Available online at www.sciencedirect.com
SCIENCE DIRECT
J. Math. Anal. Appl. 294 (2004) 62–72
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# On lineability of sets of continuous functions

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Received 9 September 2003
Available online 8 April 2004
Submitted by R.M. Aron

**Abstract**

We study the existence of vector spaces of dimension at least two of continuous functions on (subsets of) $\mathbb{R}$, every non-zero element of which admits one and only one absolute maximum.


*Keywords:* Lineability; Spaceability; Linear spaces of continuous functions

## Introduction

In [1], the authors begin by the following: "In many different settings one encounters a problem which, at first glance, appears to have no solution at all. And, in fact, it frequently happens that there is a large linear subspace of solutions to the problem."

A set $M$ in a linear topological space $X$ is said to be $n$-*lineable* (respectively *lineable*, *spaceable*) in $X$ if $M \cup \{0\}$ contains a vector space $Y$ with $\dim Y = n$ (respectively $\dim Y = \dim \mathbb{N}$, $\dim Y = \dim \mathbb{N}$ and $Y$ is closed). If the maximum cardinality of such a vector space exists it is called the *lineability* of $M$ and denoted by $\lambda(M)$. The set $M$ is said to be *totally non-lineable* or *very non-linear* if $\lambda(M) \leqslant 1$. In [1], they give number of such results of "linearity in non-linear problems" in many different fields of analysis (e.g., [3,14] concerning zeros of polynomials, [5,9] concerning hypercyclic operators, [1] concerning non-extendible holomorphic functions...). One of the first results in this spirit

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2004.01.036

is the lineability of the set of nowhere differentiable functions on $[0, 1]$, proved by the first author in [10]. This work has been intensively continued ([8,11] which prove the spaceability, [15] which proves that any separable Banach space is isometrically isomorphic to such a subspace, [12]). Recently, several papers were devoted to the study of the lineability of sets of functions on $[0, 1]$ or $\mathbb{R}$ which satisfy other special properties. For example, P. Enflo and the first author have proved in [7] that for any infinite dimensional subspace $X$ of the space $C[0, 1]$ of continuous functions on $[0, 1]$, the set of functions in $X$ having infinitely many zeros in $[0, 1]$ is spaceable in $X$ and R. Aron, J. Seoane and the first author have shown in [2] that the set of everywhere surjective functions from $\mathbb{R}$ to $\mathbb{R}$ is lineable (in fact, the lineability of this set is equal to $2^c$, the cardinality of the set of all functions from $\mathbb{R}$ to $\mathbb{R}$).

This article takes its place in that program. We study the following question: is it possible to find a vector space of dimension at least two of real-valued continuous functions with (except for the zero function) one and only one absolute maximum? The main results are the following.

**Theorem 6.** *The set $\widehat{C}[0, 1]$ of real-valued continuous functions which admit one and only one absolute maximum is very non-linear in $C[0, 1]$. In other words, $\lambda(\widehat{C}[0, 1]) = 1$.*

**Theorem 9.** *The set $\widehat{C}(\mathbb{R})$ is 2-lineable in $C(\mathbb{R})$.*

**Theorem 16.** *$\lambda(\widehat{C}_0(\mathbb{R})) = 2$, where $C_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ vanishing at infinity.*

We have some other relative results, as the spaceability of the set of continuous and bounded functions on $\mathbb{R}$ without any absolute maximum and answers to the corresponding questions for sets of sequences. Also, we can complete some results obtained in [16] concerning the lineability of the set of continuous functions which attain their supremum norm at a unique point.

We will use the following notations for a function $x$ belonging to $C(K)$ where $K$ is a subset of $\mathbb{R}$: $M(x) := \sup_{t\in K} x(t)$, $m(x) := \inf_{t\in K} x(t)$, $\|x\| := M(|x|)$, $M_x := \{t \in K\colon x(t) = M(x)\}$, $m_x := \{t \in K\colon x(t) = m(x)\}$. We will denote by $\langle x, y\rangle$ the vector space generated by $x$ and $y$, and by $|S|$ the cardinality of a set $S$.

## 1. The very non-linearity of $\widehat{C}[0, 1]$

The main tool in the proof of Theorem 6 will be the notions of *ignorability* and *fence*. Let us introduce these definitions.

**Definition 1.** Let $(x_i)_{i=1}^n$ be a finite set of functions in $C[0, 1]$. A point $t$ in $[0, 1]$ is said to be *ignorable* for $(x_i)_{i=1}^n$ if for every set $(\alpha_i)_{i=1}^n$ of strictly positive real numbers, $t \notin M_{\sum_{i=1}^n \alpha_i x_i}$. A point $t$ in $[0, 1]$ is said to be a *fence* between $t_1$ and $t_2$ in $[0, 1]$ for $(x_i)_{i=1}^n$ if $t \in \,]t_1, t_2[$ and $t$ is ignorable for $(x_i)_{i=1}^n$.

