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Integrable Many-Body Problems and Functional Equations

EUGENE GUTKIN*

University of Southern California, Los Angeles, California 90089

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0. INTRODUCTION

Since the pioneering paper of M. Kruskal *et al.* [10] on the Korteweg–DeVries equation there has been a lot of work on solvable partial differential equations and integrable models of classical and quantum mechanics. Many nonlinear equations have been explicitly solved using Lax pairs, AKNS method [1], or the inverse scattering method and its generalizations (cf. [5] and the bibliography there). It is important to notice that all these methods have to do with sufficient conditions for solvability. Namely, one exhibits an ansatz, shows that the ansatz allows integration of the equations (and how to do it) and then puts the equations in the form required by the ansatz. What is missing at this point are the necessary conditions to put the equations in the ansatz form or, more generally, necessary conditions for integrability.

The purpose of this paper is to exhibit such necessary conditions for a class of quantum many-body problems. Let me first make a few general remarks about integrable systems. In the classical Hamiltonian mechanics we have a well-known definition of complete integrability (cf. [3]) which besides its obvious esthetic attraction is very useful in studying non-integrable Hamiltonian systems (the Kolmogorov–Arnold–Moser theory, cf. [20]). The universe of Hamiltonian mechanics is so large that we cannot hope to obtain criteria of complete integrability unless we agree to consider certain classes of Hamiltonian systems. One such class consists of systems with the flat phase space $\mathbf{R}^{2n} = \{(p, q)\}$ and the Hamiltonian

$$H(p, q) = \tfrac{1}{2}\|p\|^2 + V(q) \tag{0.1}$$

determined by a potential V. An important subclass of (0.1) consists of

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N d-dimensional particles ($d = 1, 2, 3, \dots$) interacting via a pair potential v, i.e., $(p, q) = (p_1, \dots, p_N; q_1, \dots, q_N)$, $p_i, q_i \in \mathbf{R}^d$ and

$$H(p, q) = \frac{1}{2} \sum_{i=1}^{N} \|p_i\|^2 + \sum_{i<j} v(q_i - q_j). \tag{0.2}$$

Under additional assumptions on the pair potential v or the master potential V we have a good understanding of complete integrability for (0.1) and (0.2) (cf. [12, 13, 11, 22]).

Due to the space limitations we will not dwell on the subject of classical integrability and move on to the quantum Hamiltonian systems. The situation there is totally different because there is no generally accepted definition of an integrable quantum system. The present definition is operational; that is, a quantum Hamiltonian system is called integrable (or soluble) if it has been explicitly solved. This is not surprising since this is the current state of the art with the integrable partial differential equations and integrable models of statistical mechanics (cf. [4]).

The format of this paper will be as follows. We start by isolating a reasonably wide class of quantum Hamiltonian systems: N particles of the unit mass on the line interacting via a pair potential v. Then we define a property of these Hamiltonians, the nondiffractive scattering, which substitutes for the yet nonexistent definition of complete integrability. We obtain necessary conditions on the pair potential v for the N-body scattering to be nondiffractive. These necessary conditions take the form of a functional equation on the reflection coefficient of the scattering matrix for the Schroedinger operator

$$h = -\frac{1}{2} \frac{d^2}{dx^2} + v(x). \tag{0.3}$$

In other words, the N-body scattering for the Hamiltonian

$$H_N = -\sum_{i=1}^{N} \partial^2/\partial x_i^2 + \sum_{i<j} v(x_i - x_j) \tag{0.4}$$

is nondiffractive for any $N = 3, 4, \dots$ only if the two-body scattering matrix has a special form (see (5.1)).

Although we do not know all the pair potentials v with the scattering matrix (5.1), two classes of examples are known. One is the delta-potential $v(x) = c\,\delta(x)$; another is the multisoliton potentials (cf. [24]). The delta-potential is known to be nondiffractive in virtue of the famous Bethe–Yang–Lieb ansatz (cf. [6, 17, 23, 15]). The question whether the multisoliton potentials are nondiffractive remains completely open. It is not known even in the simplest case of one-soliton potential

$$v(x) = -a^2 \cosh^{-2}(ax) \tag{0.5}$$

(it is also called the Poschl–Teller potential, cf. [2]). We conclude the paper by a brief discussion of possible implications and extensions of our results.

The work grew out of correspondence and discussions with Mark Kac. This paper is a small tribute to his dear memory.

