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Minimal Normalization of Wiener–Hopf Operators in Spaces of Bessel Potentials

A. Moura Santos, F.-O. Speck, and F. S. Teixeira

Departamento de Matemática, Instituto Superior Técnico, U.T.L., Av. Rovisco Pais,

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A class of operators is investigated which results from certain boundary and transmission problems, the so-called Sommerfeld diffraction problems. In various cases these are of normal type but not normally solvable, and the problem is how to normalize the operators in a physically relevant way, i.e., not loosing the Hilbert space structure of function spaces defined by a locally finite energy norm. The present approach solves this question rigorously for the case where the lifted Fourier symbol matrix function is Hölder continuous on the real line with a jump at infinity. It incorporates the intuitive concept of compatibility conditions which is known from some canonical problems. Further it presents explicit analytical formulas for generalized inverses of the normalized operators in terms of matrix factorization.

1. INTRODUCTION

Let $H^s = H^s(\mathbb{R})$ denote the spaces of Bessel potentials of order $s \in \mathbb{R}$ [7]. Further let H^s_+ be the subspace of H^s formed by all distributions supported on $\overline{\mathbb{R}}_+$ in the sense of $\mathscr{S}' = \mathscr{S}'(\mathbb{R})$ and let $H^s(\mathbb{R}_+) = r_+ H^s$ be the space of restrictions on $\mathbb{R}_+$; see the Appendix for more details. In the vector case the same notation is used with $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$,

$$H^s = \bigtimes_{j=1}^{n} H^{s_j}, \qquad H^s_+ = \bigtimes_{j=1}^{n} H^{s_j}_+, \qquad H^s(\mathbb{R}_+) = \bigtimes_{j=1}^{n} H^{s_j}(\mathbb{R}_+). \tag{1.1}$$

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The central topic of our investigations is a class of *Wiener–Hopf operators* (WHOs) acting between spaces of Bessel potentials,

$$W = W(\Phi) = r_+ A\big|_{H^r_+} \colon H^r_+ \to H^s(\mathbb{R}_+), \qquad r, s \in \mathbb{R}^n, \tag{1.2}$$

where r_+ acts componentwise and A is a translation invariant homeomorphism between H^r and H^s. This is equivalent to writing $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$, where

$$\Phi \in L^\infty_{\mathrm{loc}}(\mathbb{R})^{n\times n}, \qquad \Phi_{jl} = \mathcal{O}(|\xi|^{r_l - s_j}), \qquad |\xi| \to \infty, \qquad j, l = 1, \ldots, n. \tag{1.3}$$

The inverse Fourier symbol matrix function Φ^{-1} also satisfies the conditions (1.3), where r and s are exchanged. Lifting the WHO into L^2 (cf. Theorem A.1), we obtain an equivalent *lifted WHO*

$$W_0 = r_+ A_0\big|_{[L^2_+]^n} \colon [L^2_+]^n \to L^2(\mathbb{R}_+)^n, \tag{1.4}$$

where $A_0 = \mathcal{F}^{-1}\Phi_0 \cdot \mathcal{F}$ and $\Phi_0 \in L^\infty(\mathbb{R})^{n\times n}$. As a general assumption let

$$\Phi_0 \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})^{n\times n} \tag{1.5}$$

for some $\nu \in]0, 1[$, i.e., the elements of Φ_0 are Hölder continuous on $\ddot{\mathbb{R}} = [-\infty, +\infty]$ and the matrix does not degenerate there. This class of operators is of particular importance in certain applications [18] (see also Sections 5 and 6).

For the sake of simplicity we shall focus first on the case where A is scalar ($n = 1$) and acts "symmetrically" ($r = s$). Let us denote by W_s the corresponding scalar WHO,

$$W_s = W_s(\Phi) = r_+ A\big|_{H^s_+} \colon H^s_+ \to H^s(\mathbb{R}_+), \tag{1.6}$$

where $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$ with $\Phi \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$ and $s \in \mathbb{R}$.

Note that the question of operators acting between spaces of different order,

$$W = r_+ A\big|_{H^{s_1}_+} \colon H^{s_1}_+ \to H^{s_2}(\mathbb{R}_+),$$

where $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$, $\tilde{\Phi}_0 = \lambda_-^{s_2}\Phi\lambda_+^{-s_1} \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$, and $\lambda^s_\pm(\xi) = (\xi \pm k_0)^s$, [cf. (2.1)], can be completely reduced to the case $s_1 = s_2 = s$ and treated analogously. A generalization to the system's case (1.2) will be described later, in Section 6.

A number $s \in \mathbb{R}$ is said to be *critical* if W_s is not normally solvable, i.e., if the image of the WHO is not closed. It is well known (cf. Corollary A.6) that these numbers are given by

$$s = s_1 + k - \frac{1}{2}, \qquad k \in \mathbb{Z}, \qquad s_1 = -\frac{1}{2\pi}\big[\arg\Phi(+\infty) - \arg\Phi(-\infty)\big]. \tag{1.7}$$

For $s - s_1 + 1/2 \notin \mathbb{Z}$, W_s is a one-sided invertible Fredholm operator with known explicit formulas for the generalized inverses in terms of a factorization of Φ [cf. (A.34)]. For the critical numbers we shall study a problem of the following type.

Normalization Problem (for bounded operators) [13, 14]. Let X_0, Y_0 be Banach spaces and $M \subset \mathscr{L}(X_0, Y_0)$ be a set of bounded linear operators. Find another pair of Banach spaces X_1, Y_1 such that

$$X_0 \cap X_1 \subset X_j, \qquad Y_0 \cap Y_1 \subset Y_j, \qquad j = 0, 1, \tag{1.8}$$

are dense, any $T \in M$ maps $X_0 \cap X_1$ into Y_1, and has a continuous extension $\overline{T}$ in the sense

$$\overline{T} = \operatorname{Ext} T\big|_{X_0 \cap X_1} \colon X_1 \to Y_1, \tag{1.9}$$

which is normally solvable. In this case, we write $(X_1, Y_1) \in \mathscr{N}(M)$; see, e.g., [6, 12, 25] for similar concepts.

Since the embeddings

$$H_+^{s_2} \subset H_+^{s_1}, \qquad H^{s_2}(\mathbb{R}_+) \subset H^{s_1}(\mathbb{R}_+), \qquad s_1 < s_2, \tag{1.10}$$

are dense, the normalization problem for

$$M_s = \left\{ W_s = W_s(\Phi) \colon \Phi \in \mathscr{G}C^{\nu}(\ddot{\mathbb{R}}), \nu \in \left]0, 1\right[, \ \operatorname{im} W_s \neq \overline{\operatorname{im} W_s} \right\} \tag{1.11}$$

can be easily solved by

$$(H_+^{s+\varepsilon}, H^{s+\varepsilon}(\mathbb{R}_+)) \in \mathscr{N}(M_s), \qquad \varepsilon \notin \mathbb{Z}. \tag{1.12}$$

This is the strategy in many papers [22, 27, 28] and it seems to be one of the most natural methods of normalization from the viewpoint of operator theory [7]. Another solution is possible with the help of Sobolev–Slobodecki spaces $W^{p,s}$, $p \neq 2$ [21].

However, sometimes it is important not to change the topologies of X_0 and Y_0 simultaneously, particularly in applications to mathematical physics, where the energy norm ($s = 1$) plays a fundamental role [18] (see also next remark). This leads to the question whether a normalization problem is solvable under one of the additional assumptions

$$X_1 = X_0, \qquad Y_1 \subset Y_0 \quad \text{or} \quad X_1 \supset X_0, \qquad Y_1 = Y_0, \tag{1.13}$$

respectively. We call each of them a *minimal normalization problem* and denote the corresponding normalized operators by

$$\overset{<}{T} = \operatorname{Rst} T \colon X_0 \to Y_1 \quad \text{and} \quad \overset{>}{T} = \operatorname{Ext} T \colon X_1 \to Y_0 \tag{1.14}$$

provided $(X_0, Y_1) \in \mathcal{N}(M)$ or $(X_1, Y_0) \in \mathcal{N}(M)$ holds, respectively. Also we speak about *image/domain normalization* in these two particular cases.

In the present paper we shall first solve these two problems for the subset of M_s defined by

$$M_{s,c} = \left\{ W_s(\Phi) \in M_s \colon \left| \frac{\Phi(+\infty)}{\Phi(-\infty)} \right| = c \right\}, \qquad s \in \mathbb{R}, c \in \mathbb{R}_+. \tag{1.15}$$

Further, one-sided inverses of $\overset{<}{W}_s$ and $\overset{>}{W}_s$ will be presented in terms of a factorization of Φ and the stability of defect numbers will be proved for the set

$$\left\{ W_s = W_s(\Phi) \colon H_+^s \to H^s(\mathbb{R}_+), s_0 < s < s_0 + 1 \right\} \cup \left\{ \overset{<}{W}_{s_0}, \overset{>}{W}_{s_0+1} \right\}, \tag{1.16}$$

where the Fourier symbol is fixed. In particular, this yields the index jump formula

$$\operatorname{Ind} \overset{>}{W}_{s_0} - \operatorname{Ind} \overset{<}{W}_{s_0} = 1.$$

A generalization to the system's case is discussed and applications are shown for the concept of image normalization in two classes of Sommerfeld diffraction problems: the impedance problem [17, 20, 27] and the oblique derivative problem [11, 22].

REMARK 1.1. The present normalization problems resulted from certain boundary and transmission problems in mathematical physics, the so-called *Sommerfeld diffraction problems* [18, 23]. There we consider, as the simplest case, a domain $\Omega = \mathbb{R}^2 \setminus \Sigma$ with $\partial\Omega = \Sigma = \{(x_1, x_2) \colon x_1 \geq 0, x_2 = 0\}$ (identified with the above-mentioned half-line $\mathbb{R}_+$) and we look for solutions of the Helmholtz equation (and other elliptic equations) in $H^1(\Omega)$ representing waves with a locally finite energy and a reasonable radiation condition at infinity. This implies layer potentials and given data in $H^s(\mathbb{R}_+)$, $s = 1/2 - k$, $k \in \mathbb{N}_0$. Actually, many problems are not well posed in the most intuitive space setting. Now we give the reasons to use our method:

1. Although it is possible and well known that the boundary pseudodifferential operators can be normalized by changing from $H^s(\mathbb{R}_+)$ to $H^{s+\varepsilon, p}(\mathbb{R}_+, \varrho)$ or $W^{s+\varepsilon, p}(\mathbb{R}_+, \varrho)$ with or without some weight function ϱ, we like to stick or return to the original parameters $(s, p) = (1/2 - k, 2)$ and $\varrho = 1$ somehow at the end, for physical reasons. This is not possible (at least directly) if we deal with *p normalization* [see, e.g., the embedding arguments after (4.8)].

2. Thinking of a change of spaces "as little as possible," we may ask for a solution $(\tilde{X}_1, \tilde{Y}_1)$ of the normalization problem for an operator class M that satisfies

$$X_0 \subset \tilde{X}_1 \subset X_1, \qquad Y_1 \subset \tilde{Y}_1 \subset Y_0$$

for every $(X_1, Y_1) \in \mathcal{N}(M)$ with $X_0 \subset X_1$, $Y_1 \subset Y_0$. This is not solvable for the operator classes under consideration [cf. (2.17)], but if we restrict on image or domain normalization (putting $\tilde{X}_1 = X_0$ or $\tilde{Y}_1 = Y_0$ *a priori*), the normalization problem turns out to be uniquely solvable. This cannot be achieved by a (pure) change of p, but only by a (temporary) change of s.

