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An Antipodal Set of a Periodic Function

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The well-known Blaschke–Süss theorem states that there are at least three pairs of antipodal points on an oval. In this paper we prove the counterpart of that result for periodic functions and plane curves.

PRELIMINARIES

We recall some results from [1, p. 202]. A plane convex curve is called an oval if it is of class C^2 with respect to arc length s and the curvature $k(s)>0$ for all $s \in R$. Two points of an oval are called an antipodal pair if the tangents at the two points are parallel and the curvatures are equal. The following result is due to W. Blaschke and W. Süss: On every oval there are at least three antipodal pairs of points.

In this paper we introduce antipodal sets for periodic functions that will be used in a generalization of the Blaschke–Süss theorem. For this purpose, we examine the real Hilbert space $L^2_{\langle 0,L\rangle}(k)$ with weight $k(s)$, where

(A) $k(s)>0$,

(B) $k(s+L)=k(s)$, for all $s \in R$,

(C) $\int_0^L k(s)\,ds=2\pi j$, for j an integer.

We define the function

$$K(s)=\begin{cases} \displaystyle\int_0^s k(t)\,dt, & \text{for} \quad s \geqslant 0 \\ \displaystyle -\int_s^0 k(t)\,dt, & \text{for} \quad s<0. \end{cases} \tag{1}$$

It can be easily noted that we have the following.

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THEOREM 1. *The sequence*

$$\frac{1}{\sqrt{2\pi j}}, \frac{1}{\sqrt{\pi j}} \cos \frac{n}{j} K(s), \frac{1}{\sqrt{\pi j}} \sin \frac{n}{j} K(s), \qquad n = 1, 2, \ldots \tag{2}$$

is an orthonormal complete system in the space $L^2_{\langle 0, L \rangle}(k)$.

It is known from [2] that the equation

$$\varphi'(s) = \frac{k(s)}{k(\varphi(s))} \tag{3}$$

has a solution given by

$$\varphi_\alpha(s) = K^{-1}(K(s) + 2\pi j \alpha), \tag{4}$$

where K^{-1} is the inverse function to K and $\alpha \geqslant 0$. The solutions (4) have the following properties:

$$\varphi_\alpha \circ \varphi_\beta(s) = \varphi_\alpha(\varphi_\beta(s)) = \varphi_{\alpha+\beta}(s), \tag{5}$$

$$\varphi_{1/n}(s + L) = \varphi_{1/n}(s) + L, \tag{6}$$

$$(\varphi_{1/n}(s))^n = \overbrace{\varphi_{1/n} \circ \varphi_{1/n} \circ \cdots \circ \varphi_{1/n}}^{n\text{-times}}(s) = s + L. \tag{7}$$

In the case $k(s) \equiv 1$ and $L = 2\pi j$, we have

$$\varphi_\alpha(s) = s + 2\pi j \alpha. \tag{8}$$

Let f be a continuous periodic function with the period L. For a fixed integer $2n$ we set

$$\beta(s) = \varphi_{1/2n}(s) \qquad \text{and} \qquad \beta^v(s) = \overbrace{\beta \circ \beta \circ \cdots \circ \beta}^{v\text{-times}}(s), \tag{9}$$

for $v = 1, 2, \ldots$ and $\beta^0(s) = s$.

DEFINITION 2. For some point $s_0 \in R$ the set

$$\{s_0, \beta(s_0), \beta^2(s_0), \ldots, \beta^{2n-1}(s_0)\} \tag{10}$$

will be called an n-antipodal set provided that the following equation holds:

$$\sum_{v=0}^{n-1} [f \circ \beta^{2v}(s_0) - f \circ \beta^{2v+1}(s_0)] = 0. \tag{11}$$

Let $k(s) \equiv 1$ and $L = 2\pi j$. In this case $\beta(s) = s + (\pi j/n)$ and the n-antipodal set is described as the set of points

$$\left\{s, s + \frac{\pi j}{n}, s + 2\frac{\pi j}{n}, ..., s + (2n-1)\frac{\pi j}{n}\right\} \tag{10'}$$

for which

$$\sum_{v=0}^{n-1}\left[f\left(s + 2v\frac{\pi j}{n}\right) - f\left(s + (2v+1)\frac{\pi j}{n}\right)\right] = 0. \tag{11'}$$

In the sequel, two n-antipodal sets are called equivalent if one of them is determined by s_0 and the second by $s_0 + iL$ for some integer i.