1. N-body Hamiltonians

Let v be a decaying as $|x| \to \infty$ even function on the line. We do not assume that v is smooth. In fact we allow v to have δ-function type singularities. For $N = 2, 3, \ldots$ consider the N-body Hamiltonian

$$H = -\sum_{i=1}^{N} \partial^2/\partial x_i^2 + \sum_{i<j} v(x_i - x_j) \tag{1.1}$$

corresponding to N quantum particles of unit mass interacting via a pair potential v (units are chosen so that the Planck constant is equal to one). To $v = 0$ corresponds the free Hamiltonian $H_0 = -\sum_{i=1}^{N} \partial^2/\partial x_i^2$ which has free eigenstates $f_0(k \mid x) = \exp(\sqrt{-1} \sum_{i=1}^{N} k_i x_i) = \exp(\sqrt{-1} \langle k \mid x \rangle)$ for any N-tuple $k = (k_1, \ldots, k_N) \in R^N$ of momenta.

Let W be the group of permutations of N indices. It naturally acts on the position space $R^N = \{x = (x_1, \ldots, x_N)\}$. Let $C_+ = \{x_1 \leqslant \cdots \leqslant x_N\}$ and for any $w \in W$ set $C_w = wC_+$. The polyhedral cones C_w partition the space $\mathbf{R}^N$ and intersect in hyperplanes $x_i = x_j$, $i \neq j$. For any $w \in W$ denote by C_w^∞ the asymptotic region $x_{w(1)} \ll \cdots \ll x_{w(N)}$ inside C_w.

2. Nondiffractive Pair Potentials

A pair potential v is called nondiffractive if for any $N \geqslant 3$ and any $k = (k_1 \neq \cdots \neq k_N)$ the Hamiltonian (1.1) has an eigenstate $f(k \mid \cdot)$ asymptotic to $\exp(\sqrt{-1} \langle k \mid x \rangle)$ in the asymptotic region C_+^∞ which we denote by

$$f(k \mid x) \sim \exp(\sqrt{-1} \langle k \mid x \rangle). \tag{2.1}$$

Condition (2.1) implies integrability of the Hamiltonian (1.1) in the intuitive sense. Indeed, the correspondence $f_0(k \mid \cdot) \to f(k \mid \cdot)$ of eigenstates of H_0 and H respectively integrates to an operator P on $L_2(\mathbf{R}^N)$ which intertwines H_0 and H, i.e.,

$$HP = PH_0. \tag{2.2}$$

Since $Pf_0(k\,|\,\cdot) = f(k\,|\,\cdot)$, it sends the higher free Hamiltonians ($n \geqslant 2$)

$$H_0^{(n)} = (-\sqrt{-1})^n \sum_{i=1}^{N} \partial^n/\partial x_i^n \tag{2.3}$$

into the higher interacting Hamiltonians

$$H^{(n)} = PH_0^{(n)}P^{-1} \tag{2.4}$$

which commute with each other. Since

$$H^{(n)}f(k\,|\,\cdot) = \left(\sum_{i=1}^{N} k_i^n\right) f(k\,|\,\cdot) \tag{2.5}$$

and in view of the asymptotics (2.1) of $f(k\,|\,\cdot)$, we have

$$H^{(n)} = H_0^{(n)} + \text{lower order terms.}$$

As we will see below the eigenstates $f(k\,|\,\cdot)$ have asymptotics

$$f(k\,|\,\cdot) \sim \sum_{w \in W} a(k, w) \exp(\sqrt{-1}\,\langle wk\,|\,x\rangle) \tag{2.6}$$

in every asymptotic region C_g^∞, $g \in W$, which implies that the incoming $f_{\text{in}}(k\,|\,\cdot)$ and the outgoing $f_{\text{out}}(k\,|\,\cdot)$ scattering states of H are related by

$$f_{\text{out}}(k\,|\,\cdot) = \sum_{w} s(w, k)\, f_{\text{in}}(wk\,|\,\cdot); \tag{2.7}$$

that is, the scattering operator S of H is nondiffractive.

N-body Hamiltonians with nondiffractive scattering have been studied mostly by Bill Sutherland [21]. Sutherland calls the property of Definition 1 the asymptotic Bethe ansatz. Following this terminology we call $f(k\,|\,\cdot)$ the asymptotic Bethe ansatz eigenstates.