3. In principle there are two concepts of minimal image normalization: a mathematical one (described before) and a physical concept formulated by compatibility conditions (in the simplest case, jumps of the layer potentials are extendable by zero). We can conclude that the two concepts lead to the same unique solution (which is also impossible to obtain by a pure change of p). Minimal domain normalization of the reduced equivalent operators $V_{\pm 1/2}$ is obtained by duality; see (4.2).

Additionally it should be pointed out that the present approach yields also generalized inverses of the minimally normalized operators by extension and restriction, respectively. All this makes the minimal normalization concept most attractive.

2. MAIN RESULT IN THE SCALAR CASE

Let k_0, $\omega \in \mathbb{C}$, $\operatorname{Im} k_0 > 0$, and

$$\lambda_{\pm}^{\omega}(\xi) = (\xi \pm k_0)^{\omega} = \exp\{\omega \log(\xi \pm k_0)\}, \qquad \xi \in \mathbb{C} \setminus \Gamma, \tag{2.1}$$

with vertical branch cuts $\Gamma_{\mp} (\Gamma = \Gamma_+ \cup \Gamma_-)$ taken from $\mp k_0$ to infinity not crossing the real line. This notation is common in diffraction theory [18].

LEMMA 2.1. *For any $\Phi \in \mathcal{G}C^{\nu}(\ddot{\mathbb{R}})$, $0 \le \nu < 1$, there exists a unique number $\omega \in \mathbb{C}$ and a function $\Psi \in \mathcal{G}C^{\nu}(\dot{\mathbb{R}})$ such that*

$$\Phi = \left(\frac{\lambda_-}{\lambda_+}\right)^{\omega} \Psi, \qquad \operatorname{ind} \Psi = \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Psi = 0, \qquad \Psi(+\infty) = \Phi(+\infty), \tag{2.2}$$

and vice versa.

Proof. Inspired by a similar notation [4, p. 48], let

$$\omega = \frac{1}{2\pi i} \int_{\mathbb{R}} d \log \Phi. \tag{2.3}$$

Then we write $\omega = \sigma + i\tau$ with real and imaginary parts given by

$$\sigma = \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Phi, \qquad \tau = \frac{1}{2\pi} \log \left| \frac{\Phi(-\infty)}{\Phi(+\infty)} \right|, \tag{2.4}$$

respectively. From the first formula of (2.4) it is possible to conclude that the argument increase along $\mathbb{R}$ of $(\lambda_-/\lambda_+)^\omega$ coincides with that of Φ, and from definition (2.1),

$$\lim_{\xi\to\pm\infty}\left|\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^\omega\right| = \lim_{\xi\to\pm\infty}\exp\left\{-\tau\arg\left(\frac{\xi-k_0}{\xi+k_0}\right)\right\}$$
$$= \begin{cases} 1, & \text{at } +\infty, \\ e^{2\pi\tau}, & \text{at } -\infty. \end{cases} \tag{2.5}$$

This implies the representation (2.2), and the inverse conclusion is obvious. ∎

The following notation is needed. The *Bessel potential operators*

$$\Lambda_\pm^\omega = \mathscr{F}^{-1}\lambda_\pm^\omega \cdot \mathscr{F}\colon H^s \to H^{s-\operatorname{Re}\omega} \tag{2.6}$$

are bounded invertible for $s \in \mathbb{R}$, $\omega \in \mathbb{C}$, as well as the operators

$$\Lambda_+^\omega\big|_{H_+^s}\colon H_+^s \to H_+^{s-\operatorname{Re}\omega}, \qquad r_+\Lambda_-^\omega \ell^{(s)}\colon H^s(\mathbb{R}_+) \to H^{s-\operatorname{Re}\omega}(\mathbb{R}_+), \tag{2.7}$$

where $\ell^{(s)}\varphi$ denotes any extension of $\varphi \in H^s(\mathbb{R}_+)$ to $\ell^{(s)}\varphi \in H^s$ (see [7, Theorem 4.4 and Lemma 4.6]). For the particular orders $s = \pm 1/2$ we shall need the spaces

$$\tilde{H}_s(\mathbb{R}_+) = r_+H_+^s \tag{2.8}$$

as subspaces of $H^s(\mathbb{R}_+)$, but equipped with the topology of H_+^s. The spaces $\tilde{H}_s(\mathbb{R}_+)$ are continuously embedded into $H^s(\mathbb{R}_+) = r_+H^s$ and represent proper dense subspaces [cf. (A.9) and (A.10)]. These facts justify the following notation.

DEFINITION 2.1. For every $\omega \in \mathbb{C}$, let

$$\overset{<}{H}{}^\omega(\mathbb{R}_+) = r_+\Lambda_-^{-\omega-1/2}H_+^{-1/2} \subset H^{\operatorname{Re}\omega}(\mathbb{R}_+), \tag{2.9}$$

equipped with the norm induced by $H_+^{-1/2}$. This means

$$\overset{<}{H}{}^\omega(\mathbb{R}_+) = \left\{\psi \in H^{\operatorname{Re}\omega}(\mathbb{R}_+)\colon \varphi = r_+\Lambda_-^{\omega+1/2}\ell^{(\operatorname{Re}\omega)}\psi \in \tilde{H}_{-1/2}(\mathbb{R}_+)\right\} \tag{2.10}$$

i.e., φ is extendable by zero from $H^{-1/2}(\mathbb{R}_+)$ into $H^{-1/2}$ and

$$\|\psi\|_{\overset{<}{H}{}^\omega(\mathbb{R}_+)} = \|\varphi\|_{\tilde{H}_{-1/2}(\mathbb{R}_+)} = \|\ell_0\varphi\|_{H^{-1/2}}. \tag{2.11}$$

Further, we define for $\omega \in \mathbb{C}$,

$$\overset{>}{H}{}_+^\omega = \operatorname{clos}\left\{\psi \in H_+^{\operatorname{Re}\omega}\colon \|\psi\|_{\overset{>}{H}{}_+^\omega} = \left\|r_+\Lambda_+^{\omega-1/2}\psi\right\|_{H^{1/2}(\mathbb{R}_+)}\right\}. \tag{2.12}$$

Note that from definition (2.9) the space $\overset{<}{H}{}^{-1/2}(\mathbb{R}_+)$ coincides with $\tilde{H}_{-1/2}(\mathbb{R}_+)$ and from (2.12), $\overset{>}{H}{}^{1/2}_{+}$ is the closure of $H^{1/2}_{+}$ in the norm induced by $H^{1/2}(\mathbb{R}_+)$. It is evident that all these spaces do not depend on k_0, as different numbers k_0 with $\operatorname{Im} k_0 > 0$ generate equivalent norms.

COROLLARY 2.2. *For any $\omega \in \mathbb{C}$, the embeddings*

$$\overset{<}{H}{}^{\omega}(\mathbb{R}_+) \subset H^{\operatorname{Re}\omega}(\mathbb{R}_+), \qquad \overset{>}{H}{}^{\omega}_{+} \supset H^{\operatorname{Re}\omega}_{+} \tag{2.13}$$

are proper, dense, and continuous. Further, for $k \in \mathbb{N}_0$,

$$\overset{<}{H}{}^{-k-1/2}(\mathbb{R}_+) = r_+ H_+^{-k-1/2}, \qquad \overset{>}{H}{}^{k+1/2}_{+} \cong H_0^{k+1/2}(\mathbb{R}_+), \tag{2.14}$$

where the last space is the closure of $\mathcal{D}(\mathbb{R}_+) = C_0^\infty(\mathbb{R}_+)$ with respect to the norm of $H^{k+1/2}(\mathbb{R}_+)$ [cf. (A.6)].

Proof. We prove the properties of the first relation in (2.13). For the second embedding we can use duality arguments. By definition (2.9) it is obvious that if $\psi \in \overset{<}{H}{}^{\omega}(\mathbb{R}_+)$, then $\psi \in H^{\operatorname{Re}\omega}(\mathbb{R}_+)$. To show that the inclusion is proper, let $\psi \in H^{\operatorname{Re}\omega}(\mathbb{R}_+)$. Thus $\ell^{(\operatorname{Re}\omega)}\psi \in H^{\operatorname{Re}\omega}$, which implies

$$\ell^{(\operatorname{Re}\omega)}\psi = \Lambda_-^{-\omega-1/2}\Lambda_+^{\omega+1/2}\ell^{(\operatorname{Re}\omega)}\psi = \Lambda_-^{-\omega-1/2} g, \quad \text{i.e., } \psi = r_+\Lambda_-^{-\omega-1/2} g,$$

with $g \in H^{-1/2}$. Comparing with (2.9) we conclude that

$$\overset{<}{H}{}^{\omega}(\mathbb{R}_+) \subset H^{\operatorname{Re}\omega}(\mathbb{R}_+) = r_+\Lambda_-^{-\omega-1/2} H^{-1/2},$$

where the inclusion is strict. The continuity is easily proved from (2.10):

$$\begin{aligned}
\left\| \tilde{\ell}^{(\operatorname{Re}\omega)}\psi \right\|_{H^{\operatorname{Re}\omega}} &= \left\| \tilde{\ell}^{(\operatorname{Re}\omega)} r_+ \Lambda_-^{-\omega-1/2}\ell_0\varphi \right\|_{H^{\operatorname{Re}\omega}} \\
&= \left\| \Lambda_+^{\omega+1/2}\tilde{\ell}^{(\operatorname{Re}\omega)} r_+ \Lambda_-^{-\omega-1/2}\ell_0\varphi \right\|_{H^{-1/2}} \\
&\le M \|\psi\|_{\overset{<}{H}{}^{\omega}(\mathbb{R}_+)},
\end{aligned}$$

where $\tilde{\ell}^{(\operatorname{Re}\omega)}\psi \in H^{\operatorname{Re}\omega}$ is any extension of ψ.

For the density we show that every $\psi \in H^{\operatorname{Re}\omega}(\mathbb{R}_+)$ is the limit of a sequence $\psi_n \in \overset{<}{H}{}^{\omega}(\mathbb{R}_+)$, i.e.,

$$\left\| \psi - \psi_n \right\|_{H^{\operatorname{Re}\omega}(\mathbb{R}_+)} \to 0, \qquad n \to \infty.$$

By definition of the norm of $H^{\operatorname{Re}\omega}(\mathbb{R}_+)$ and (2.9),

$$\|\psi - \psi_n\|_{H^{\operatorname{Re}\omega}(\mathbb{R}_+)} = \inf_{\ell^{(\operatorname{Re}\omega)}} \left\| \ell^{(\operatorname{Re}\omega)}(\psi - \psi_n) \right\|_{H^{\operatorname{Re}\omega}} = \inf_{\ell^{(\operatorname{Re}\omega)}} \left\| f - \Lambda_-^{-\omega-1/2}\varphi_n \right\|_{H^{\operatorname{Re}\omega}}$$

with $f \in H^{\operatorname{Re}\omega}$, $\varphi_n \in H_+^{-1/2}$, $n \in \mathbb{N}$. Since we can approximate each $\varphi_n \in H_+^{-1/2}$ by $\varphi_\varepsilon \in \ell_0 C_0^\infty(\mathbb{R}_+)$ (see [7, Lemma 4.3]), there exists an order $p \in \mathbb{N}$, such that for all $n > p$,

$$\|\varphi_n - \varphi_\varepsilon\|_{H_+^{-1/2}} \to 0, \qquad \varepsilon \to 0.$$

Therefore, the following estimate holds for any $n > p$,

$$\|\psi - \psi_n\|_{H^{\operatorname{Re}\omega}(\mathbb{R}_+)} \le \left\|f - \Lambda_-^{-\omega-1/2}\varphi_\varepsilon\right\|_{H^{\operatorname{Re}\omega}} + \left\|\Lambda_-^{-\omega-1/2}(\varphi_\varepsilon - \varphi_n)\right\|_{H^{\operatorname{Re}\omega}},$$

where in the right-hand side both norms converge to zero. Note that $f \in H^{\operatorname{Re}\omega}$ can also be approximated by $\{\psi_\varepsilon\} \subset C_0^\infty$ in the norm of $H^{\operatorname{Re}\omega}$ (see [7, Theorem 4.1]).

