 2$. The values $n_1 = 7$, $t_1 = 7$ were used to obtain n_2 and t_2; the validity of $n_1 = 7$ and $t_1 = 7$ was shown by Straight [8]. Algebraic manipulation to eliminate r shows

$$t_{i+1} = t_i + 1 + 2\lfloor -\tfrac{1}{2}(t_i + 1) + \tfrac{1}{2}\sqrt{(t_i + 3)^2 + 4n_i} \rfloor.$$

The next two lemmas follow by replacing the binomials by their expansions as power series and then replacing the series, which are alternating after the first few terms, by the first two, three, or four terms of those series. Since the preceding recursion formulas do not arise from a complete solution to the tree packing problem, it is sufficient to find rough bounds for t_i in terms of n_i from the recursions. In particular, the bounds on n_i in the next two lemmas could be improved, as could the coefficients of $n_i^{2/3}$ in Theorem 8, but the improvements would not be interesting.

Lemma 5. *If $n_i \geq 8000$, then*

$$(1.9)^2 n_i^{4/3}(1 + n_i^{-1/3})^{4/3} - (1.9)^2 n_i^{4/3}\left(1 + \frac{4}{3.61} n_i^{-1/3}\right) - (11.4)n_i^{2/3} - 9 > 0.$$

Lemma 6. *If $n_i \geq 27$, then*

$$n_i^{4/3}(1 + 4n_i^{-1/3}) + 6n_i^{2/3} + 5 - n_i^{4/3}(1 + 2n_i^{-1/3})^{4/3} - 4n_i^{2/3}(1 + 2n_i^{-1/3})^{2/3} > 0.$$

Theorem 8. *If t_i is the number of trees packed into K_{n_i} by the preceding construction, then $(1.9)n_i^{2/3} > t_i > n_i^{2/3}$ for $i \geq 2$.*

Proof. Using the recursion formulas found in the preceding work, the result of this theorem has been directly verified for $2 \leq i \leq 36$. Further, $n_{36} = 8120$. Suppose this theorem is true for one value of i such that $n_i \geq 8000$. Then $t_{i+1} \leq \sqrt{(t_i+3)^2 + 4n_i}$. Hence

$$\begin{aligned} t_{i+1}^2 &\leq t_i^2 + 4n_i + 6t_i + 9 \\ &< (1.9)^2 n_i^{4/3} + 4n_i + 6(1.9)n_i^{2/3} + 9 \\ &= (1.9)^2 n_i^{4/3}\left(1 + \frac{4}{3.61} n_i^{-1/3}\right) + (11.4)n_i^{2/3} + 9 \\ &\leq (1.9)^2 n_i^{4/3}(1 + n_i^{-1/3})^{4/3}, \quad \text{by Lemma 5,} \\ &\leq (1.9)^2 n_i^{2/3} \end{aligned}$$

because $n_{i+1} = n_i + t_i + 3 \geq n_i + n_i^{2/3} = n_i^{2/3}(1 + n_i^{-1/3})$. Thus $t_{i+1} < (1.9)n_{i+1}^{2/3}$.
Also $t_{i+1} + 2 > \sqrt{(t_i+3)^2 + 4n_i}$, so

$$\begin{aligned} (t_{i+1}+2)^2 &> t_i^2 + 4n_i + 6t_i + 9 \\ &> n_i^{4/3} + 4n_i + 6n_i^{2/3} + 9 \\ &> 4 + n_i^{4/3}(1 + 2n_i^{-1/3})^{4/3} + 4n_i^{2/3}(1 + 2n_i^{-1/3})^{2/3}, \end{aligned}$$

by Lemma 6. But $n_{i+1} = n_i + t_i + 3 < n_i + 2n_i^{2/3}$ if $t_i + 3 < 2n_i^{2/3}$, which occurs for $n_i \geq 165$. Hence the preceding formula is greater than $4 + n_{i+1}^{4/3} + 4n_{i+1}^{2/3} = (n_{i+1}^{2/3} + 2)^2$. Thus $t_{i+1} > n_{i+1}^{2/3}$. □

There are many edges not used in K_{t_i+3} and K_{n_i} in the preceding packing. We attempted to take advantage of these edges by using Turan's Theorem to find a set of vertices with no colored edges between them, usually in K_{n_i}, and then have two complete graphs overlap on those vertices. However, because the number of trees that can be inserted in the complete bipartite graph is thereby reduced, there was little improvement over the results described in the preceding paragraphs.

Acknowledgments

The authors are indebted to Dr. Fan Chung and Dr. A. M. Odlyzko of Bell Laboratories and Prof. Neal Carothers of Texas A & M University for ideas used in proving Theorem 8.

The authors wish to thank Texas A & M University students Marilyn Helton and Sue Ellen Slack for patiently withstanding the authors' enthusiasm during the research for this paper.

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