1. Existence of Antipodal Sets

Now we pass to the main result of the paper which is a generalization of the Blaschke–Süss theorem.

THEOREM 3. *Let f be a periodic continuous function with the period $L > 0$. If there exists a continuous function g such that*

1° $g(t + 2\pi j) = g(t)$, *for all* $t \in R$,

2° $g(t - (\pi j/n)) = -g(t)$, *for all* $t \in R$,

3° $g(t) > 0$, *for* $0 < t < \pi j/n$,

4° *the function f is orthogonal to the function $g \circ K$ in the space* $L^2_{\langle 0, L\rangle}(k)$,

then there are at least three non-equivalent n-antipodal sets for the function f.

Proof. Consider the function

$$\lambda(s) = \sum_{v=0}^{n-1} \int_{\beta^{2v}(s)}^{\beta^{2v+1}(s)} f(t)\, k(t)\, dt. \tag{12}$$

In view of Eqs. (3), (5), and (7), we have

$$\beta^i(s) = \varphi_{i/2n}(s) \qquad \text{and} \qquad k(s) = k \circ \beta^i(s)(\beta^i)'(s), \tag{13}$$

for $i = 1, 2, ..., 2n-1$. From the above relations we obtain the derivative of $\lambda(s)$, namely,

$$\lambda'(s) = k(s) \sum_{v=0}^{n-1} [f \circ \beta^{2v+1}(s) - f \circ \beta^{2v}(s)]. \tag{14}$$

Let us define

$$\sigma(s)=\sum_{v=0}^{n-1}[f\circ\beta^{2v}(s)-f\circ\beta^{2v+1}(s)].$$

By (7), it is easy to verify that $\sigma\circ\beta(s)=-\sigma(s)$ for all $s\in R$. Consequently there exists a point s_0 such that $\sigma(s_0)=0$ and then s_0 determines an n-antipodal set. Now we examine the following integral:

$$H=\int_{s_0}^{\beta(s_0)}\sigma(s)\,g\circ K(s)\,k(s)\,ds.$$

Applying the definition of $\sigma(s)$, we rewrite H in the form

$$H=\sum_{v=0}^{n-1}\left[\int_{s_0}^{\beta(s_0)}f\circ\beta^{2v}(s)\,g\circ K(s)\,k(s)\,ds\right.$$
$$\left.-\int_{s_0}^{\beta(s_0)}f\circ\beta^{2v+1}(s)\,g\circ K(s)\,k(s)\,ds\right].$$

Next we observe that

$$K\circ\beta^i(s)=K\circ\varphi_{i/2n}(s)=K\left(K^{-1}\left(K(s)+2\pi j\frac{i}{2n}\right)\right)$$
$$=K(s)+\frac{\pi ij}{n},\qquad i=1,2,\dots,2n-1.$$

Therefore

$$g\circ K(s)=g\left(K\circ\beta^i(s)-\frac{\pi ij}{n}\right)\overset{2^0}{=}(-1)^i g\circ K\circ\beta^i(s),$$

for $i=1,2,\dots,2n-1$, and

$$H=\sum_{v=0}^{n-1}\left[\int_{s_0}^{\beta(s_0)}f\circ\beta^{2v}(s)(-1)^{2v}\,g\circ K\circ\beta^{2v}(s)k\circ\beta^{2v}(s)(\beta^{2v})'(s)\,ds\right.$$
$$\left.-\int_{s_0}^{\beta(s_0)}f\circ\beta^{2v+1}(s)(-1)^{2v+1}\,g\circ K\circ\beta^{2v+1}(s)k\circ\beta^{2v+1}(s)(\beta^{2v+1})'(s)\,ds\right].$$