We demonstrate now the first relation in (2.14). For $k = 0$, this is trivial by definition (2.9). For $k \in \mathbb{N}$, the equality

$$r_+\Lambda_-^k\ H_+^{-1/2} = r_+H_+^{-k-1/2}$$

holds, since the operator $\Lambda_-^k = \mathscr{F}^{-1}(\xi - k_0)^k \cdot \mathscr{F}$ has a polynomial symbol, which means that Λ_-^k is a differential operator and preserves the support of $\varphi \in H_+^{-1/2}$.

The second relation in (2.14) follows from definition (2.12) and from the properties of $\Lambda_+^k = \mathscr{F}^{-1}(\xi + k_0)^k \cdot \mathscr{F}$, $k \in \mathbb{N}_0$ (is trivial for $k = 0$). ∎

REMARK 2.3. In general, the spaces $\overset{<}{H}{}^\omega(\mathbb{R}_+)$ and $\overset{>}{H}{}_+^\omega(\mathbb{R}_+)$ depend on $\tau = \operatorname{Im}\omega$. For instance, $\overset{<}{H}{}^{i\tau}(\mathbb{R}_+)$ is a proper dense subspace of $L^2(\mathbb{R}_+)$ for every $\tau \in \mathbb{R}$. All this can be seen by considerations similar to those in the last proof, but also directly as follows. Consider the operator

$$U_\omega = r_+\Lambda_-^{-\omega-1/2}\Lambda_+^{1/2}\colon L_+^2 \to H^{\operatorname{Re}\omega}(\mathbb{R}_+)$$

and the corresponding lifted operator

$$\begin{aligned} U_{\omega 0} &= r_+\Lambda_-^{\operatorname{Re}\omega}\ell r_+\Lambda_-^{-\omega-1/2}\Lambda_+^{1/2} \\ &= r_+\Lambda_-^{-i\tau}\big(\Lambda_-^{-1/2}\Lambda_+^{1/2}\big)\colon L_+^2 \to L^2(\mathbb{R}_+). \end{aligned}$$

This last operator is not normally solvable since the jump condition (A.20) is violated (cf. Theorem A.3), i.e., the equivalent operator U_ω satisfies $\operatorname{im} U_\omega \ne \overline{\operatorname{im} U_\omega}$, which yields that $\overset{<}{H}{}^\omega(\mathbb{R}_+)$ is a proper subspace of $H^{\operatorname{Re}\omega}(\mathbb{R}_+)$. Moreover, it is dense, because $\operatorname{im} U_{\omega 0}$ is dense in $L^2(\mathbb{R}_+)$; see the formula (A.31) for $\beta(U_{\omega 0})$ in Corollary A.6.

COROLLARY 2.4. *For any $\omega_1, \omega_2 \in \mathbb{C}$ the following operators are homeomorphisms*:

$$\begin{aligned} \operatorname{Rst} r_+\Lambda_-^{\omega_1-\omega_2}\ell^{(\operatorname{Re}\omega_1)}\colon \overset{<}{H}{}^{\omega_1}(\mathbb{R}_+) &\to \overset{<}{H}{}^{\omega_2}(\mathbb{R}_+), \\ \operatorname{Ext}\Lambda_+^{\omega_1-\omega_2}\big|_{H_+^{\operatorname{Re}\omega_1}}\colon \overset{>}{H}{}_+^{\omega_1} &\to \overset{>}{H}{}_+^{\omega_2}. \end{aligned} \tag{2.15}$$

Proof. The operator

$$r_+ \Lambda_-^{\omega_1 - \omega_2} \ell^{(\operatorname{Re}\omega_1)}\colon H^{\operatorname{Re}\omega_1}(\mathbb{R}_+) \to H^{\operatorname{Re}\omega_2}(\mathbb{R}_+)$$

is well defined and bounded by (2.7). By definition (2.9) and Corollary 2.2, the spaces $\overset{<}{H}{}^{\omega_j}(\mathbb{R}_+)$, $j = 1, 2$, are proper dense subspaces of $H^{\omega_j}(\mathbb{R}_+)$, $j = 1, 2$. Thus, the restricted operator given by the first formula in (2.15) is a bijection with inverse

$$\operatorname{Rst} r_+ \Lambda_-^{-\omega_1 + \omega_2} \ell^{(\omega_2)}\colon \overset{<}{H}{}^{\omega_2}(\mathbb{R}_+) \to \overset{<}{H}{}^{\omega_1}(\mathbb{R}_+)$$

continuous by the same arguments. The second relation in (2.15) can be proved analogously. ∎

THEOREM 2.5 (Main theorem). *Let* $\Phi \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$, $\nu \in]0, 1[$, *and* $\omega = \sigma + i\tau$ *defined by* (2.3). *Then* W_s *defined by* (1.6) *is not normally solvable iff*

$$\kappa = s + \sigma + \tfrac{1}{2} \in \mathbb{Z}. \tag{2.16}$$

In this case

$$\big(H_+^s, \overset{<}{H}{}^{s-i\tau}(\mathbb{R}_+)\big), \qquad \big(\overset{>}{H}{}_+^{s-i\tau}, H^s(\mathbb{R}_+)\big) \in \mathcal{N}(M_{s,c}), \tag{2.17}$$

where $c = |\exp(-2\pi\tau)|$; *see* (1.15) *and* (2.5). (*The question of whether* W_s *is normally solvable depends on* s *and* σ; *the solution of the normalization problem depends on* s *and* τ, *provided* $s + \sigma + 1/2 \in \mathbb{Z}$, *but not on the particular integer* $\kappa = s + \sigma + 1/2$.)

Each of the normalized operators [*cf.* (1.14)]

$$\begin{aligned} \overset{<}{W}_s &= \operatorname{Rst} W_s\colon H_+^s \to \overset{<}{H}{}^{s-i\tau}(\mathbb{R}_+), \\ \overset{>}{W}_s &= \operatorname{Ext} W_s\colon \overset{>}{H}{}_+^{s-i\tau} \to H^s(\mathbb{R}_+), \end{aligned} \tag{2.18}$$

is left or right invertible with index

$$\operatorname{Ind} \overset{<}{W}_s = -\kappa, \qquad \operatorname{Ind} \overset{>}{W}_s = -\kappa + 1, \tag{2.19}$$

respectively. Generalized inverses (*which are one-sided inverses*) *can be obtained by extension/restriction from the generalized inverses of* $W_{s\pm\varepsilon}$, *for any* $\varepsilon \in]0, 1[$, *which are constructed by factorization of* Φ,

$$\begin{aligned} \overset{<}{W}{}_s^- &= \operatorname{Ext} W_{s+\varepsilon}^-\colon \overset{<}{H}{}^{s-i\tau}(\mathbb{R}_+) \to H_+^s, \\ \overset{>}{W}{}_s^- &= \operatorname{Rst} W_{s-\varepsilon}^-\colon H^s(\mathbb{R}_+) \to \overset{>}{H}{}_+^{s-i\tau}, \end{aligned} \tag{2.20}$$

where $W_{s\pm\varepsilon}^-$ *are given in the Appendix* [*cf.* (A.34)].

Proof. Condition (2.16) is known, see Corollary A.6. The rest is proved by construction of one-sided inverses (2.20) as follows. In Section 3, we show that the normalization makes sense, i.e., the formulas (2.18) represent continuous operators, and that the problem can be reduced to the consideration of

$$V_{\pm 1/2} = W_{\pm 1/2}\left(\left(\frac{\lambda_-}{\lambda_+}\right)^{\kappa} \Psi\right) \tag{2.21}$$

with Ψ defined as in (2.2) [and $W_s(\Phi)$ in (1.16)]. In Section 4, the reduced case is treated and the results are assembled to complete the proof.

3. REDUCTION TO $W_s = W_s(\Psi_\kappa)$, $\Psi_\kappa \in \mathcal{G}C^\nu(\dot{\mathbb{R}})$, $s = \pm\frac{1}{2}$

Under the assumption of Theorem 2.5 about Φ we have from Lemma 2.1,

$$\Phi = \left(\frac{\lambda_-}{\lambda_+}\right)^{\sigma+i\tau} \Psi. \tag{3.1}$$

Let $B = \mathcal{F}^{-1}\Psi_\kappa \cdot \mathcal{F}$ with

$$\Psi_\kappa = \left(\frac{\lambda_-}{\lambda_+}\right)^{\kappa} \Psi, \qquad \kappa = s + \sigma + \frac{1}{2}, \qquad \text{wind}\, \Psi_\kappa = \text{ind}_2 \Psi_\kappa = \kappa, \tag{3.2}$$

where the 2-index of Ψ_κ is defined according to [21] [cf. (A.29) and (7.36)]. In the critical case, $\Psi_\kappa \in \mathcal{G}C^\nu(\dot{\mathbb{R}})$ since κ is an integer [cf. (2.16)]. Using the notation (3.1) and (3.2) we obtain the factorization of $W_s = W_s(\Phi)$ in (1.6),

$$\begin{aligned} W_s &= r_+ \Lambda_-^{\sigma+i\tau} \mathcal{F}^{-1} \Psi \cdot \mathcal{F} \Lambda_+^{-(\sigma+i\tau)} \\ &= \big(r_+ \Lambda_-^{-(s+1/2-i\tau)} \ell^{(-1/2)}\big)(r_+ B) \Lambda_+^{s+1/2-i\tau} \colon \\ &\qquad H^s(\mathbb{R}_+) \underset{\text{bij.}}{\longleftarrow} H^{-1/2}(\mathbb{R}_+) \underset{V_{-1/2}}{\longleftarrow} H_+^{-1/2} \underset{\text{bij.}}{\longleftarrow} H_+^s, \end{aligned} \tag{3.3}$$

where we used Eskin's formulas (2.7). Thus, W_s is equivalent [up to bijective operators of the form (2.7)] to

$$V_{-1/2} = W_{-1/2}(\Psi_\kappa) \colon H_+^{-1/2} \to H^{-1/2}(\mathbb{R}_+). \tag{3.4}$$