**Definition 2.** A pair of functions $\{x, y\}$ in $C[0,1]$ is said to be *canonical* if $\exists t_x \in M_x$, $\exists t_y \in M_y$, $\exists \tilde{t} \in ]t_x, t_y[$: $m_x = \{\tilde{t}\}$ or $m_y = \{\tilde{t}\}$.

Obviously, we have

**Lemma 3.** *In the canonical situation of Definition* 2, $\tilde{t}$ *is a fence for* $\{x, y\}$ *between* $t_x$ *and* $t_y$.

**Proof.** Let us suppose that $m_x = \{\tilde{t}\}$. Then, $x(\tilde{t}) < x(t_y)$, $y(\tilde{t}) \leqslant y(t_y)$ and $\tilde{t} \notin M_{\alpha x+\beta y}$ for every strictly positive real numbers $\alpha$ and $\beta$. □

A canonical pair of functions cannot be the basis of a two-dimensional vector space $V$ such that $V \setminus \{0\}$ is contained in $\widehat{C}[0,1]$. Indeed,

**Proposition 4.** *For any canonical pair of functions* $\{x, y\}$ *in* $C[0,1]$ *there exist two positive real numbers* $\alpha$ *and* $\beta$ *such that the function* $\alpha x + \beta y$ *has at least two absolute maxima.*

In order to prove this proposition, we will need the following.

**Lemma 5.** *If* $\Phi$ *is a continuous map from* $[0,1]$ *to* $C[0,1]$ *such that for every* $\alpha$ *in* $[0,1]$, $M_{\Phi_\alpha}$ *is a singleton* $\{t_\alpha\}$, *then the map* $\mu$ *defined from* $[0,1]$ *to* $[0,1]$ *by* $\mu(\alpha) = t_\alpha$ *is continuous.*

**Proof of Lemma 5.** Let us suppose that $\alpha \to \alpha_0$ and, by contradiction, let us suppose that $(t_\alpha)$ does not converge to $t_{\alpha_0}$. Since $[0,1]$ is compact, up to a subsequence, $(t_\alpha)$ converges to a point $\tilde{t} \in [0,1]$. We have: $|\Phi_\alpha(t_\alpha) - \Phi_{\alpha_0}(\tilde{t})| \leqslant \|\Phi_\alpha - \Phi_{\alpha_0}\| + |\Phi_{\alpha_0}(t_\alpha) - \Phi_{\alpha_0}(\tilde{t})| \to 0$ when $\alpha \to \alpha_0$. But we have also $M(\Phi_\alpha) = \Phi_\alpha(t_\alpha) \to M(\Phi_{\alpha_0}) = \Phi_{\alpha_0}(t_{\alpha_0})$. Then, $\Phi_{\alpha_0}(\tilde{t}) = \Phi_{\alpha_0}(t_{\alpha_0}) = M(\Phi_{\alpha_0})$ and since $M(\Phi_{\alpha_0}) = \{t_{\alpha_0}\}$, we have $\tilde{t} = t_{\alpha_0}$. This concludes the proof. □

**Proof of Proposition 4.** Let us suppose that there exists a canonical pair of functions $\{x, y\}$ such that for every $(\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, $M_{\alpha x+\beta y}$ is a singleton. Let us consider the map $\Phi$ defined from $[0,1]$ to $C[0,1]$ by $\Phi_\alpha = (1-\alpha)x + \alpha y$ and the map $\mu$ defined from $[0,1]$ to $[0,1]$ by $\mu(\alpha) = t_\alpha$ where $\{t_\alpha\} = M_{(1-\alpha)x+\alpha y}$. By Lemma 5, $\mu$ is continuous and by the intermediate value property (Weierstrass theorem) $\mu$ takes all the values between $\mu(0) = t_0$ and $\mu(1) = t_1$ where $\{t_0\} = M_x$ and $\{t_1\} = M_y$. This is in contradiction with Lemma 3 which asserts that there exists a fence between $t_0$ and $t_1$. This concludes the proof of Proposition 4. □

We can now prove the very non-linearity of $\widehat{C}[0,1]$.