Changing the variables according to the formula

$$t=\beta^i(s),\qquad i=1,2,\dots,2n-1,$$

we obtain

$$H=\sum_{v=0}^{n-1}\left[\int_{\beta^{2v}(s_0)}^{\beta^{2v+1}(s_0)} f(t)\, g\circ K(t)\, k(t)\, dt + \int_{\beta^{2v+1}(s_0)}^{\beta^{2v+2}(s_0)} f(t)\, g\circ K(t)\, k(t)\, dt\right]$$

$$=\int_{s_0}^{\beta^{2n}(s_0)} f(t)\, g\circ K(t)\, k(t)\, dt \overset{(7)}{=} \int_{s_0}^{s_0+L} f(t)\, g\circ K(t)\, k(t)\, dt$$

$$=\int_0^L f(t)\, g\circ K(t)\, k(t)\, dt.$$

Finally, since f and $g\circ K$ are orthogonal by 4°, we have

$$H=\int_{s_0}^{\beta(s_0)} \sigma(s)\, g\circ K(s)\, k(s)\, ds=\int_0^L f(t)\, g\circ K(t)\, k(t)\, dt=0.$$

The same arguments as in the proof of the Blaschke–Süss theorem guarantee the existence of two further zeros s_1, s_2 lying between s_0 and $\beta(s_0)$ which determine two further n-antipodal sets. Obviously $\beta(s_0)-s_0<L$, and thus those n-antipodal sets are not equivalent. Q.E.D.

COROLLARY 4. *If s_0, s_1, s_2 determine the n-antipodal sets for f by the way described in the proof of Theorem* 3, *then $\lambda(s)$ given by* (12) *has the extremum at each of the points s_0, s_1, s_2.*

Proof. Obviously, $\lambda'=k\sigma$ and the function changes its sign at s_0, s_1, s_2. Q.E.D.

Now we give some examples of functions g satisfying the conditions 1°, 2°, 3°, 4° of Theorem 3.

Let f be a periodic continuous function with the period L and let the Fourier series with respect to the orthonormal and complete system (2) for the function f be given by

$$f(s)=\frac{A_0}{2}+\sum_{n=1}^{\infty} A_n\cos\frac{n}{j}K(s)+B_n\sin\frac{n}{j}K(s). \tag{15}$$

THEOREM 5. *If $B_n=0$, then f given by* (15) *has at least three non-equivalent n-antipodal sets.*

Proof. It is easy to see that the function $g(t)=\sin(n/j)t$ satisfies conditions 1°, 2°, 3° of Theorem 3. Moreover the assumption

$$B_n=\frac{1}{\sqrt{\pi j}}\int_0^L f(s)\, k(s)\sin\frac{n}{j}K(s)\, ds=0$$

implies that f is orthogonal to $g \circ K(s) = \sin(n/j)\, K(s)$. Thus f has at least three non-equivalent n-antipodal sets. Q.E.D.

Now we consider a function f for which $5B_1 + B_3 = 0$. Note that the function $g(t) = 5 \sin t + \sin 3t$ satisfies conditions 1°, 2°, 3° of Theorem 3 for $n = 1$ and $j = 1$. Since $5B_1 + B_3 = 0$, the function f is orthogonal to $g \circ K(s) = 5 \sin K(s) + \sin 3K(s)$ and f has at least three non-equivalent 1-antipodal sets. This example shows that in some special cases assumption 4° of Theorem 3 can be expressed in the following form.

If f is a function for which a certain linear combination vanishes, i.e.,

$$\sum_{v=1}^{m} b_v B_v = 0$$

and if

$$g(t) = \sum_{v=1}^{m} b_v \sin \frac{v}{j} t$$

satisfies conditions 1°, 2°, 3° of Theorem 3, then there exist at least three non-equivalent n-antipodal sets for the function f.