In the same way we find for $W_{s+1} = W_{s+1}(\Phi)$,

$$\begin{aligned} W_{s+1} &= \big(r_+ \Lambda_-^{-(s+1/2-i\tau)} \ell^{(1/2)}\big)(r_+ B)\, \Lambda_+^{s+1/2-i\tau} \colon \\ &\qquad H^{s+1}(\mathbb{R}_+) \underset{\text{bij.}}{\longleftarrow} H^{1/2}(\mathbb{R}_+) \underset{V_{1/2}}{\longleftarrow} H_+^{1/2} \underset{\text{bij.}}{\longleftarrow} H_+^{s+1}, \end{aligned} \tag{3.5}$$

which operator is equivalent to

$$V_{1/2} = W_{1/2}(\Psi_\kappa) \colon H_+^{1/2} \to H^{1/2}(\mathbb{R}_+). \tag{3.6}$$

LEMMA 3.1. *If* $\Psi_\kappa \in \mathcal{G}C^\nu(\dot{\mathbb{R}})$, *then*

$$\operatorname{im} W_{-1/2}(\Psi_\kappa) \subset \tilde{H}_{-1/2}(\mathbb{R}_+) = \overset{<}{H}{}^{-1/2}(\mathbb{R}_+), \tag{3.7}$$

and

$$\operatorname{Rst} W_{-1/2}(\Psi_\kappa)\colon H_+^{-1/2} \to \tilde{H}_{-1/2}(\mathbb{R}_+) \tag{3.8}$$

is a bounded operator. Further $W_{1/2}(\Psi_\kappa)$ *has a continuous extension*

$$\operatorname{Ext} W_{1/2}(\Psi_\kappa)\colon \overset{>}{H}{}_+^{1/2} \to H^{1/2}(\mathbb{R}_+) \tag{3.9}$$

to

$$\overset{>}{H}{}_+^{1/2} = \operatorname{clos}\left\{\psi \in H_+^{1/2}\colon \|\psi\|_{\overset{>}{H}{}_+^{1/2}} = \|r_+\psi\|_{H^{1/2}(\mathbb{R}_+)}\right\}. \tag{3.10}$$

Proof. Since $B = aI + B_{-\nu}$, where $a = \Psi_\kappa(+\infty)$ and $B_{-\nu}$ is a smoothing operator of order $-\nu$, i.e., $B_\nu H^r \subset H^{r+\nu}$ for every $r \in \mathbb{R}$, we have

$$\begin{aligned} r_+ B H_+^{-1/2} &\subset r_+ H_+^{-1/2} + r_+ H_+^{-1/2+\nu} \\ &= \tilde{H}_{-1/2}(\mathbb{R}_+) + H^{-1/2+\nu}(\mathbb{R}_+) = \tilde{H}_{-1/2}(\mathbb{R}_+), \end{aligned}$$

where the embedding is continuous and $r_+\colon H_+^{-1/2} \to \tilde{H}_{-1/2}(\mathbb{R}_+)$ is isometric. In the second case, $W_{1/2}$ is equivalent to

$$\tilde{W}_{1/2} = r_+(aI + B_{-\nu})\ell_0\colon \tilde{H}_{1/2}(\mathbb{R}_+) \to H^{1/2}(\mathbb{R}_+).$$

Since the embeddings

$$\tilde{H}_{1/2}(\mathbb{R}_+) \subset H^{1/2}(\mathbb{R}_+), \qquad H^{1/2}(\mathbb{R}_+) \subset H^{1/2-\nu}(\mathbb{R}_+) = \tilde{H}_{1/2-\nu}(\mathbb{R}_+)$$

are continuous and $r_+ B_{-\nu}\ell_0\colon \tilde{H}_{1/2-\nu}(\mathbb{R}_+) \to H^{1/2}(\mathbb{R}_+)$ as well, the operator $r_+(aI + B_{-\nu})\ell_0$ has a continuous extension to $H^{1/2}(\mathbb{R}_+)$. This, by definition (3.10), implies the continuity of (3.9). ∎

4. NORMALIZATION OF V_s, $s = \pm 1/2$

Continuing the proof of Theorem 2.5 we have to normalize operators of the form (2.21), i.e.,

$$V_s = W_s(\Psi_\kappa)\colon H_+^s \to H^s(\mathbb{R}_+), \tag{4.1}$$

where $\Psi_\kappa \in \mathcal{G}C^\nu(\dot{\mathbb{R}})$, $\operatorname{wind} \Psi_\kappa = \kappa \in \mathbb{Z}$, and $s = \pm 1/2$. We recall that by definitions (1.11) and (1.15), the operators $V_{\pm 1/2}$ in (4.1) belong to the classes $M_{\pm 1/2, 1}$ of not normally solvable operators.

PROPOSITION 4.1. *Under the above assumptions, the operators*

$$\begin{aligned} \overset{<}{V}_{-1/2} &= \operatorname{Rst} W_{-1/2}(\Psi_\kappa)\colon H_+^{-1/2} \to \tilde{H}_{-1/2}(\mathbb{R}_+), \\ \overset{>}{V}_{1/2} &= \operatorname{Ext} W_{1/2}(\Psi_\kappa)\colon \overset{>}{H}{}_+^{1/2} \to H^{1/2}(\mathbb{R}_+) \end{aligned} \tag{4.2}$$

due to Lemma 3.1 *are Fredholm and one-sided invertible operators with*

$$\operatorname{Ind} \overset{<}{V}_{-1/2} = \operatorname{Ind} \overset{>}{V}_{1/2} = -\kappa. \tag{4.3}$$

One-sided inverses are then given by

$$\begin{aligned} \overset{<}{V}{}_{-1/2}^{-} &= \operatorname{Ext} V_s^-\colon \tilde{H}_{-1/2}(\mathbb{R}_+) \to H_+^{-1/2}, \\ \overset{>}{V}{}_{1/2}^{-} &= \operatorname{Rst} V_s^-\colon H^{1/2}(\mathbb{R}_+) \to \overset{>}{H}{}_+^{1/2}, \end{aligned} \tag{4.4}$$

where $V_s V_s^- V_s = V_s$ for all $|s| < 1/2$ and V_s^- is represented by factorization of Ψ_κ (cf. Corollary A.7).

Proof. Because of definition (4.1) and the injections (1.10), which are dense and continuous, we have

$$\ker V_{s_2} \subset \ker V_{s_1}, \qquad \operatorname{im} V_{s_2} \subset \operatorname{im} V_{s_1}, \qquad s_1 < s_2, \tag{4.5}$$

and the functions from $\mathbb{R}$ into $\mathbb{Z}$ (cf. Corollary A.6),

$$\begin{aligned} \alpha(V_s) = \dim \ker V_s, \qquad -\beta(V_s) &= -\dim H^s(\mathbb{R}_+)/\overline{\operatorname{im} V_s} \\ \operatorname{Ind} V_s = \alpha(V_s) &- \beta(V_s), \end{aligned} \tag{4.6}$$

are monotonically decreasing. Moreover, the index formula

$$\operatorname{Ind} V_s = -\operatorname{ind}_2 \Psi_\kappa = -\operatorname{wind} \Psi_\kappa = -\kappa = \begin{cases} \alpha(V_s), & \text{if } \kappa \le 0, \\ \beta(V_s), & \text{if } \kappa \ge 0, \end{cases} \tag{4.7}$$

holds for $|s| < 1/2$. That is, the family $\{V_s\colon |s| < 1/2\}$ consists of Fredholm operators with constant defect numbers. We will show that this is true also for the enlarged family $\{V_s\colon |s| < 1/2\} \cup \{\overset{<}{V}_{-1/2}, \overset{>}{V}_{1/2}\}$. To this end let first $\kappa \le 0$. Then V_s is surjective if $s < 1/2$ and $s - 1/2 \notin \mathbb{Z}$. To show that $\overset{<}{V}_{-1/2}$ is surjective, we try to solve, for any given $g \in \tilde{H}_{-1/2}(\mathbb{R}_+)$, the equation

$$V_s f = r_+(aI + B_{-\nu}) f = g \tag{4.8}$$

in $H_+^{-1/2}$, putting $B = \mathcal{F}^{-1}\Psi_\kappa \cdot \mathcal{F} = aI + B_{-\nu}$ as in Section 3. From the embeddings

$$g \in \tilde{H}_{-1/2}(\mathbb{R}_+) \subset H^{-1/2}(\mathbb{R}_+) \subset H^{-1/2-\nu/2}(\mathbb{R}_+)$$

we know that there exists a solution $f \in H_+^{-1/2-\nu/2}$ since $V_{-1/2-\nu/2}$ is surjective. Substituting this solution in (4.8) we obtain, since $a \neq 0$,

$$r_+ f = \frac{1}{a}[g - r_+ B_{-\nu} f] \in \tilde{H}_{-1/2}(\mathbb{R}_+) + H^{-1/2+\nu/2}(\mathbb{R}_+) \subset \tilde{H}_{-1/2}(\mathbb{R}_+)$$

and $f \in H_+^{-1/2}$.

A similar argument [put $g = 0$ in (4.8)] implies that $\ker \overset{<}{V}_{-1/2} \subset \ker V_{-1/2+\nu}$, which has the dimension $-\kappa$. Hence $\overset{<}{V}_{-1/2}$ is Fredholm and right invertible with the same defect numbers of all V_s, $|s| < 1/2$.

Second, let $\kappa > 0$. As before we see that $\ker \overset{<}{V}_{-1/2} \subset \ker V_{-1/2+\nu} = \{0\}$, i.e., $\overset{<}{V}_{-1/2}$ is injective. Also $V_{-1/2\pm\nu/2}$ are injective (even left invertible) and

$$\begin{gathered} \operatorname{im} V_{-1/2+\nu/2} \subset \operatorname{im} \overset{<}{V}_{-1/2} \subset \operatorname{im} V_{-1/2-\nu/2}, \\ \kappa = \beta(V_{-1/2+\nu/2}) \geq \beta(\overset{<}{V}_{-1/2}) \geq \beta\big(V_{-1/2-\nu/2}\big) = \kappa - 1 \end{gathered} \tag{4.9}$$

according to Lemma 3.1, (4.5)–(4.7). Thus, we can find $g_1, \ldots, g_\kappa \in H^{-1/2+\nu/2}(\mathbb{R}_+)$ such that

$$\begin{aligned} H^{-1/2+\nu/2}(\mathbb{R}_+) &= \operatorname{im} V_{-1/2+\nu/2} \dotplus \operatorname{span}\{g_1, \ldots, g_\kappa\}, \\ H^{-1/2-\nu/2}(\mathbb{R}_+) &= \operatorname{im} V_{-1/2-\nu/2} \dotplus \operatorname{span}\{g_1, \ldots, g_{\kappa-1}\}. \end{aligned} \tag{4.10}$$