The classical analog of conditions (2.1) and (2.7) is as follows. Let N interacting particles $q_1(t), ..., q_N(t)$ separate as $t \to \pm\infty$ (on almost every trajectory) and acquire asymptotic velocities $p_i(\pm\infty)$. The interaction is nondiffractive if the sets of asymptotic velocities $\{p_i(-\infty), i = 1, ..., N\}$ and $\{p_i(+\infty), i = 1, ..., N\}$ coincide; that is, as the result of the interaction the particles merely exchange their velocities. The only known examples of classical nondiffractive potentials are $v(x) = 1/x^2$ and $v(x) = 1/\sinh^2 x$ [19].

Although a good deal is known about the classical many-particle scattering (cf. [11, 13]) very little is known about the classical nondiffractive scattering (the only paper known to me is [16], where nondiffractive potentials are called reflectionless).

Now the intuition behind Definition 1 is explained and we return to the study of quantum nondiffractive potentials. Let s_{12} be the transposition of x_1 and x_2 and consider the union $D_{1,2}$ of C_+ and $s_{12}C_+ = \{x_2 \leqslant x_1 \leqslant x_3 \leqslant \cdots \leqslant x_N\}$. The asymptotic part $D_{1,2}^{\infty}$ of $D_{1,2}$ is given by $\{x_1, x_2 \ll x_3 \ll \cdots \ll x_N\}$. Rewrite the Hamiltonian (1.1) as

$$H = -\tfrac{1}{2}(\partial/\partial x_1 - \partial/\partial x_2)^2 + v(x_1 - x_2) - \tfrac{1}{2}(\partial/\partial x_1 + \partial/\partial x_2)^2 + \left[-\sum_{i=3}^{N} \partial^2/\partial x_i^2 + \sum_{i<j}' v(x_i - x_j) \right], \tag{2.8}$$

where $\sum'$ is the sum over pairs $(i, j) \neq (1, 2)$. In $D_{1,2}^{\infty}$ we can disregard the terms in $\sum'$. Setting $x_1 - x_2 = x$ and $x_1 + x_2 = y$ we have asymptotically in $D_{1,2}^{\infty}$

$$H \sim \left[-\frac{1}{2} d^2/dx^2 + v(x) \right] - \frac{1}{2} d^2/dy^2 + \sum_{i=3}^{N} \partial^2/\partial x_i^2. \tag{2.9}$$

Denote $(k_1 - k_2)/2$ by k, then in C_+^{∞}

$$f(k_1, ..., k_N \mid x_1, ..., x_N) \sim e^{-\sqrt{-1}\,kx} e^{\sqrt{-1}\,(k_1+k_2)\,y/2} \exp\left(\sqrt{-1} \sum_{i=3}^{N} k_i x_i \right). \tag{2.10}$$

We conclude from (2.9) and (2.10) that in the asymptotic region $D_{1,2}^{\infty}$ the variables $x, y, x_i, i = 3, ..., N$, separate and the eigenfunction $f(k_1, ..., k_N \mid \cdot)$ factors into the product of a Jost eigenfunction $\varphi(k \mid x)$ of the Schroedinger operator

$$h = -\tfrac{1}{2} d^2/dx^2 + v(x) \tag{2.11}$$

and the plane wave $\exp(\sqrt{-1}\,[(k_1 + k_2)\,y/2 + \sum_{i=3}^{N} k_i x_i]$. The Jost function has the asymptotics $(k \mid x) \sim e^{\sqrt{-1}\,kx}$ as $x \to -\infty$.

Recall that for every real $k \neq 0$ the Jost eigenfunctions $\varphi(k \mid x)$ and $\varphi(-k \mid x)$ of h have the asymptotics at $x \to +\infty$ given by

$$\begin{aligned} \varphi(k \mid x) &\sim a(k)\, e^{\sqrt{-1}\,kx} + b(k)\, e^{-\sqrt{-1}\,kx} \\ \varphi(-k \mid x) &\sim b(-k)\, e^{\sqrt{-1}\,kx} + a(-k)\, e^{-\sqrt{-1}\,kx}, \end{aligned} \tag{2.12}$$