2. Applications to Geometry of Plane Curves

Let $f(s) > 0$ be a continuous periodic function with the period $L > 0$. We define the plane curve $\mathbf{r}_f(s) = (x(s), y(s))$ by

$$x(s) = \begin{cases} \int_0^s f(t)\, k(t) \cos K(t)\, dt, & \text{for } s \geqslant 0, \\ -\int_s^0 f(t)\, k(t) \cos K(t)\, dt, & \text{for } s < 0, \end{cases}$$
$$y(s) = \begin{cases} \int_0^s f(t)\, k(t) \sin K(t)\, dt, & \text{for } s \geqslant 0, \\ -\int_s^0 f(t) \sin K(t)\, dt, & \text{for } s < 0. \end{cases} \tag{16}$$

If f belongs to the C^1-class, then $\mathbf{r}_f(s)$ is a curve of the C^2-class with the curvature $\hat{k}(s)$ equal to $1/f(s)$. Since $\hat{k}(s) > 0$, $\mathbf{r}_f$ is a locally convex curve.

In the case when $f(s) = 1/k(s)$, then Eqs. (16) give the well-known representation of the plane C^2-curve with the curvature $k(s)$ (see for instance [3]). If the curve $\mathbf{r}_{1/k}(s)$ is closed, then for $j=1$ the curve $\mathbf{r}_{1/k}(s)$ is an oval [3], and for $j>1$ the curve is a rosette [2].

Putting $k(s)\equiv 1$ and $L=2\pi j$ in (16) we obtain

$$x(s)=\begin{cases}\displaystyle\int_0^s f(t)\cos t\,dt, & \text{for} \quad s\geqslant 0\\[2ex] \displaystyle-\int_s^0 f(t)\cos t\,dt, & \text{for} \quad s<0\end{cases}$$
$$y(s)=\begin{cases}\displaystyle\int_0^s f(t)\sin t\,dt, & \text{for} \quad s\geqslant 0\\[2ex] \displaystyle-\int_s^0 f(t)\sin t\,dt, & \text{for} \quad s<0.\end{cases} \tag{17}$$

Equations (17) are well known as the angle-integral representation of the plane curves. On the other hand, with the aid of the standard change of variable rule $u=K(s)$ in (16), we obtain the angle-integral representation of the curve $\mathbf{r}_f$ in the form

$$x(u)=\begin{cases}\displaystyle\int_0^\theta r(u)\cos u\,du, & \text{for} \quad \theta\geqslant 0\\[2ex] \displaystyle-\int_\theta^0 r(u)\cos u\,du, & \text{for} \quad \theta<0\end{cases}$$
$$y(u)=\begin{cases}\displaystyle\int_0^\theta r(u)\sin u\,du, & \text{for} \quad \theta\geqslant 0\\[2ex] \displaystyle-\int_\theta^0 r(u)\sin u\,du, & \text{for} \quad \theta<0,\end{cases} \tag{18}$$

where $r(u)=f(K^{-1}(u))$ and $\theta=K(s)$.

This means that formulae (16) and (18) give the same class of curves in a plane whenever f and r are continuous.

In view of arguments from the proof of Lemma 17 [3], it is not cumbersome to prove the following.

THEOREM 6. *For every function f with the period $L>0$ the curve $\mathbf{r}_f(s)$ is closed if and only if*

$$\int_0^L f(s)\,k(s)\cos K(s)\,ds=\int_0^L f(s)\,k(s)\sin K(s)\,ds=0,$$

where $k(s)$ satisfies conditions (A), (B), (C). *Moreover, the length of the curve* $\mathbf{r}_f(s)$ *is equal to*

$$\int_0^L f(s)\,k(s)\,ds.$$

Evidently, Theorem 6 can be rewritten in terms of the Fourier coefficients of f. Indeed,

$$A_j=\frac{1}{\sqrt{\pi j}}\int_0^L f(s)\,k(s)\cos K(s)\,ds, \qquad B_j=\frac{1}{\sqrt{\pi j}}\int_0^L f(s)\,k(s)\sin K(s)\,ds;$$

thus we get the following result.