Now we show that

$$\operatorname{im} \overset{<}{V}_{-1/2} = C := \operatorname{clos}\big\{\operatorname{im} V_{-1/2+\nu/2} \colon \|\cdot\|_{\tilde{H}_{-1/2}(\mathbb{R}_+)}\big\} \tag{4.11}$$

in two steps. First we demonstrate that this closure is contained in $\operatorname{im} \overset{<}{V}_{-1/2}$. For the proof let $g \in C$ and solve

$$V_{-1/2-\nu/2} f = r_+(aI + B_{-\nu}) f = g \tag{4.12}$$

in $H_+^{-1/2-\nu/2}$ uniquely [see (4.9)], because $g \in \operatorname{im} V_{-1/2-\nu/2}$ and the operator is left invertible. Equation (4.12) implies $r_+ f \in C + r_+ H_+^{-1/2+\nu/2} \subset \tilde{H}_{-1/2}(\mathbb{R}_+)$, i.e., $g = \overset{<}{V}_{-1/2} f \in \tilde{H}_{-1/2}(\mathbb{R}_+)$ due to Lemma 3.1. The second step is to see that $g_j \notin \operatorname{im} \overset{<}{V}_{-1/2}$, which is evident for $j = 1, \ldots, \kappa - 1$ from (4.9) and (4.10). For $j = \kappa$ let us assume that g_κ (or a linear combination with the others) belongs to $\operatorname{im} \overset{<}{V}_{-1/2}$. Then there is an $f \in H_+^{-1/2}$ such that $r_+(aI + B_{-\nu}) f = g_\kappa$, which yields $r_+ f \in H^{-1/2+\nu/2}(\mathbb{R}_+) + r_+ H_+^{-1/2+\nu}$ and contradicts (4.10). This completes the proof of (4.11) and therefore $\overset{<}{V}_{-1/2}$

is left invertible with $\beta(\overset{<}{V}_{-1/2}) = \beta(V_s) = \kappa$, $|s| < 1/2$. Altogether we have the diagram

$$\begin{array}{llcll} V_{-1/2+\nu/2}\colon & H_+^{-1/2+\nu/2} & \longrightarrow & \operatorname{im} V_{-1/2+\nu/2} & \dot{+}\ \operatorname{span}\{g_1, \dots, g_\kappa\} \\ & \cap \text{ dense} & & \cap \text{ dense} & \\ \overset{<}{V}_{-1/2}\colon & H_+^{-1/2} & \longrightarrow & C & \dot{+}\ \operatorname{span}\{g_1, \dots, g_\kappa\}, \end{array}$$

where, dropping the finite-dimensional components, we have bounded invertible operators and know the inverse of the first one. Therefore the inverse of the second (acting onto C) is just the extension of that first inverse. Returning to the full spaces (including the spans) we obtain the first formula of (4.4), in the case $\kappa > 0$. The corresponding conclusion for $\kappa \leq 0$ with a diagram analogous to the preceding one is evident.

The proof of the second part due to $s = 1/2$ makes use of similar arguments and therefore is omitted. ∎

Completion of the proof of Theorem 2.5. The formulas (3.3) and (3.5), respectively, have shown that for the critical numbers s,

$$W_s = EV_{-1/2}F, \qquad W_{s+1} = EV_{1/2}F, \tag{4.13}$$

where $E = r_+\Lambda_-^{-(s+1/2-i\tau)}\ell^{(\mp 1/2)}$ and $F = \Lambda_+^{s+1/2-i\tau}$, respectively, are invertible operators (bijections in the original space setting) and $V_{\mp 1/2}$ are not normally solvable but admit a normalization described in Proposition 4.1.

In the first case, $s = -1/2$, we have an image normalization, where

$$\begin{aligned} \operatorname{dom} \overset{<}{V}_{-1/2} &= H_+^{-1/2} = \operatorname{dom} V_{-1/2}, \\ \operatorname{im} \overset{<}{V}_{-1/2} &= r_+H_+^{-1/2} = \tilde{H}_{-1/2}(\mathbb{R}_+) \subset H^{-1/2}(\mathbb{R}_+). \end{aligned} \tag{4.14}$$

The new image space of $\overset{<}{W}_s$ induced by that normalization in (4.13) is $\overset{<}{H}{}^{s-i\tau}(\mathbb{R}_+)$ since a comparison of (3.3) with definition (2.9) implies $\omega = s - i\tau$. The restricted operator

$$\overset{<}{E} = \operatorname{Rst} E\colon r_+H_+^{-1/2} \to \overset{<}{H}{}^{s-i\tau}(\mathbb{R}_+) \tag{4.15}$$

is also bounded invertible according to Corollary 2.4. Consequently, the composition

$$\overset{<}{W}_s = \overset{<}{E}\,\overset{<}{V}_{-1/2}F\colon \overset{<}{H}{}^{s-i\tau}(\mathbb{R}_+) \leftarrow \tilde{H}_{-1/2}(\mathbb{R}_+) \leftarrow H_+^{-1/2} \leftarrow H_+^s \tag{4.16}$$

makes sense. In (4.16), $\overset{<}{E}$ and F are bijections and $\overset{<}{V}_{-1/2}$ is generalized invertible, by $\overset{<}{V}{}^-_{-1/2}$ say [see (4.4)]. Hence a generalized inverse of $\overset{<}{W}_s$ reads

$$\overset{<}{W}{}_s^- = F^{-1}\overset{<}{V}{}^-_{-1/2}\overset{<}{E}{}^{-1} \tag{4.17}$$

with

$$\overset{<}{E}^{-1} = \text{Rst } E^{-1}\colon \overset{<}{H}^{s-i\tau}(\mathbb{R}_+) \to \tilde{H}_{-1/2}(\mathbb{R}_+), \qquad E^{-1} = r_+ \Lambda_-^{s-i\tau+1/2} \ell^{(s)}.$$

The rest of the statement of Theorem 2.5 concerning the image normalization, namely, the index formula [see (2.19)], is an obvious consequence of (4.3).

The proof of the second part due to domain normalization makes use of similar arguments and therefore is omitted. ∎

5. FURTHER CONCLUSIONS AND SOME APPLICATIONS

From the proof of Theorem 2.5 it is clear that the solutions of the minimal normalization problems for $M_{s,c}$ are unique up to norm equivalence in Y_1 or X_1, respectively. So let us take the "canonical image normalized operators" $\overset{<}{W}_s$ in (4.16) and the analogous domain normalized operators $\overset{>}{W}_s$, provided $\Phi \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$, and define

$$\overset{<}{W}_s = W_s = \overset{>}{W}_s \tag{5.1}$$

if W_s is normally solvable. There are several results which now hold for all $s \in \mathbb{R}$.

COROLLARY 5.1. *For all* $s \in \mathbb{R}$, *the operators* $\overset{<}{W}_s$ *and* $\overset{>}{W}_s$ *are Fredholm and one-sided invertible with*

$$\text{Ind } \overset{<}{W}_s = -\left[s + \sigma + \tfrac{1}{2}\right], \qquad \text{Ind } \overset{>}{W}_s = -\left[-\left(s + \sigma - \tfrac{1}{2}\right)\right], \tag{5.2}$$

where the brackets denote the integer part of a real number and $\sigma = \text{Re}\,\omega$ *is defined in* (2.4). *Consequently*

$$\text{Ind } \overset{>}{W}_s - \text{Ind } \overset{<}{W}_s = \begin{cases} 1, & \text{if } s + \sigma + \frac{1}{2} \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

Moreover, the kernels of $\overset{<}{W}_s$ *are generated by a sequence of elements in the manner*

$$\ker \overset{<}{W}_s = \text{span}\{\varphi_1, \ldots, \varphi_\alpha\} \tag{5.4}$$

for $s \in [-\alpha - \sigma - 1/2, -\alpha - \sigma + 1/2[$, *where*

$$\varphi_j \in \bigcap_{\varepsilon > 0} H_+^{-j-\sigma+1/2-\varepsilon} \setminus H_+^{-j-\sigma+1/2}. \tag{5.5}$$

In particular, $\ker \overset{<}{W}_s \neq \{0\}$ *for* $s < -\sigma - 1/2$ *and* $\text{Ind } \overset{<}{W}_s = \alpha(\overset{<}{W}_s) = \max\{0, -[s + \sigma + 1/2]\}$, $s \in \mathbb{R}$, *which is right continuous.*

The complements of the image of $\overset{<}{W}_s$ *are characterized by*

$$H^s(\mathbb{R}_+)/\overline{\operatorname{im}\overset{<}{W}_s} = \operatorname{span}\{\psi_1, \dots, \psi_\beta\} \tag{5.6}$$

for $s \in]\beta - \sigma - 1/2,\ \beta - \sigma + 1/2]$, *where*

$$\psi_j \in \bigcap_{\varepsilon>0} H^{j-\sigma-1/2}(\mathbb{R}_+)\Big\backslash H^{j-\sigma-1/2+\varepsilon}(\mathbb{R}_+). \tag{5.7}$$

This means that $H^s(\mathbb{R}_+)/\overline{\operatorname{im}\overset{<}{W}_s} \neq \{0\}$ *for* $s > -\sigma + 1/2$ *and* $\operatorname{Ind}\overset{<}{W}_s = -\beta(\overset{<}{W}_s) = \max\{0, [-(s + \sigma - 1/2)]\}$, $s \in \mathbb{R}$, *which is left continuous*.

Proof. Formulas (5.2) and (5.3) follow directly from definition (5.1) and known results for W_s in the case where it is normally solvable (cf. Corollary A.6). Based on Lemma 2.1 and Theorem A.3, we factorize $\Phi \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$,

$$\Phi = \Psi_-\left(\frac{\lambda_-}{\lambda_+}\right)^{\omega}\Psi_+ = \tilde{\Psi}_-\left(\frac{\lambda_-}{\lambda_+}\right)^{[s+\sigma+1/2]}\tilde{\Psi}_+, \tag{5.8}$$

where $\omega = \sigma + i\tau$ is defined by (2.3), $\Psi_\pm \in \mathcal{G}C^\nu(\dot{\mathbb{R}})$, and $\tilde{\Psi}_\pm \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$ [cf. (3.1) and (3.2)]. This yields

$$\ker\overset{<}{W}_s = \mathcal{F}^{-1}\tilde{\Psi}_+^{-1}\operatorname{span}\left\{\lambda_+^{-1},\ \lambda_+^{-2}, \dots, \lambda_+^{-\alpha}\right\},$$

i.e., $\varphi_1 \in H^s_+$ for $s < -\sigma - 1/2$, $\varphi_2 \in H^s_+$ for $s < -\sigma - 3/2, \dots, \varphi_\alpha \in H^s_+$ for $s < -\alpha - \sigma - 1/2$. In other words, for $s \in [-\sigma - 3/2, -\sigma - 1/2[$ we have $\ker\overset{<}{W}_s = \{\varphi_1\}$ with

$$\varphi_1 \in \bigcap_{\varepsilon>0} H_+^{-\sigma-1/2-\varepsilon}\Big\backslash H_+^{-\sigma-1/2};$$

for $s \in [-\sigma - 5/2, -\sigma - 3/2[$, $\ker\overset{<}{W}_s = \{\varphi_1, \varphi_2\}$, where φ_1 is defined as before and

$$\varphi_2 \in \bigcap_{\varepsilon>0} H_+^{-\sigma-3/2-\varepsilon}\Big\backslash H_+^{-\sigma-3/2};$$

and so on. For $s \in [-\alpha - \sigma - 1/2, -\alpha - \sigma + 1/2[$, the kernel of $\overset{<}{W}_s$ is given by (5.4) with φ_j, $j = 1, \dots, \alpha$, defined in (5.5).