**Theorem 6.** $\lambda(\widehat{C}[0,1]) = 1$.

**Proof.** We want to prove that for any pair of linearly independent functions $\{x, y\}$ in $C[0,1]$ there exists $(\alpha, \beta)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that the function $\alpha x + \beta y$ admits at least two absolute maxima. Let us suppose that it is not true and consider $x$ and $y$ in $C[0,1]$ such that for every $(\alpha,\beta)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$, $M_{\alpha x+\beta y}$ is a singleton. Let us define $\epsilon(x,y) := M_x \cup M_y \cup m_x \cup m_y$. Obviously, $\epsilon(x,y)$ contains at most four points: $|\epsilon(x,y)| \leqslant 4$. We have to consider two cases:

(1) If $|\epsilon(x,y)| \geqslant 3$, one of the four pairs of functions $\{x,y\}$, $\{x,-y\}$, $\{-x,y\}$ or $\{-x,-y\}$ is canonical and, by Proposition 4, we have a contradiction.
(2) If $|\epsilon(x,y)| = 2$. Let us fix $x$ and, if $M_x = M_y$ and $m_x = m_y$, let us replace $y$ by $-y$. Using Lemma 5 as in the proof of Proposition 4, we can find $\alpha \in {]0,1[}$ such that $M_{(1-\alpha)x+\alpha y}$ is different from $M_x$ and $m_x$. So, $|\epsilon(x,(1-\alpha)x+\alpha y)| \geqslant 3$ and the first case gives the contradiction. □

**Remark 7.** Let us note that we can deduce from [16] that $\lambda(\widehat{C}[0,1]) \leqslant 2$ and, actually, even more: the subset $\|\widehat{C}[0,1]\|$ of $C[0,1]$ of functions which attain their supremum norm at a unique point is very non-linear. This approach is connected with the existence of alternating elements in subspaces of $C[0,1]$.

## 2. The lineability of $\widehat{C}(\mathbb{R})$

We will prove that the situation of a close interval of the previous section is rather different from the situation of open or semi-open intervals.

**Proposition 8.** *$\widehat{C}([0,2\pi[)$ is 2-lineable.*

**Proof.** Let us consider the trigonometric functions sine and cosine defined on the semi-open interval $[0,2\pi[$. We have: $\forall(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\exists\theta \in [0,\pi]$: $\alpha\cos+\beta\sin = \sqrt{\alpha^2+\beta^2}\cos(\cdot+\theta)$. Since the function cosine admits one and only one maximum on $[0,2\pi[$, this proves that $\langle\sin,\cos\rangle \setminus \{0\} \subset \widehat{C}([0,2\pi[)$ and concludes the proof. □

We can now easily prove the

**Theorem 9.** *$\widehat{C}(\mathbb{R})$ is 2-lineable.*

**Proof.** The functions $x$ and $y$ defined on $\mathbb{R}$ by

$$x(t) := \mu(t)\cos(4\arctan|t|) \quad \text{and} \quad y(t) := \mu(t)\sin(4\arctan|t|),$$

where $\mu$ is the real-valued continuous function defined on $\mathbb{R}$ by

$$\mu(t) := \begin{cases} \exp t & \text{if } t \leqslant 0, \\ 1 & \text{if } t \geqslant 0, \end{cases}$$

are two linearly independent functions of $C(\mathbb{R})$ such that for every $(\alpha,\beta)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$: $M_{\alpha x+\beta y}$ is a singleton. □

**Remarks.**

(1) We do not know if the set $\widehat{C}(\mathbb{R})$ is $n$-lineable for $n > 3$, lineable or even spaceable. In the next section, we give a negative answer for vanishing functions.

(2) The two-dimensional subspace constructed in this proof is isometric to $\ell_2(2)$. It is impossible to find such a subspace isometric to $\ell_1(2)$. In order to prove that we need the notion of *$\epsilon$-Rademacher* sequence.

A finite sequence $\tilde{e} = (e_1, \ldots, e_n)$ in $C(\mathbb{R})$ is said to be *$\epsilon$-Rademacher* ($\epsilon \geqslant 0$) if there exist $2^n$ distinct points $t_1, \ldots, t_{2^n}$ in $\mathbb{R}$ such that

(a) $\forall i \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, 2^n\}$:

$$\|e_i\| = 1 \quad \text{and} \quad \left|e_i(t_j)\right| \in [1 - \epsilon, 1],$$

(b) $\forall \eta = (\eta_1, \ldots, \eta_n)$ with $\eta_i = \pm 1$, $\exists j \in \{1, \ldots, 2^n\}$ such that

$$\bigl(\operatorname{sign} e_1(t_j), \ldots, \operatorname{sign} e_n(t_j)\bigr) = \eta.$$

If $\tilde{e}$ is $\epsilon$-Rademacher for each $\epsilon \gneq 0$ then $\tilde{e}$ is said to be *almost-Rademacher*. And, if $\tilde{e}$ is 0-Rademacher then $\tilde{e}$ is simply said *Rademacher*.

It is easy to prove that a sequence $\tilde{e} = (e_1, \ldots, e_n)$ in $C(\mathbb{R})$ is isometrically equivalent to the unit basis of $\ell_1(n)$ if and only if $\tilde{e}$ is almost-Rademacher.