THEOREM 7. *Let $k(s)$ satisfy conditions* (A), (B), (C). *The curve* $\mathbf{r}_f(s)$ *is closed if and only if the Fourier coefficients A_j and B_j for f vanish, i.e.,*

$$A_j=B_j=0.$$

Note that

$$\mathbf{r}_f(s+2L)-2\mathbf{r}_f(s+L)+\mathbf{r}_f(s)=0. \tag{19}$$

This means that for every fixed parameter s the points $\mathbf{r}_f(s+nL)$, $n=0, \pm 1, \pm 2, \ldots$ form an arithmetical progression, i.e.,

$$\mathbf{r}_f(s+nL)=\mathbf{r}_f(s)+n\cdot\mathbf{W}, \tag{20}$$

where $W=(\sqrt{\pi j}\,A_j, \sqrt{\pi j}\,B_j)$.

Hence one can construct the graph of $\mathbf{r}_f$ in the following way. Let r_L be the part of the graph of $\mathbf{r}_f(s)$ for $0\leqslant s\leqslant L$. Then the whole graph can be obtained translating r_L by the vectors $n\cdot\mathbf{W}$, $n=0, \pm 1, \pm 2, \ldots$.

Now we define an n-antipodal set on a curve $\mathbf{r}_f(s)$.

DEFINITION 8. Let f be a continuous function of the period $L>0$. We shall call an n-antipodal set on $\mathbf{r}_f$ any set of points on the curve $\mathbf{r}_f$ with coordinates

$$\mathbf{r}_f(s),\ r_f\circ\beta(s),\ \ldots,\ r_f\circ\beta^{2n-1}(s), \tag{21}$$

for some $s\in R$ such that Eq. (10) holds true, i.e.,

$$\sum_{v=0}^{n-1}\left[f\circ\beta^{2v}(s)-f\circ\beta^{2v+1}(s)\right]=0.$$

In particular, if $k(s) \equiv 1$ and $L = 2\pi j$, and j is an integer, then the n-antipodal set on the curve $\mathbf{r}_f(s)$ defined by (17) has a simple form

$$\mathbf{r}_f(s),\ \mathbf{r}_f\left(s + \frac{\pi j}{n}\right),\ \mathbf{r}_f\left(s + 2\,\frac{\pi j}{n}\right),\ \ldots,\ \mathbf{r}_f\left(s + (2n-1)\,\frac{\pi j}{n}\right)$$

and the function f satisfies Eq. (11′).

In the above way the notion of an n-antipodal set for a closed curve represented by (18) was introduced in [4].

From Definitions 2 and 9 and Theorem 7, we can derive

THEOREM 9. *If the curve $\mathbf{r}_f(s)$ is closed, then two equivalent n-antipodal sets of the function f generate the same n-antipodal set on the curve $\mathbf{r}_f(s)$. If the curve $\mathbf{r}_f(s)$ is not closed, then there exists a one-to-one correspondence between antipodal sets of the function f and antipodal sets on the curve $\mathbf{r}_f(s)$.*

Now we shall present a geometrical interpretation of an n-antipodal set on the closed convex simple curve $\mathbf{r}_f(s)$. In this case the function $k(s)$ satisfies conditions (A), (B), and (C) for $j = 1$. Moreover its Fourier coefficients A_1 and B_1 vanish (see Theorem 7).