Analogously, we prove (5.6) and (5.7) from the factorization (5.8), which gives

$$H^s(\mathbb{R}_+)/\overline{\operatorname{im}\overset{<}{W}_s} = r_+\mathcal{F}^{-1}\tilde{\Psi}_-\operatorname{span}\left\{\lambda_+^{-1}, \lambda_+^{-2}, \dots, \lambda_+^{-\beta}\right\}.$$

∎

REMARK 5.2. Similar statements hold for the kernels of $\overset{>}{W}_s$ and for the complements of the images of $\overset{>}{W}_s$. Therefore, the elements of the kernel of $\overset{>}{W}_s$, defined as in (5.4) read as

$$\varphi_j \in \bigcap_{\varepsilon>0} H_+^{-j-\sigma-1/2} \backslash H_+^{-j-\sigma-1/2+\varepsilon} \tag{5.9}$$

for $s \in]-\alpha-\sigma-1/2, -\alpha-\sigma+1/2]$, and the elements of the complement of the image corresponding to (5.6) read as

$$\psi_j \in \bigcap_{\varepsilon>0} H^{j-\sigma+1/2-\varepsilon}(\mathbb{R}_+) \backslash H^{j-\sigma+1/2}(\mathbb{R}_+) \tag{5.10}$$

for $s \in [\beta-\sigma-1/2, \beta-\sigma+1/2[$. In particular, $\ker \overset{>}{W}_s \neq \{0\}$ for $s \leq -\sigma-1/2$ and $\operatorname{Ind} \overset{>}{W}_s = \alpha(\overset{>}{W}_s) = \max\{0, [-(s+\sigma-1/2)]\}$, which is left continuous. $H^s(\mathbb{R}_+)/\operatorname{im} \overset{>}{W}_s \neq \{0\}$ for $s \geq -\sigma+1/2$ and $\operatorname{Ind} \overset{>}{W}_s = -\beta(\overset{>}{W}_s) = \max\{0, [s+\sigma+1/2]\}$, which is right continuous.

COROLLARY 5.3. *There is a unique* $s_1 \in \mathbb{R}$, *namely*, $s_1 = -\sigma$ *such that*:

(i) *for* $s \in [s_1 - 1/2, s_1 + 1/2[$, $\overset{<}{W}_s$ *is invertible*;

(ii) *for* $s \in [s_1 - k - 1/2, s_1 - k + 1/2[$, $k \in \mathbb{N}$, $\overset{<}{W}_s$ *is right invertible and* $\alpha(\overset{<}{W}_s) = k$;

(iii) *for* $s \in [s_1 + k - 1/2, s_1 + k + 1/2[$, $k \in \mathbb{N}$, $\overset{<}{W}_s$ *is left invertible and* $\beta(\overset{<}{W}_s) = k$.

The one-sided inverses are given by formula (2.20), *where* $W_{s+\varepsilon}^-$, $\varepsilon \in]0,1[$, *is constructed as in* (A.34) *if* $s = k - \sigma - 1/2$, $k \in \mathbb{Z}$ (*critical number*), *and straightforwardly by formula* (A.34) *in the other cases*. *Analogous statements hold for* $\overset{>}{W}_s$.

COROLLARY 5.4 (Extended shift theorem). *For each* $k \in \mathbb{Z}$ *the "shifted operator"* $\overset{<}{W}_{s+k}(\Phi)$ *satisfies*

$$\overset{<}{W}_{s+k}(\Phi) = r_+ \Lambda_-^{-k} \ell^{(s)} \big(r_+ \Lambda_-^k A \Lambda_+^{-k}\big) \Lambda_+^k \tag{5.11}$$

and is therefore equivalent to $\overset{<}{W}_s((\lambda_-/\lambda_+)^k \Phi)$. *Thus*, *if* V *is a generalized inverse of the last mentioned operator, then*

$$\overset{<}{W}_{s+k}(\Phi)^- = \Lambda_+^{-k} V r_+ \Lambda_-^k \ell^{(s+k)} \tag{5.12}$$

is a generalized inverse of the first one.

We apply now the concept of image normalization to some scalar Sommerfeld diffraction problems that are not normally solvable. Let us consider first the well-known *impedance problem* in the scalar case, which is equivalent to the WHO

$$W_{\mathcal{J}} = r_+ A_1\big|_{H_+^{-1/2}} \colon H_+^{-1/2} \to H^{-1/2}(\mathbb{R}_+), \qquad A_1 = \mathcal{F}^{-1}(1 - ipt^{-1}) \cdot \mathcal{F}, \tag{5.13}$$

where $p \in \mathbb{C}$ is the face impedance number and $t(\xi) = (\xi^2 - k_0^2)^{1/2}$ is abbreviated by t [20]. If $W_{\mathcal{J}}$ is of normal type, i.e., $(1 - ipt^{-1}) \in \mathcal{G}C(\dot{\mathbb{R}})$, then according to definitions (1.11) and (1.15) it belongs to the class $M_{-1/2,\,1}$. We have the same assumptions of Proposition 4.1 with $\kappa = s + \sigma + 1/2 = 0$. The image normalized operator

$$\overset{<}{W}_{\mathcal{J}} = \operatorname{Rst} W_{\mathcal{J}} \colon H_+^{-1/2} \to \tilde{H}_{-1/2}(\mathbb{R}_+) \tag{5.14}$$

is even invertible, since $\operatorname{Ind} \overset{<}{W}_{\mathcal{J}} = 0$. This case of image normalization is also known as *normalization by compatibility conditions* [23].

The representation of the inverse of (5.13) follows from the fact that it is invertible and we know the inverse [21] of

$$W_{\mathcal{J}_0} = r_+ A_1\big|_{L_+^2} \colon L_+^2 \to L^2(\mathbb{R}_+),$$

where A_1 has the same Fourier symbol as (5.13). So we know the inverse of $W_{\mathcal{J}}$ on a dense subspace $L^2(\mathbb{R}_+)$ of the image $\tilde{H}_{-1/2}(\mathbb{R}_+)$ of $\overset{<}{W}_{\mathcal{J}}$. Thus $\overset{<}{W}_{\mathcal{J}}$ is invertible by

$$\overset{<}{W}{}_{\mathcal{J}}^{-1} = \operatorname{Ext} W_{\mathcal{J}_0}^{-1} \colon \tilde{H}_{-1/2}(\mathbb{R}_+) \to H_+^{-1/2} \tag{5.15}$$

with

$$\begin{aligned} W_{\mathcal{J}_0}^{-1} &= A_{1+}^{-1} \ell_0 r_+ A_{1-}^{-1} \ell_0, \qquad A_{1\pm}^{-1} = \mathcal{F}^{-1} \Phi_{1\pm}^{-1} \cdot \mathcal{F}, \\ \Phi_{1\pm} &= \exp\left\{\tfrac{1}{2}(I \pm S_{\mathbb{R}}) \log(1 - ipt^{-1})\right\}, \end{aligned} \tag{5.16}$$

where $S_{\mathbb{R}}$ is the Cauchy operator on $\mathbb{R}$.

Another example from mathematical physics, the *Sommerfeld diffraction problem with oblique derivatives* [22], gives rise to scalar WHOs when the boundary conditions are equal on both faces of the screen. The so-called main problem decomposes into two scalar problems, each of them being equivalent to

$$\begin{aligned} W_{\mathcal{OD}} &= r_+ A_2|_{H_+^{-1/2}} \colon H_+^{-1/2} \to H^{-1/2}(\mathbb{R}_+), \\ A_2 &= \mathcal{F}^{-1}(\alpha + i\beta\xi t^{-1}) \cdot \mathcal{F}, \end{aligned} \tag{5.17}$$

where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ are coefficients and $t(\xi) = (\xi^2 - k_0^2)^{1/2}$ as before. From (5.17) we see that $W_{\mathcal{OD}}$ is of normal type iff $\alpha/(i\beta) \neq \xi t^{-1}$ and belongs to $M_{-1/2,c}$ with $c = |(\alpha + i\beta)/(\alpha - i\beta)|$ if $\kappa = \sigma = (1/2\pi)\arg(\alpha + i\beta)/(\alpha - i\beta) \in \mathbb{Z}$ (critical number). Writing in the same way as (3.3),

$$\begin{aligned} W_{\mathcal{OD}} &= \big(r_+\Lambda_-^{i\tau}\ell^{(-1/2)}\big)(r_+B)\Lambda_+^{-i\tau} \\ &= EV_{-1/2}F\colon H^{-1/2}(\mathbb{R}_+) \leftarrow H^{-1/2}(\mathbb{R}_+) \leftarrow H_+^{-1/2} \leftarrow H_+^{-1/2}, \end{aligned} \tag{5.18}$$

we reduce the normalization of (5.17) to the image normalization of $V_{-1/2} = W_{-1/2}(\Psi_\kappa)$ with $\kappa = \sigma \in \mathbb{Z}$. This can be done as in Section 4. Therefore, the image normalized operator is given by

$$\overset{<}{W}_{\mathcal{OD}} = \overset{<}{E}\,\overset{<}{V}_{-1/2}F\colon \overset{<}{H}{}^{-1/2-i\tau}(\mathbb{R}_+) \leftarrow \tilde{H}_{-1/2}(\mathbb{R}_+) \leftarrow H_+^{-1/2} \leftarrow H_+^{-1/2}, \tag{5.19}$$

where $\tau = (1/2\pi)\log|(\alpha - i\beta)/(\alpha + i\beta)|$, which is left invertible with index

$$\operatorname{Ind}\overset{<}{W}_{\mathcal{OD}} = -\sigma = \frac{1}{2\pi}\arg\frac{\alpha - i\beta}{\alpha + i\beta}. \tag{5.20}$$

From Proposition 4.1, a generalized inverse of $\overset{<}{V}_{-1/2}$ is given by the first formula (4.4). Take then $s = 0$ and construct a factorization of $\Phi = (\lambda_-/\lambda_+)^\eta\Psi$ with $\eta = [\sigma + 1/2] + \delta$, $|\delta| < 1/2$,

$$\Phi = \Psi_-\lambda_-^\delta\left(\frac{\lambda_-}{\lambda_+}\right)^{[\sigma+1/2]}\lambda_+^{-\delta}\Psi_+ = \Phi_{2-}\left(\frac{\lambda_-}{\lambda_+}\right)^{[\sigma+1/2]}\Phi_{2+}, \tag{5.21}$$

which yields

$$\overset{<}{V}{}_{-1/2}^{-} = \operatorname{Ext} V_0^-\colon \tilde{H}_{-1/2}(\mathbb{R}_+) \to H_+^{-1/2} \tag{5.22}$$

with

$$\begin{aligned} V_0^- &= A_{2+}^{-1}\ell_0 r_+ C^{-1}\ell_0 r_+ A_{2-}^{-1}\ell_0, \qquad A_{2\pm}^{-1} = \mathcal{F}^{-1}\Phi_{2\pm}^{-1}\cdot\mathcal{F}, \\ C &= \mathcal{F}^{-1}\left(\frac{\lambda_-}{\lambda_+}\right)^{[\sigma+1/2]}\cdot\mathcal{F}. \end{aligned} \tag{5.23}$$

Therefore a generalized inverse of $\overset{<}{W}_{\mathcal{OD}}$ reads as

$$\overset{<}{W}{}_{\mathcal{OD}}^{-} = F^{-1}\overset{<}{V}{}_{-1/2}^{-}\overset{<}{E}{}^{-1},$$

where $F^{-1} = \Lambda_+^{i\tau}$, $\overset{<}{V}{}_{-1/2}^{-}$ is given by (5.22) and (5.23), and

$$\overset{<}{E}{}^{-1} = \operatorname{Rst} r_+\Lambda_-^{-i\tau}\ell^{(-1/2)}\colon \overset{<}{H}{}^{-1/2-i\tau}(\mathbb{R}_+) \to \tilde{H}_{-1/2}(\mathbb{R}_+).$$

Note that (5.19) can also be interpreted as a compatibility condition for the scalar oblique derivative problem, since it represents the extendability of the given data from the half-line onto the full real line [see definition (2.10)].