If we suppose that there exists a two-dimensional subspace $E$ of $C(\mathbb{R})$ with an almost-Rademacher basis $\tilde{e} = (e_1, e_2)$ such that $E \setminus \{0\} \subset \widehat{C}(\mathbb{R})$, then there are two cases:

(a) $\tilde{e}$ is Rademacher and then one of the four functions $-e_1$, $e_1$, $-e_2$ or $e_2$ has at least two maxima, which is a contradiction.

(b) $\tilde{e}$ is almost-Rademacher but not Rademacher. There exist $t_1$ and $t_2$ in $\mathbb{R}$ such that $e_i(t_i) = 1 = \max_{t \in \mathbb{R}} e_i(t)$, $i = 1, 2$. If $t_1 = t_2$ we define $e := e_1 - e_2$, if not $e := e_1 + e_2$. Since $\tilde{e}$ is almost-Rademacher, for each $\epsilon > 0$ there exists $t \in \mathbb{R}$ such that $e(t) \in [2 - \epsilon, 2[$. But, since $e_1$ and $e_2$ admit one and only one maximum: $\forall t \in \mathbb{R},\ e(t) < 2$. That means that the function $e$ has no maximum and gives a contradiction.

## 3. The 2-lineability of $\widehat{C}_0(\mathbb{R})$

In this paragraph we will prove that there exists a two-dimensional vector subspace $F$ of $C_0(\mathbb{R})$ such that $F \setminus \{0\} \subset \widehat{C}_0(\mathbb{R})$ and that it is impossible to construct such a $n$-dimensional vector subspace for $n > 2$.

Let us recall the notion of inclination.

**Definition 10.** Let $P$ and $Q$ be two closed subspaces of a Banach space $(X, \|.\|)$. The *inclination* of $P$ on $Q$ is defined by

$$\bigl(\widehat{P, Q}\bigr) := \inf\bigl\{d(x, Q)\colon\ x \in P,\ \|x\| = 1\bigr\},$$

where $d(x, Q) := \inf\{\|x - q\|\colon\ q \in Q\}$.

**Remark 11.** Clearly, if $P = \langle x \rangle$ and $Q = \langle y \rangle$ where $x$ and $y$ are linearly independent in $X$, then $(\widehat{P, Q})$ and $(\widehat{Q, P})$ are strictly positive. Moreover, if $(\widehat{P, Q}) = \delta > 0$ and $z = \alpha x + \beta y$ with $x \in P$, $y \in Q$ and $\|x\| = \|y\| = 1$ then $|\alpha| \leqslant \|z\|/\delta$.

**Definition 12.** A real-valued function $x$ defined on a set $K$ is said to be *alternating* if there exist $t_1$ and $t_2$ in $K$ such that $f(t_1) < 0$ and $f(t_2) > 0$. A set of functions is said to be alternating if every non-zero function is alternating.

**Proposition 13.** *It is impossible to find an alternating two-dimensional vector subspace $A$ of $C_0(\mathbb{R})$ such that $A \setminus \{0\} \subset \widehat{C}_0(\mathbb{R})$.*

**Proof.** Let us suppose that there exist $x$ and $y$ two linearly independent functions such that $\langle x, y \rangle \setminus \{0\} \subset \widehat{C}_0(\mathbb{R})$ and $\langle x, y \rangle \setminus \{0\}$ is alternating. Let us consider the set $Z := \{z = \alpha x + \beta y\colon \|z\| = 1\}$. By Remark 11, there exists $\delta > 0$ such that if $z = \alpha x + \beta y \in Z$ then $\alpha$ and $\beta$ belong to $[-1/\delta, 1/\delta]$. Let us put, for every $z = \alpha x + \beta y \in Z$, $m_{\alpha\beta} := \inf\{(\alpha x + \beta y)(t)\colon t \in \mathbb{R}\}$ and $M_{\alpha\beta} := \sup\{(\alpha x + \beta y)(t)\colon t \in \mathbb{R}\}$. We have $\sup\{m_{\alpha\beta}\colon z = \alpha x + \beta y \in Z\} < 0$. Indeed, if not: $\exists (\alpha_n)_{n \geqslant 1}$, $(\beta_n)_{n \geqslant 1} \subset [-1/\delta, 1/\delta]$, $\forall \epsilon > 0$, $\exists n_0 \geqslant 1$, $\forall n \geqslant n_0$: $-\epsilon \leqslant m_{\alpha_n \beta_n} \leqslant 0$. Up to a subsequence, we can assume that $\alpha_n \to \tilde{\alpha}$ and $\beta_n \to \tilde{\beta}$. Since $m_{\alpha_n \beta_n} \to m_{\tilde{\alpha}\tilde{\beta}}$ we have $m_{\tilde{\alpha}\tilde{\beta}} = 0$. That means that $\tilde{z} = \tilde{\alpha} x + \tilde{\beta} y$ is positive which contradicts the fact that $\tilde{z}$ is alternating. In the same way, $\inf\{M_{\alpha\beta}\colon z = \alpha x + \beta y \in Z\} > 0$. Thus, let $N > 0$ be such that: $\forall z \in Z$, $m(z) < -N < 0 < N < M(z)$. Since $x$ and $y$ belong to $C_0(\mathbb{R})$ and since $z = \alpha x + \beta y \in Z$ implies $\alpha, \beta \in [-1/\delta, 1/\delta]$, there exists $T > 0$ such that if $|t| \geqslant T$ and $z \in Z$ then $z(t) \in [-N, N]$. This implies that every $t \in \mathbb{R}$ such that $|t| \geqslant T$ is ignorable for $z \in Z$. So, the problem is reduced on $[-T, T]$: we have $\langle x, y \rangle \setminus \{0\} \subset \widehat{C}([-T, T])$, which contradicts Theorem 6. □