where the mondromy matrix

$$T(k)=\begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} \tag{2.13}$$

is completely determined by the potential v and satisfies the equations

$$a(-k)=\overline{a(k)}, \qquad b(-k)=\overline{b(k)} \tag{2.14}$$

and

$$\det T(k)=a(k)\,a(-k)-b(k)\,b(-k)=|a(k)|^2-|b(k)|^2=1. \tag{2.15}$$

The monodromy matrix determines the scattering matrix $S(k)$ of h

$$S(k)=\begin{bmatrix} t(k) & r(k) \\ -\bar{r}(k) & \bar{t}(k) \end{bmatrix} \tag{2.16}$$

by

$$t(k)=1/a(-k), \qquad r(k)=-b(k)/a(-k), \tag{2.17}$$

where $t(k)$ and $r(k)$ are respectively the transmission and the reflection coefficients of potential v (cf. [9] for details).

We conclude from the discussion above that in the asymptotic region $s_{1,2}C_+^\infty=\{x_2 \ll x_1 \ll x_3 \ll \cdots \ll x_N\}$ we have

$$\begin{aligned} &f(k_1, ..., k_N \mid x_1, ..., x_N) \\ &\quad \sim a\left(\frac{k_1-k_2}{2}\right)\exp(\sqrt{-1}\,[k_1x_1+k_2x_2+\cdots+k_Nx_N]) \\ &\quad\quad +b\left(\frac{k_1-k_2}{2}\right)\exp(\sqrt{-1}\,[k_2x_1+k_1x_2+k_3x_3+\cdots+k_Nx_N)]. \end{aligned} \tag{2.18}$$

The element $s_{1,2}\in W$ is the reflection across the wall $W_{1,2}=\{x\in \mathbf{R}^N$: $x_1=x_2\}$ separating regions C_+ and $s_{1,2}C_+$ and we have obtained (2.18) by matching the asymptotics of $f(k\mid x)$ in C_+^∞ and $s_{1,2}C_+^\infty$. Continuing the process of matching across the walls we obtain that the asymptotics of $f(k\mid x)$ in each asymptotic region C_w^∞ has the form

$$f(k\mid x)|_{C_w^\infty} \sim \sum_{g\in W} c(k, w, g)\exp(\sqrt{-1}\,\langle gk\mid x\rangle), \tag{2.19}$$

where $c(k, w, g)$ are constants depending on the pair potential v. A sequence of regions C_{w_i}, $i=1, ..., n$, is called a gallery if for any i, C_{w_i} and

$C_{w_{i+1}}$ are separated by a wall. By nature of the matching process, the constants $c(k, w, g)$ in (2.19) depend a priori on the sequence of matchings across the walls, i.e., on the choice of a gallery leading from C_+ to C_w. Since the eigenfunction $f(k \mid x)$ has no more than one asymptotic expansion (2.19) in C_w^∞, we immediately come to a necessary condition for the asymptotic Bethe ansatz.

PROPOSITION 1. *A pair potential v is nondiffractive (for $N \geqslant 3$) only if the constants $c(k, w, g)$ of the asymptotic expansion* (2.19) *are determined by w and do not depend on the choice of the sequence of matchings.*

3. Combinatorics of the Permutation Group

The walls $W_{p,q} = \{x : x_p = x_q\}$ which appear in the process of matching for the asymptotic expansions of $f(k \mid x)$ correspond to the transpositions $s_{p,q} \in W$. In what follows we call them (the transpositions) reflections. A gallery leading to C_w determines a sequence $s_1, ..., s_n$ of reflections such that $w = s_n \cdots s_1$ (see [14] for details). Two different galleries leading to the same region C_w determine two different sequences of reflections $s_1, ..., s_n$ and $s'_1, ..., s'_m$ such that