6. THE SYSTEM'S CASE

Let us return to the WHO

$$W = W_{r,s} = r_+ A\big|_{H_+^r} \colon H_+^r \to H^s(\mathbb{R}_+), \tag{6.1}$$

where $r, s \in \mathbb{R}^n$ and $A\colon H^r \to H^s$ is a translation invariant homeomorphism. The corresponding lifted operator

$$W_0 = r_+ A_0\big|_{[L_+^2]^n} \colon [L_+^2]^n \to L^2(\mathbb{R}_+)^n \tag{6.2}$$

is assumed to have a Fourier symbol

$$\Phi_0 \in \mathscr{G}C^\nu(\ddot{\mathbb{R}})^{n\times n} \tag{6.3}$$

for some $\nu \in]0,1[$, i.e., it is Hölder continuous of order ν at any $\xi \in \mathbb{R}$ and satisfies the conditions

$$(\Phi_0(\xi) - \Phi_0(\pm\infty))_{jl} = \mathscr{O}(|\xi|^{-\nu}) \quad \text{as } |\xi| \to \infty, j, l = 1, \dots, n, \tag{6.4}$$

$$\left|\det \Phi_0(\xi)\right| \neq 0, \qquad \xi \in \ddot{\mathbb{R}}. \tag{6.5}$$

Suppose now that $\mu_1, \dots, \mu_m$ $(m \leq n)$ are the eigenvalues of the jump at infinity of the Fourier symbol with regard to their multiplicities, i.e., if $l_1, \dots, l_m$ are the lengths of the corresponding chains of associated vectors, then $\sum_{j=1}^m l_j = n$. The following notation will be used for the diagonal matrix,

$$\operatorname{diag}(\tilde{\mu}_1, \dots, \tilde{\mu}_n) = \operatorname{diag}\big(\underbrace{\mu_1, \dots, \mu_1}_{l_1 \text{ times}}, \dots, \underbrace{\mu_m, \dots, \mu_m}_{l_m \text{ times}}\big) \tag{6.6}$$

or, after introducing

$$\tilde{\mu}_j = \exp(2\pi i \tilde{\omega}_j), \qquad \operatorname{Re} \tilde{\omega}_j \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right[, \qquad j = 1, \dots, n \tag{6.7}$$

(or $\operatorname{Re} \tilde{\omega}_j \in]-1/2, 1/2]$ alternatively) with

$$\omega = (\tilde{\omega}_1, \dots, \tilde{\omega}_n) = \big(\underbrace{\omega_1, \dots, \omega_1}_{l_1 \text{ times}}, \dots, \underbrace{\omega_m, \dots, \omega_m}_{l_m \text{ times}}\big) \in \mathbb{C}^n, \tag{6.8}$$

we write briefly

$$\operatorname{diag}(\tilde{\mu}_1, \dots, \tilde{\mu}_n) = \operatorname{diag}\big(\exp(2\pi i \tilde{\omega}_j)\big), \qquad \operatorname{Re} \omega_j \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right[, \quad j = 1, \dots, n. \tag{6.9}$$

For the class of symbols described by (6.3)–(6.5) the jump at infinity can be written in the normal Jordan form (see, e.g. [8])

$$\Phi_0^{-1}(+\infty)\Phi_0(-\infty) = T^{-1} J_{\Phi_0} T, \tag{6.10}$$

where $T \in \mathscr{G}\mathbb{C}^{n\times n}$ and the quasidiagonal matrix

$$J_{\Phi_0} = \operatorname{diag}(J_1, \dots, J_m) \tag{6.11}$$

has Jordan blocks of size $l_j \times l_j$ in the diagonal given by

$$J_j = \begin{bmatrix} \mu_j & 1 & 0 & \cdots & 0 \\ 0 & \mu_j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \mu_j \end{bmatrix}, \qquad j = 1, \dots, m. \tag{6.12}$$

LEMMA 6.1. *Let* (6.3) *be satisfied. Then the representation* (6.10) *is equivalent to*

$$\Phi_0(\xi) = \Phi_0(-\infty)T^{-1}\left(\operatorname{diag}\left(\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\tilde{\omega}_j}\right)J_{\Phi_0} + \Psi_0(\xi)\right)T, \tag{6.13}$$

where the elements of Ψ_0 *satisfy* $\Psi_0 \in C^\nu(\dot{\mathbb{R}})^{n\times n}$ *and*

$$\Psi_{0jl}(\xi) = \mathscr{O}(|\xi|^{-\nu}) \quad \textit{as } |\xi| \to \infty. \tag{6.14}$$

Proof. Since $\operatorname{diag}((\lambda_-/\lambda_+)^{\tilde{\omega}_j}) \in C^\infty(\ddot{\mathbb{R}})$ and

$$\lim_{\xi\to\pm\infty} \operatorname{diag}\left(\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\tilde{\omega}_j}\right) = \begin{cases} I, & \text{at } +\infty, \\ \operatorname{diag}\big(\exp(-2\pi i\tilde{\omega}_j)\big), & \text{at } -\infty, \end{cases} \tag{6.15}$$

where I denotes the identity $n \times n$ matrix, from the representation (6.13) one gets

$$\Phi_0(+\infty) = \Phi_0(-\infty)T^{-1}\big(J_{\Phi_0} + \Psi_0(+\infty)\big)T.$$

Moreover $\Psi_0(+\infty) = 0$ and we conclude the normal Jordan form (6.10). ∎

PROPOSITION 6.2. *Under the assumptions* (6.1)–(6.3) *the operator W is normally solvable iff*

$$\operatorname{Re} \tilde{\omega}_j \neq -\tfrac{1}{2}, \qquad j = 1, \dots, n \tag{6.16}$$

(*or* $\operatorname{Re} \tilde{\omega}_j \neq 1/2$ *alternatively*).

Proof. By Theorem A.1 and Lemma 6.1 the operator W is equivalent to the lifted operator W_0 with a Fourier symbol Φ_0 given by (6.13). Hence the condition for W_0 to be normally solvable can be written in the form (6.16) [cf. (A.20)]. ∎

Now let us suppose that W is not normally solvable, i.e., (6.16) is violated or, after ordering the components by a permutation of the columns of T,

$$\operatorname{Re}\tilde{\omega}_j = -\tfrac{1}{2} \quad \text{iff } j = 1, \ldots, n', \tag{6.17}$$

where $1 \le n' \le n$ (alternatively $\operatorname{Re}\tilde{\omega}_j = 1/2$, $j = 1, \ldots, n'$).

THEOREM 6.3. *Let* (6.1)–(6.3) *and* (6.17) *be satisfied. Then the first minimal normalization problem* (*image normalization*) *is solvable by*

$$Y_1 = r_+ \Lambda_-^{-s} T l^{(0)} \{ \overset{<}{H}{}^{-i(\tau_1,\ldots,\tau_{n'})}(\mathbb{R}_+) \times L^2(\mathbb{R}_+)^{n-n'} \}, \tag{6.18}$$

where

$$\begin{aligned} \overset{<}{H}{}^{-i(\tau_1,\ldots,\tau_{n'})}(\mathbb{R}_+) &= \bigtimes_{j=1}^{n'} r_+ \Lambda_-^{-i\tau_j} \Lambda_-^{-1/2} \Lambda_+^{1/2} L_+^2, \\ \tau_j = \operatorname{Im}\tilde{\omega}_j &= -\frac{1}{2\pi} \int_{\mathbb{R}} d \log |\Psi_{jj}(\xi)|, \qquad j = 1, \ldots, m. \end{aligned} \tag{6.19}$$

Further, there exists an $\varepsilon_0 > 0$ *such that* $W_{r',s'}$ *is generalized invertible for* $r' = (r_1 + \varepsilon, \ldots, r_n + \varepsilon)$, $s' = (s_1 + \varepsilon, \ldots, s_n + \varepsilon)$, $0 < \varepsilon < \varepsilon_0$. *A generalized inverse of*

$$\overset{<}{W} = \operatorname{Rst} W \colon H_+^r = X_0 \to Y_1 \tag{6.20}$$

is obtained by extension of any generalized inverse $W_{r',s'}^-$ *of* $W_{r',s'} = \operatorname{Rst} W\colon H_+^{r'} \to H^{s'}(\mathbb{R}_+)$; *in short*,

$$\overset{<}{W}{}^- = \operatorname{Ext} W_{r',s'}^- \colon Y_1 \to H_+^r, \qquad 0 < \varepsilon < \varepsilon_0. \tag{6.21}$$

Proof. The proof is a modification of the argumentation in the scalar case. First W maps into Y_1, since [see (6.13) and (2.9)]

$$\begin{gathered} r_+ \Lambda_-^{-1/2+i\tau_j} \Lambda_+^{1/2-i\tau_j} \colon L_+^2 \to \overset{<}{H}{}^{-i\tau_j}(\mathbb{R}_+), \\ r_+ \mathcal{F}^{-1} \Psi_0 \cdot \mathcal{F} \colon [L_+^2]^n \to H^\nu(\mathbb{R}_+)^n \to Y_1 \end{gathered}$$

are continuous operators. Therefore the restricted operator in (6.20) is well defined with a closed image (6.18) and (6.19). Moreover $\overset{<}{W}$ is generalized invertible and has a complemented (finite-dimensional) kernel and image. The representation (6.21) for a generalized inverse follows from a density argument similar to that used in the proof of Proposition 4.1. ∎

THEOREM 6.4. *Let* (6.1)–(6.5) *and the alternative of* (6.17) *be satisfied, i.e.*, $\operatorname{Re}\tilde{\omega}_j = 1/2$, $j = 1, \ldots, n'$. *Then the second minimal normalization problem* (*domain normalization*) *is solved by*

$$X_1 = \Lambda_+^r T \{ \overset{>}{H}{}_+^{i(\tau_1,\ldots,\tau_{n'})} \times [L_+^2]^{n-n'} \}, \tag{6.22}$$

where the numbers τ_j *are defined as in* (6.19). *A generalized inverse of*

$$\overset{>}{W} = \operatorname{Ext} W\colon X_1 \to H^s(\mathbb{R}_+) = Y_0 \tag{6.23}$$

is obtained by restriction of a generalized inverse of $W_{r',s'} = \operatorname{Ext} W\colon H_+^{r'} \to H^{s'}(\mathbb{R}_+)$, *where* $r' = (r_1 - \varepsilon, \dots, r_n - \varepsilon)$, $s' = (s_1 - \varepsilon, \dots, s_n - \varepsilon)$ *for suitable* $0 < \varepsilon < \varepsilon_0$, *i.e.*,

$$\overset{>}{W}{}^{-} = \operatorname{Rst} W_{r',s'}^{-}\colon H^s(\mathbb{R}_+) \to X_1. \tag{6.24}$$

Proof. By analogy, noting that [see (6.13) and (2.12)]

$$\Lambda_+^{i\tau_j}\colon \overset{>}{H}{}_+^{i\tau_j} \to L_+^2,$$

$$r_+\mathcal{F}^{-1}\Psi_0 \cdot \mathcal{F}\colon \overset{>}{H}{}_+^{i\tau_j} \to H_+^{-\nu} \to L^2(\mathbb{R}_+)$$

are continuous, a continuous extension of W as defined in (6.23) is possible. Hence we can use arguments similar to former considerations and conclude the representation (6.24) for a generalized inverse. ∎

Remark 6.5. The system's case admits eventually a mixed image/-domain normalization in different components, i.e., a simultaneous change of both spaces X_0 and Y_0 (see the Introduction). Although the solution of the normalization is not unique up to isomorphy, we can also speak of a minimal normalization in view of the density of $X_0 \subset X_1$ and $Y_1 \subset Y_0$.