**Proposition 14.** *Every $n$-dimensional $(n > 2)$ vector space of functions contains an $(n-1)$-dimensional alternating subspace.*

In order to prove this proposition we need the following algebraic lemma.

**Lemma 15.** *Let $V$ be an $n$-dimensional $(n \geqslant 2)$ vector space of real-valued functions on a set $K$. There exist $n$ points $(t_j)_{j=1}^n$ in $K$ such that for every $(y_{ij}) \in \mathbb{R}^{n \times n}$, there exist $n$ functions $(Y_i)_{i=1}^n$ of $V$ such that $\forall i, j \in \{1, \dots, n\}$: $Y_i(t_j) = y_{ij}$.*

**Proof of Lemma 15.** Clearly, if $\dim V = n$ then $K$ contains at least $n$ points.

(1) Let us begin by proving by induction that: if $\{X_i\}_{i=1}^n$ is a basis of $V$ then there exist $n$ points $\{t_i\}_{i=1}^n$ in $K$ such that the $n$ vectors $(X_1(t_j))_{j=1}^n, \dots, (X_n(t_j))_{j=1}^n$ are linearly independent.
*For* $n = 2$. Let us suppose, by contradiction, that for every $t_1$, $t_2$ in $K$ the vectors $(X_1(t_1), X_1(t_2))$ and $(X_2(t_1), X_2(t_2))$ are linearly dependent. We can suppose that

there exist $t_0$ in $K$ and $\alpha$ in $\mathbb{R}$ such that $X_2(t_0) = \alpha X_1(t_0) \neq 0$ (if not, the assertion is trivial). Then, we have:

$$\forall t \in K,\ \exists \beta_t \in \mathbb{R}, \quad \big(\beta_t X_1(t_0), \beta_t X_1(t)\big) = \big(X_2(t_0), X_2(t)\big).$$

The equality of the first components implies that for every $t$ in $K$, $\beta_t = \alpha$ and then we have: $\forall t \in K,\ X_2(t) = \alpha X_1(t)$ which contradicts the fact that $X_1$ and $X_2$ are linearly independent in $V$.

*Let us suppose that the assertion is true for* $n = k \geqslant 2$ *and let us prove that it is longer true for* $n = k+1$. Again, by contradiction, let us suppose that for every $\{t_j\}_{j=1}^{k+1} \subset K$, the vectors $(X_1(t_j))_{j=1}^{k+1}, \dots, (X_{k+1}(t_j))_{j=1}^{k+1}$ are linearly dependent. Since the assertion is true for $n = k$, there exist $\{t_1, \dots, t_k\} \subset K$ such that the span of the $k$ vectors $(X_1(t_j))_{j=1}^{k}, \dots, (X_k(t_j))_{j=1}^{k}$ is equal to $\mathbb{R}^k$. Then, there exists an unique sequence $\{\alpha_i\}_{i=1}^{k} \subset \mathbb{R}$ such that

$$\left(\sum_{i=1}^{k} \alpha_i X_i(t_j)\right)_{j=1}^{k} = \big(X_{k+1}(t_j)\big)_{j=1}^{k}.$$

Indeed, since the rank of $(X_i(t_j))_{i,j=1}^{k}$ is equal to $k$, $(\alpha_i)_{i=1}^{k}$ is the unique solution of the system

$$\left(\sum_{i=1}^{k} \beta_i X_i(t_j)\right)_{j=1}^{k} = \big(X_{k+1}(t_j)\big)_{j=1}^{k}.$$