$$w = s_n \cdots s_1 = s'_m \cdots s'_1, \tag{3.1}$$

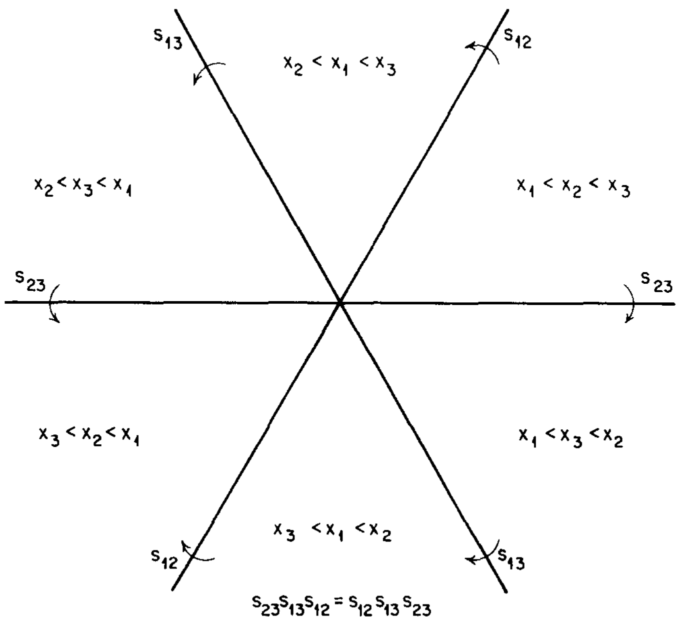

FIGURE 1

i.e., a relation in W. All basic relations in W can be obtained this way. The determining relations in W are very simple (because W is a Coxeter group, cf. [7]) and have the following form. Let $1 \leqslant i < i+1 < i+2 \leqslant N$ be any three consecutive indices and let s_{ij} be the transposition of i and j. The determining relations in W are

$$s_{i+1,i+2} s_{i,i+2} s_{i,i+1} = s_{i,i+1} s_{i,i+2} s_{i+1,i+2}. \tag{3.2}$$

In order to study these relations we can assume without loss of generality that $i=1$. Relation (3.2) becomes

$$s_{23} s_{13} s_{12} = s_{12} s_{13} s_{23} \tag{3.3}$$

which corresponds to the two natural galleries leading from C_+ to C_w, where $w(1, 2, 3) = (3, 2, 1)$ (see Fig. 1).

4. Functional Equation

Now we take $N=3$ and write down two asymptotic expansions (2.19) of $f(k\,|\,\cdot)$ in C_w corresponding to the two galleries leading to C_w (see Fig. 1). First we compute the asymptotic expansion (2.19) obtained by going from C_+ to C_w counterclockwise. By (2.18), we have in $s_{1,2}C_+^\infty$

$$f(k\,|\,x)|_{s_{12}C_+^\infty} \sim a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\,\langle k|x\rangle} + b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\,\langle s_{12}k|x\rangle}. \tag{4.1}$$

We apply the argument of Section 2 to obtain from (4.1) the asymptotic expansion of $f(k\,|\,\cdot)$ in the next asymptotic region $s_{13}s_{12}C_+^\infty$ (see Fig. 1). We have

$$\begin{aligned} f(k\,|\,x)|_{s_{13}s_{12}C_+^\infty} \sim{}& a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\,\langle k|x\rangle} \\ &+ b\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\,\langle s_{13}k|x\rangle} \\ &+ a\left(\frac{k_2-k_3}{2}\right) b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\,\langle s_{12}k|x\rangle} \\ &+ b\left(\frac{k_2-k_3}{2}\right) b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\,\langle s_{13}s_{12}k|x\rangle}. \end{aligned} \tag{4.2}$$

On the next step we obtain the asymptotic expansion of $f(k|\cdot)$ in $s_{23}s_{13}s_{12}C_+^\infty = C_w^\infty$ (see Fig. 1)

$$\begin{aligned} f(k|x)|_{C_w^\infty} \sim\ & a\left(\frac{k_2-k_3}{2}\right) a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle k|x\rangle} \\ & + b\left(\frac{k_2-k_3}{2}\right) a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{23}k|x\rangle} \\ & + a\left(\frac{k_2-k_1}{2}\right) b\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{13}k|x\rangle} \\ & + b\left(\frac{k_2-k_1}{2}\right) b\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{23}s_{13}k|x\rangle} \\ & + a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_2-k_3}{2}\right) b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{12}k|x\rangle} \\ & + b\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_2-k_3}{2}\right) b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{23}s_{12}k|x\rangle} \\ & + a\left(\frac{k_1-k_2}{2}\right) b\left(\frac{k_2-k_3}{2}\right) b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{13}s_{12}k|x\rangle} \\ & + b\left(\frac{k_1-k_2}{2}\right) b\left(\frac{k_2-k_3}{2}\right) b\left(\frac{k_1-k_2}{2}\right) e^{\sqrt{-1}\langle s_{23}s_{13}s_{12}k|x\rangle}. \end{aligned} \tag{4.3}$$