Consider any n-antipodal set on the curve $\mathbf{r}_f(s)$. If f is a C^1-class function, then the curvature of $\mathbf{r}_f(s)$ is equal to $\hat{k}(s) = 1/f(s)$. Hence Eq. (10) can be rewritten as an equality for sums of curvature radii

$$\sum_{v=0}^{n-1} \frac{1}{\hat{k} \circ \beta^{2v}(s)} = \sum_{v=0}^{n-1} \frac{1}{\hat{k} \circ \beta^{2v+1}(s)} \tag{22}$$

in the points defined by (21).

But for each integer $n > 1$ the points (21) are points of tangency of the circumscribed polygon on $\mathbf{r}_f(s)$ with $2n$ sides and the interior angles equal to π/n.

In fact, by (4) and (9) we have

$$\beta(s) = \varphi_{1/2n}() = K^{-1}\left(K(s) + \frac{\pi}{n}\right).$$

Moreover, the tangent vector $(\mathbf{r}_f \circ \beta)'\,(s)$ in the point $\beta(s)$ can be easily computed, namely,

$$(\mathbf{r}_f \circ \beta)'\,(s) = k(s) f \circ \beta(s) \left(\cos\left(K(s) + \frac{\pi}{n}\right), \sin\left(K(s) + \frac{\pi}{n}\right)\right).$$

Denoting by V the rotation by the angle π/n we obtain

$$(\mathbf{r}_f \circ \beta)'\,(s) = \frac{f \circ \beta(s)}{f(s)}\, V(\mathbf{r}_f'(s))$$

and

$$(\mathbf{r}_f \circ \beta^i)'(s) = \frac{f \circ \beta^i(s)}{f \circ \beta^{i-1}(s)} V(r_f' \circ \beta^{i-1})(s),$$

for $i = 2, 3, ..., 2n-1$.

Now we examine the case $n = 1$ and $f = 1/k$.

Clearly, $\mathbf{r}_{1/k}(s)$ is then an oval with curvature $\hat{k}(s) = k(s)$, so the vector tangent to $\mathbf{r}_{1/k}(s)$ is given by

$$r_f'(s) = (\cos K(s), \sin K(s)).$$

Hence

$$r_{1/k}' \circ \beta(s) = (\cos K \circ \beta(s), \sin K \circ \beta(s))$$
$$= (\cos(K(s) + \pi), \sin(K(s) + \pi)) = -r_{1/k}'(s)$$

for all $s \in R$. Thus vectors $\mathbf{r}_{1/k}'(s)$ and $\mathbf{r}_{1/k}' \circ \beta(s)$ are parallel.

Now let s_0 determine a 1-antipodal set. Substituting $n = 1$ in (22) we find that

$$\frac{1}{k(s_0)} = \frac{1}{k \circ \beta(s_0)}.$$

Thus a 1-antipodal set is an antipodal pair [1].

Finally, we can formulate the following counterpart of the Blaschke–Süss theorem for closed curves $\mathbf{r}_f(s)$.

THEOREM 10. *If f is a continuous periodic function with the period $L > 0$ for which Fourier coefficients A_j and B_j vanish, then there exist at least three n-antipodal sets on the closed curve $\mathbf{r}_f(s)$.*

This theorem is a simple consequence of Theorems 5 and 9 and Definitions 2 and 8.

Remark. Obviously, the length of the arc of $\mathbf{r}_f$ between s_1, s_2 is equal to

$$\int_{s_1}^{s_2} f(s)\, k(s)\, ds.$$

Therefore the function $\lambda(s)$ given by (12) is the sum of the lengths of arcs between $\mathbf{r}_f \circ \beta^{2v}(s)$, $\mathbf{r}_f \circ \beta^{2v+1}(s)$, $v = 0, 1, ..., 2n-1$ (without common ends). Hence our Corollary 4 can be reformulated as follows.

COROLLARY 11. *Let $\mathbf{r}_f(s)$ be positively oriented. In this case the sum of the lengths of arcs between the points*

$$r_f \circ \beta^{2v}(s), \qquad r_f \circ \beta^{2v+1}(s), \qquad v = 0, 1, 2, ..., 2n-1,$$

has at least three points of extremum.

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