For every $t$ in $K$ the $k+1$ vectors

$$\Big(\big(X_1(t_j)\big)_{j=1}^{k}, X_1(t)\Big), \dots, \Big(\big(X_{k+1}(t_j)\big)_{j=1}^{k}, X_{k+1}(t)\Big)$$

are linearly dependent. Then, for every $t$ in $K$ there exists $(\gamma_i)_{i=1}^{k} \subset \mathbb{R}$ such that

$$\left(\left(\sum_{i=1}^{k} \gamma_i X_i(t_j)\right)_{j=1}^{k}, \sum_{i=1}^{k} \gamma_i X_i(t)\right) = \Big(\big(X_{k+1}(t_j)\big)_{j=1}^{k}, X_{k+1}(t)\Big).$$

The equality of the $k$ first components implies that $\{\gamma_i\}_{i=1}^{k} = \{\alpha_i\}_{i=1}^{k}$ and then we have: $\forall t \in K$, $X_{k+1}(t) = \sum_{i=1}^{k} \alpha_i X_i(t)$ which contradicts the fact that $(X_i)_{i=1}^{k+1}$ are linearly independent in $V$.

(2) Let us suppose that $\dim V = n$ and let us denote by $\{X_i\}_{i=1}^{n}$ a basis of $V$. By the previous step, there exists $(t_j)_{j=1}^{n} \in K$ such that the vectors $(X_1(t_j))_{j=1}^{n}, \dots, (X_n(t_j))_{j=1}^{n}$ are linearly independent. Let us consider the matrix $(y_{ij}) \in \mathbb{R}^{n \times n}$. We have:

$$\forall i \in \{1, \dots, n\},\ \exists \{\alpha_{il}\}_{l=1}^{n} \subset \mathbb{R}: \quad \sum_{l=1}^{n} \alpha_{il} \big(X_l(t_j)\big)_{j=1}^{n} = (y_{ij})_{j=1}^{n}.$$

Then, the functions $\{Y_i\}_{i=1}^{n} \subset V$ defined by $Y_i = \sum_{l=1}^{n} \alpha_{li} X_l$ are such that $Y_i(t_j) = y_{ij}$. $\square$

**Proof of Proposition 14.** Let $V$ be an $n$-dimensional ($n > 2$) vector space of functions on $K$ and let us consider the vector $(1, 1, \dots, 1) \in \mathbb{R}^n$. Clearly, the orthogonal complement of this vector in $\mathbb{R}^n$ is an alternating vector subspace of $\mathbb{R}^n$ of dimension $n - 1$. Let $(y_{1j})_{j=1}^n, \dots, (y_{(n-1)j})_{j=1}^n$ be a basis of this subspace of $\mathbb{R}^n$. By Lemma 15, there exist $n$ points $\{t_j\}_{j=1}^n \subset K$ and $n$ functions $\{Y_i\}_{i=1}^n \subset V$ such that $Y_i(t_j) = y_{ij}$. So, $W = \langle Y_i \rangle_{i=1}^n$ is an alternating subspace of $V$ of dimension $n - 1$. □

We can now easily prove the announced

**Theorem 16.** $\lambda(\widehat{C}_0(\mathbb{R})) = 2$.

**Proof.** We have $\langle \sin, 1 - \cos \rangle \setminus \{0\} \subset \widehat{C}([0, 2\pi[)$ and then, as in the proof of Theorem 9, we have that $\widehat{C}_0(\mathbb{R})$ is 2-lineable. The fact that $\widehat{C}_0(\mathbb{R})$ is not $n$-lineable for $n > 2$ is a straightforward consequence of Propositions 13 and 14. □

If we denote by $C_L(\mathbb{R})$ the set of functions defined on $\mathbb{R}$ such that the limits $\lim_{t\to-\infty} f(t)$ and $\lim_{t\to+\infty} f(t)$ exist, we have the following corollary of Theorem 16:

**Corollary 17.** $\lambda(\widehat{C}_L(\mathbb{R})) = 2$.

**Remark 18.** Using the very non-linearity of $\|\widehat{C}[0, 1]\|$ (see Remark 7) instead of Theorem 6 in the proof of Proposition 13, we can prove that: it is impossible to find an alternating two-dimensional vector subspace $A$ of $C_0(\mathbb{R})$ such that $A \setminus \{0\} \subset \|\widehat{C}_0(\mathbb{R})\|$ (where $\|\widehat{C}_0(\mathbb{R})\|$ is the subset of $C_0(\mathbb{R})$ which attains their supremum norm at a unique point). So, Proposition 14 implies: $\lambda(\|\widehat{C}_0(\mathbb{R})\|) \leqslant 2$. We do not know if this set is 2-lineable or very non-linear.

Surprisingly, the corresponding result for the space of convergent sequences is different: the set $\hat{c}_0$ of vanishing real sequences with an unique maximum is very non-linear.