The same way we compute the asymptotic expansion of $f(k|\cdot)$ in C_w^∞ obtained via the clockwise gallery going from C_+ to C_w (see Fig. 1). Omitting the two intermediate steps we write down the answer

$$\begin{aligned} f(k|x)|_{C_w^\infty} \sim\ & a\left(\frac{k_1-k_2}{2}\right) a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_2-k_3}{2}\right) e^{\sqrt{-1}\langle k|x\rangle} \\ & + b\left(\frac{k_1-k_2}{2}\right) a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_2-k_3}{2}\right) e^{\sqrt{-1}\langle s_{12}k|x\rangle} \\ & + a\left(\frac{k_3-k_2}{2}\right) b\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_2-k_3}{2}\right) e^{\sqrt{-1}\langle s_{13}k|x\rangle} \\ & + b\left(\frac{k_3-k_2}{2}\right) b\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_2-k_3}{2}\right) e^{\sqrt{-1}\langle s_{12}s_{13}k|x\rangle} \\ & + a\left(\frac{k_1-k_3}{2}\right) a\left(\frac{k_1-k_2}{2}\right) b\left(\frac{k_2-k_3}{2}\right) e^{\sqrt{-1}\langle s_{23}k|x\rangle} \end{aligned}$$

$$+b\left(\frac{k_1-k_3}{2}\right)a\left(\frac{k_1-k_2}{2}\right)b\left(\frac{k_2-k_3}{2}\right)e^{\sqrt{-1}\langle s_{12}s_{23}k|x\rangle}$$

$$+a\left(\frac{k_2-k_3}{2}\right)b\left(\frac{k_1-k_2}{2}\right)b\left(\frac{k_2-k_3}{2}\right)e^{\sqrt{-1}\langle s_{13}s_{23}k|x\rangle}$$

$$+b\left(\frac{k_2-k_3}{2}\right)b\left(\frac{k_1-k_2}{2}\right)b\left(\frac{k_2-k_3}{2}\right)e^{\sqrt{-1}\langle s_{12}s_{13}s_{23}k|x\rangle}. \quad (4.4)$$

Denote $(k_1-k_2)/2$ by u and $(k_2-k_3)/2$ by v. The right-hand sides of (4.3) and (4.4) must coincide when we rewrite them in the form (2.19). This yields 6 equations corresponding to 6 elements of the permutation group S_3†

$$a(u)\,a(v)\,a(u+v)=a(u)\,a(v)\,a(u+v) \quad (4.5)$$

$$b(u)\,a(v)\,a(u+v)=b(u)\,a(v)\,a(u+v) \quad (4.6)$$

$$a(u)\,b(v)\,a(u+v)=a(u)\,b(v)\,a(u+v) \quad (4.7)$$

$$a(u)\,b(-u)\,b(u+v)+a(u)\,b(u)\,b(v)=a(u)\,b(v)\,b(u+v) \quad (4.8)$$

$$a(v)\,b(u)\,b(u+v)=a(v)\,b(u+v)\,b(-v)+a(v)\,b(u)\,b(v) \quad (4.9)$$

$$a(u)\,a(-u)\,b(u+v)+b^2(u)\,b(v)=a(v)\,a(-v)\,b(u+v)+b^2(v)\,b(u). \quad (4.10)$$

Equations (4.5), (4.6), and (4.7) are idenities, Eqs. (4.8) and (4.9) transform one into another by the change of variables $u\to v$, thus we are left with a system of two equations, (4.8) and (4.10). In virtue of (2.14) and (2.15), $|a(u)|\geqslant 1$ for u real so we divide (4.8) by $a(u)$ and obtain

$$b(v)-b(-u)=\frac{b(u)\,b(v)}{b(u+v)}. \quad (4.11)$$

Since the right-hand side of (4.11) is symmetric in u and v we have

$$b(v)-b(-u)=b(u)-b(-v) \quad (4.12)$$

which implies

$$b(u)+b(-u)=b(v)+b(-v)=\text{const.} \quad (4.13)$$

By the standard asymptotic properties of the scattering matrix

$$\lim_{u\to\infty} a(u)=1, \qquad \lim_{u\to\infty} b(u)=0 \quad (4.14)$$

Eq. (4.13) implies

$$b(u)+b(-u)=0. \quad (4.15)$$

† *Note added in proof.* It is interesting that similar functional equations arise in the study of representations of Hecke algebras [26].