**Proposition 19.** $\lambda(\hat{c}_0) = 1$.

**Proof.** Let us suppose, by contradiction, that there exist two linearly independent elements $x = (x_n)_{n\geqslant 1}$ and $y = (y_n)_{n\geqslant 1}$ of $c_0$ such that for every $(\alpha, \beta)$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$, $\alpha x + \beta y$ admits one and only one maximum. Without loss of generality we can suppose that $\max_{i\geqslant 1} x_i = x_{i_0} = 1$, $y_{i_0} = 0$ and that there exists $j_0 \neq i_0$ such that $y_{j_0} > 0$. Let $\lambda_{j_0} \in \mathbb{R}_0^+$ be such that $x_{j_0} + \lambda_{j_0} y_{j_0} = 1$ and let us consider $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1/(1 + \lambda_{j_0})$. Since the sequences $x$ and $y$ converge to 0: $\exists N > j_0$, $\forall i \geqslant N$, $\max\{|x_i|, |y_i|\} < \epsilon$. Let us consider $\{y_{i_k}\}_{k=1}^m \subset \{y_i\}_{i=1}^{N-1}$ such that $\forall k \in \{1, \dots, m\}$, $y_{i_k} > 0$ and $\{\lambda_k\}_{k=1}^m \subset \mathbb{R}_0^+$ such that $x_{i_k} + \lambda_k y_{i_k} = 1$. So, $\lambda_0 := \min\{\lambda_k\}_{k=1}^m > 0$. Let us define the sequence $z := x + \lambda_0 y$. It is such that $\max_{i\geqslant 1} z_i = 1$, $z_{i_0} = x_{i_0} = 1$ and $\forall k \in \{1, \dots, m\}$ such that $\lambda_k = \lambda_0$: $z_{i_k} = 1$. Then $z$ has at least two maxima, which is a contradiction. □

The following proposition is proved in [16]. We give here a proof of the same result based on the proof of Proposition 19.

**Proposition 20.** *Let $L \subset c_0$ be a subspace with* $\dim L = n \in \mathbb{N} \setminus \{0\}$. *Then there exists $x \in L$ such that $\|x\|_\infty = 1$ and* $\left|\{i\colon |x_i| = 1\}\right| \geqslant n$.

In particular, this proposition implies that the subset $\|\hat{c}_0\|$ of $c_0$ of sequences which attain their norm at a unique point is very non-linear:

**Corollary 21.** $\lambda(\|\hat{c}_0\|) = 1$.

Since the sup-norm of $c_0$ is Gâteaux-differentiable at $x$ if and only if $t \to |x(t)|$ attains its supremum over $\mathbb{N}$ at a single point $t_0$ and $|x(t_0)| > \sup\{|x(t)|\colon t \in \mathbb{N} \setminus \{t_0\}\}$ (cf. [6]), we have

**Corollary 22.** *The set of points of Frechet-differentiability of the supremum norm of $c_0$ is very non-linear.*

**Proof of Proposition 20.** Let $\{x^1, \ldots, x^n\}$ be a basis of $L$. Let us proceed by induction on the dimension $n$ of $L$. *The case* $n = 1$ is trivial.

*The case* $n = 2$. Let us suppose, by contradiction, that for every $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\alpha x^1 + \beta x^2$ attains its norm at a unique point. Without loss of generality we can suppose that $\|x^1\|_\infty = x^1_{i_0} = 1$, $x^2_{i_0} = 0$ and that there exists $j_0 \neq i_0$ such that $x^2_{j_0} \neq 0$. Let us define the positive real number $\lambda_{j_0}$ such that $x^1_{j_0} + \lambda_{j_0} x^2_{j_0} = \operatorname{sign} x^2_{j_0}$ and consider $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1/(1 + \lambda_{j_0})$. Since the sequences $x^1$ and $x^2$ converge to 0: $\exists N > j_0$, $\forall i \geqslant N$: $\max\{|x^1_i|, |x^2_i|\} < \epsilon$. For $i \in \{1, \ldots, N-1\}$ and such that $x^2_i \neq 0$, let us define $\lambda_i \in \mathbb{R}^+_0$ such that $x^1_i + \lambda_i x^2_i = \operatorname{sign} x^2_i$. Let us consider $\Lambda_0 = \min\{\lambda_i\} > 0$ and the sequence $w^0 = x^1 + \Lambda_0 x^2$. We have $\|w^0\|_\infty = 1$, $w^0_{i_0} = x^1_{i_0} = 1$ and for all $i \in \{1, \ldots, N-1\}$ such that $\lambda_i = \Lambda_0$: $w^0_i = \operatorname{sign} x^2_i$. Then $w^0$ attains its norm at at least two distinct points, a contradiction.