Now (4.11) becomes

$$b(u)+b(v)=\frac{b(u)\,b(v)}{b(u+v)} \tag{4.16}$$

for arbitrary u and v. This is equivalent to

$$b^{-1}(u)+b^{-1}(v)=b^{-1}(u+v) \tag{4.17}$$

unless $b=0$ which also satisfies (4.16). The general solution of (4.17) is

$$b^{-1}(u)=\lambda u, \tag{4.18}$$

where λ is any complex number. For $\lambda=0$ we have $b=0$, and λ must be purely imaginary in view of (2.14). Thus the general solution for b is

$$b(k)=\sqrt{-1}\,ck^{-1}, \qquad c\in R. \tag{4.19}$$

Rewrite Eq. (4.10) as

$$a(u)\,a(-u)-a(v)\,a(-v)=[b(v)-b(-u)]\,\frac{b(u)\,b(v)}{b(u+v)} \tag{4.20}$$

which, in view of (4.16) becomes

$$a(u)\,a(-u)-a(v)\,a(-v)=b^2(v)-b^2(u). \tag{4.21}$$

Using (4.15) we rewrite (4.21) as

$$a(u)\,a(-u)-a(v)\,a(-v)=b(u)\,b(-u)-b(v)\,b(-v) \tag{4.22}$$

which is an identity, by (2.15). Thus the general solution of Eqs. (4.5)–(4.10) which takes (2.14) and (2.15) into account is given by (4.19).

If a pair potential v gives for some $N\geqslant 3$ the N-body Hamiltonian (1.1) satisfying conditions of Definition 1, then, obviously, Eqs. (4.5)–(4.10) hold. On the other hand, since relations in the permutation group S_N are generated by the relations of the type (3.2), it is clear that for $N>3$ the matching argument above does not yield new equations on $a(k)$ and $b(k)$. We state the result as a theorem.

THEOREM 1. *Let $v(x)$ be a nondiffractive pair potential. Then the offdiagonal coefficient $b(k)$ of the monodromy matrix $T(k)$ of the Schroedinger operator h is equal to $\sqrt{-1}\,ck^{-1}$ where c can be any real number.*

5. Discussion

Remark 1. *The case of delta-potential.* Let $v(x) = c\,\delta(x)$ be the delta-potential. An elementary computation shows that the monodromy coefficients of the corresponding Schroedinger operator (2.11) are given by

$$a(k) = 1 - \sqrt{-1}\,ck^{-1}, \qquad b(k) = \sqrt{-1}\,ck^{-1} \tag{5.1}$$

and it is well known that the δ-potential is nondiffractive (cf. [6]). The N-body eigenstates $f(k\,|\,\cdot)$ of Definition 1 are called in this case the Bethe ansatz eigenstates; they were constructed from a different viewpoint and in a more general situation in [15].

Since the δ-potential is suported on the union of hyperplanes $\{x_i = x_j, i < j\}$ one can replace in the argument above the asymptotic regions C_w^∞ by C_w and the matching conditions (4.5)–(4.10) become necessary and sufficient conditions, thus the matching argument of Theorem 1 also proves that $v(x) = c\,\delta(x)$ is a nondiffractive potential. Our argument is reminiscent of the argument C. N. Yang used in [23] to construct symmetric Bethe ansatz eigenstates and Eqs. (4.8), (4.10) are analogous to the Yang–Baxter equations (cf. [4, 25]).

Remark 2. *Multisoliton potentials.* An interesting special case of Eq. (4.19) is $b(u) = 0$ which means, by (2.17), that the reflection coefficient of the Schroedinger operator (2.11) is equal to zero. These are the reflectionless potentials in the theory of the Schroedinger operator (2.11) and they are explicitly known (cf. [24]). These potentials v are famous because of the connection between the Schroedinger operator (2.11) and the Korteweg–DeVries equation (cf. [18]). They correspond to the multisoliton solutions of the Korteweg–DeVries equation. It is remarkable that solitons which are in some sense responsible for the integrability of the KDV equation appear in the study of quantum integrability.

It is not known at this point whether multisoliton potentials are indeed nondiffractive. It is not known even for the simplest case of single soliton

$$v(x) = -a^2 \cosh^{-2}(ax), \tag{5.2}$$

although this potential has been studied in the literature under the name of the Poeschl–Teller potential. Soliton potentials (5.2) also appear in the quantum integrable many-body problem with 2 species of interacting particles studied by Calogero [8].

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