*The case* $n = 3$. Let us suppose, by contradiction, that the proposition is false for $n = 3$. Thus, for $w^0$ defined in the previous step, there exists only one $i_1 \in \mathbb{N}$ such that $\lambda_{i_1} = \Lambda_0$. Without loss of generality we can suppose that $x^3_{i_0} = x^3_{i_1} = 0$ and that there exists $j_1 \notin \{i_0, i_1\}$ such that $x^3_{j_1} \neq 0$. Let us define the positive real number $\lambda_{j_1}$ such that $w^0_{j_1} + \lambda_{j_1} x^3_{j_1} = \operatorname{sign} x^3_{j_1}$ and consider $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1/(1 + \lambda_{j_1})$. Since the sequences $w^0$ and $x^3$ converge to 0: $\exists N > j_1$, $\forall i \geqslant N$: $\max\{|w^0_i|, |x^3_i|\} < \epsilon$. For $i \in \{1, \ldots, N-1\}$ and such that $x^3_i \neq 0$, let us define $\lambda_i \in \mathbb{R}^+_0$ such that $w^0_i + \lambda_i x^3_i = \operatorname{sign} x^3_i$. Let us consider $\Lambda_1 = \min\{\lambda_i\} > 0$ and the sequence $w^1 = w^0 + \Lambda_1 x^3$. We have $\|w^1\|_\infty = 1$, $w^1_{i_0} = w^0_{i_0} = 1$, $w^1_{i_1} = w^0_{i_1} = \operatorname{sign} x^2_{i_1}$ and for all $i \in \{1, \ldots, N-1\}$ such that $\lambda_i = \Lambda_1$: $w^1_i = \operatorname{sign} x^3_i$. Then $w^1$ attains its norm at at least three distinct points, a contradiction.

We can now use the same idea to perform the step $n = 4$ and so on. □

Let us remark that a statement similar to Proposition 20 which would say that if $L$ is an $n$-dimensional subspace of $c_0$ then there exists $x \in L$ such that $\|x\|_\infty = 1$ and $|\{i\colon x_i = 1\}| \geqslant n$, is false. Indeed, in [16] the author give the following example: $L = \langle (1, 1, -2), (1, -2, 1) \rangle \subset \mathbb{R}^3$ is such that it is impossible to find $x \in L$, $\|x\|_\infty = 1$ which has the value 1 in two coordinates.

## 4. The spaceability of $\widetilde{CB(\mathbb{R})}$

Let us consider the set $\widetilde{CB}(\mathbb{R})$ (respectively $\|\widetilde{CB}(\mathbb{R})\|$) of continuous and bounded real-valued functions defined on $\mathbb{R}$ which do not attain their supremum (respectively their supremum norm).

**Theorem 23.** *$\widetilde{CB}(\mathbb{R})$ and $\|\widetilde{CB}(\mathbb{R})\|$ are spaceable.*

By linear interpolations and symmetrisation, this theorem is a straightforward corollary of the corresponding following result concerning sequences:

**Proposition 24.** *$\widetilde{\ell_\infty}$ and $\|\widetilde{\ell_\infty}\|$ are spaceable.*

**Proof.** Let us consider the set of sequences $\{e_n\}_{n\geqslant 1} \subset \ell_\infty$, defined by

$$e_n := \sum_{i=1}^{+\infty} (-1)^i \left(1 - 1/2^i\right) b_{(p_n)^i},$$

where $\{b_n\}_{n\geqslant 1}$ denotes the canonical basis of $\ell_1$ and $p_n$ the $n$th prime number. For every $N \geqslant 1$, we have:

$$\left\| \sum_{n\geqslant 1}^{N} \alpha_n e_n \right\|_\infty = \max_{1\leqslant n\leqslant N} |\alpha_n|,$$

which implies that $\{e_n\}_{n\geqslant 1}$ is a (monotone) basic sequence. Obviously, we have:

$$\forall i \geqslant 1, \quad \left| \left( \sum_{n=1}^{+\infty} \alpha_n e_n \right)_i \right| < \left\| \sum_{n=1}^{+\infty} \alpha_n e_n \right\|_\infty = \sup_{n\geqslant 1} |\alpha_n| = \sup_{i\geqslant 1} \left( \sum_{n=1}^{+\infty} \alpha_n e_n \right)_i.$$

This proves that $E = \langle e_n \rangle_{n\geqslant 1}$ is an infinite dimensional closed vector subspace of $\ell_\infty$ such that $E \setminus \{0\} \subset \widetilde{\ell_\infty} \cap \|\widetilde{\ell_\infty}\|$. □

The idea to use sequences $e_n$ with pairwise disjoint support was suggested by the referee of the paper. The idea to use a basic sequence such that the (easily described) closed linear span generated by it satisfies a given property already appears in [4,13].


## Acknowledgments

The authors thank the referee for his helpful remarks. This work was initiated during a visit of the second author at Kent State University. He thanks the members of the Department of Mathematics for their wonderful hospitality. This visit was supported by the grant "Bourse de Voyages 2002" of the Ministère de la Recherche Scientifique en Communauté Française de Belgique.

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