Remark 6.6. In the case where *the jump at infinity is diagonalizable*, i.e.,

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = T^{-1} \operatorname{diag}(\mu_1, \dots, \mu_n)T, \tag{6.25}$$

where

$$\mu_j = \exp(2\pi i \omega_j), \qquad \operatorname{Re}\, \omega_j \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right[, \qquad j = 1, \dots, n$$

(or $\operatorname{Re} \omega_j \in]-1/2, 1/2]$ alternatively) with $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{C}^n$, are eigenvalues of the jump at infinity, we have the representation formula [instead of (6.13)]

$$\Phi_0(\xi) = T^{-1}\left(\operatorname{diag}\left(c_j\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\omega_j}\right) + \Psi_0(\xi)\right)T \tag{6.26}$$

with $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, $c_j \neq 0$,

$$\operatorname{diag} c_j = T\Phi_0(+\infty)T^{-1}, \tag{6.27}$$

and where the elements of Ψ_0 satisfy (6.14). Now under the assumptions (6.1)–(6.3) and (6.25) the operator W is normally solvable iff

$$\operatorname{Re} \omega_j \neq -\tfrac{1}{2}, \qquad j = 1, \dots, n \tag{6.28}$$

(or $\operatorname{Re}\omega_j \neq 1/2$ alternatively). Thus, for not normally solvable WHOs the statements of Theorems 6.3 and 6.4 hold, where condition (6.17) is substituted by

$$\operatorname{Re}\omega_j = -\tfrac{1}{2} \quad \text{iff } j = 1, \ldots, n', 1 \leq n' \leq n \tag{6.29}$$

(alternatively $\operatorname{Re}\omega_j = 1/2$, $j = 1, \ldots, n'$). The diagonalizable case, as we shall see next, is relevant in many applications from mathematical physics.

THEOREM 6.7. *If* (6.1)–(6.3), (6.25), *and* (6.28) *hold and*

$$\nu > \tfrac{1}{2} + \max\{\operatorname{Re}\,\omega_j \colon j = 1, \ldots, n\}, \tag{6.30}$$

then the intermediate space Z in a factorization of A_0 (*cf. Corollary* A.4) *due to a generalized factorization of Φ_0 is* (*up to a perturbation of the components by T*) *the fractional Sobolev space*

$$Z = H^{\operatorname{Re}\omega} = \bigtimes_{j=1}^{n} H^{\operatorname{Re}\omega_j}. \tag{6.31}$$

For the proof, see [24, proof of Theorem 4.2] and [3].

Let us briefly analyze the system impedance problem [17, 20, 23, 27], which corresponds to

$$\begin{aligned} W_{\mathcal{J}} &= r_+ A_1\big|_{H_+^r} \colon H_+^r \to H^s(\mathbb{R}_+), \qquad r = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \qquad s = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right), \\ W_{\mathcal{J}_0} &= r_+ A_{1,0}\big|_{[L_+^2]^2} \colon [L_+^2]^2 \to L^2(\mathbb{R}_+)^2, \end{aligned} \tag{6.32}$$

with Fourier symbols

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} -(t - ip) & -iqt^{-1} \\ iq & 1 - ipt^{-1} \end{bmatrix}, \\ \Phi_{1,0} &= \begin{bmatrix} -(1 - ipt^{-1}) & -iqt^{-1}\left(\dfrac{\lambda_-}{\lambda_+}\right)^{-1/2} \\ iqt^{-1} & (1 - ipt^{-1})\left(\dfrac{\lambda_-}{\lambda_+}\right)^{-1/2} \end{bmatrix}, \end{aligned} \tag{6.33}$$

where $t(\xi) = (\xi^2 - k_0^2)^{1/2}$, $\xi \in \mathbb{R}$, $p, q \in \mathbb{C} \setminus \{0\}$. The coefficient q appears here due to different impedance numbers on each face of the half-plane. These operators are not normally solvable. Assuming $Y_1 = H^{-1/2}(\mathbb{R}_+) \times \tilde{H}_{-1/2}(\mathbb{R}_+)$ as the image of $\overset{<}{W}_{\mathcal{J}} = \operatorname{Rst} W_{\mathcal{J}}$ and $Y_1 = L^2(\mathbb{R}_+) \times \overset{<}{H}{}^0(\mathbb{R}_+)$ as the image of $\overset{<}{W}_{\mathcal{J}_0} = \operatorname{Rst} W_{\mathcal{J}_0}$, we solve the image normalization problem (by compatibility conditions as in [23]) for this case. On the other hand, from [27] we know already an inverse of

$$W_\varepsilon = \operatorname{Rst} W_{\mathcal{J}_0} \colon [H_+^\varepsilon]^2 \to H^\varepsilon(\mathbb{R}_+)^2$$

for a given $\varepsilon \in]0, 1/2[$, which can be used straightforwardly to define $\overset{<}{W}{}^{-}_{\mathcal{J}_0}$ as in Theorem [3].

For the Sommerfeld diffraction problem with oblique derivatives in the general case [11, 22] we have

$$\begin{aligned} W_{\mathcal{OD}} &= r_+ A_2|_{H^r_+} \colon H^r_+ \to H^s(\mathbb{R}_+), \qquad r = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \\ s &= \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right), \\ W_{\mathcal{OD},0} &= r_+ A_2|_{[L^2_+]^2} \colon [L^2_+]^2 \to L^2(\mathbb{R}_+)^2 \end{aligned} \tag{6.34}$$

with Fourier symbols

$$\begin{aligned} \Phi_2 &= -\frac{1}{2}\begin{bmatrix} \alpha t + i\beta\xi & -(\gamma + i\delta\xi t^{-1}) \\ \gamma t + i\delta\xi & -(\alpha + i\beta\xi t^{-1}) \end{bmatrix}, \\ \Phi_{2,0} &= -\frac{1}{2}\begin{bmatrix} \alpha + i\beta\xi t^{-1} & -(\gamma + i\delta\xi t^{-1})\left(\dfrac{\lambda_-}{\lambda_+}\right)^{-1/2} \\ \gamma + i\delta\xi t_{-1} & -(\alpha + i\beta\xi t^{-1})\left(\dfrac{\lambda_-}{\lambda_+}\right)^{-1/2} \end{bmatrix}, \end{aligned} \tag{6.35}$$

where $\gamma, \delta \in \mathbb{C}\backslash\{0\}$ appear due to the difference of the parameters on each face of the screen (in the main problem $\gamma = \delta = 0$). These operators are also not normally solvable for all parameters $\alpha, \beta, \gamma, \delta$ up to some exceptional cases.

The image normalization of $W_{\mathcal{OD}}$ (provided $W_{\mathcal{OD}}$ is of normal type) can here be obtained by considering the data on the half-plane in

$$Y_1 = r_+ \Lambda^s_- T\ell^{(0)}\left\{\overset{<}{H}{}^{-i\tau}(\mathbb{R}_+) \times L^2(\mathbb{R}_+)\right\}, \tag{6.36}$$

where

$$\tau = -\frac{1}{2\pi}\log\left|\sqrt{\frac{\alpha^2 + \beta^2 - \gamma^2 - \delta^2}{(\alpha + i\delta)^2 + (\beta - i\gamma)^2}}\right|.$$

This represents the compatibility conditions for the oblique derivative problem. Furthermore, we can use a previous result [15] to give a representation for a generalized inverse of the image normalized operator

$$\overset{<}{W}_{\mathcal{OD}} = \operatorname{Rst} W_{\mathcal{OD}} \colon H^r_+ = X_0 \to Y_1. \tag{6.37}$$

In [15], a representation for a generalized inverse of $W_{r',s'}$, $r' = (1/2 + \varepsilon, -1/2 + \varepsilon)$, $s' = (-1/2 + \varepsilon, -1/2 + \varepsilon)$, $2\varepsilon \notin \mathbb{N}_0$, was presented explicitly from a rather sophisticated factorization process. Thus, by extension of $W^-_{r',s'}$, we now conclude the result also in the sense of minimal normalization without losing the finite energy norm.

APPENDIX

We present briefly some relevant details of the notation of spaces of Bessel potentials and known results about WHOs in appropriate formulation. Starting with the Schwarz test function space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions, the dual space $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ of tempered distributions, and the Fourier transformation

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{ix\xi}\varphi(x)\,dx \tag{A.1}$$

on $\mathcal{S}$ and $\mathcal{S}'$, respectively, we define

$$H^s = H^s(\mathbb{R}) = \left\{f \in \mathcal{S}' \colon \lambda^s \mathcal{F} f \in L^2\right\}, \qquad s \in \mathbb{R}, \tag{A.2}$$

where $\lambda(\xi) = (\xi^2 + 1)^{1/2}$ for $\xi \in \mathbb{R}$. This is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_s = \int_{\mathbb{R}} \lambda^s \mathcal{F} f \cdot \lambda^s \overline{\mathcal{F} g} \tag{A.3}$$

and can be considered as a subspace of L^2 for $s \geq 0$.

Denote by H^s_+ the subspace of H^s distributions f supported on $\overline{\mathbb{R}}_+$, i.e.,

$$f(\varphi) = 0 \quad \text{for } \varphi \in \mathcal{S} \text{ and } \operatorname{supp}\varphi \subset \overline{\mathbb{R}}_- \tag{A.4}$$

with the norm induced by H^s. $H^s(\mathbb{R}_+)$ represents the restrictions $g = r_+ f$ of H^s distributions on $\mathbb{R}_+$, i.e.,

$$\langle g, \varphi\rangle = \langle r_+ f, \varphi\rangle = \langle f, \ell_0\varphi\rangle \tag{A.5}$$

for $\varphi \in \mathcal{S}(\mathbb{R}_+)$, the C^∞ functions on $\mathbb{R}$ which admit a zero extension $\ell_0\varphi \in \mathcal{S}$. $H^s(\mathbb{R}_+)$ is equipped with the infimum norm

$$\|g\|_{H^s(\mathbb{R}_+)} = \inf\left\{\|f\|_{H^s} \colon r_+ f = g\right\},$$

which is a Hilbert space as well. These two scales of spaces H^s_+ and $H^s(\mathbb{R}_+)$, $s \in \mathbb{R}$, are sufficient for the definition and discussion of many properties of the WHOs (1.6) and, moreover, of pseudodifferential operators (PDOs), see [7, 29]. However, for various reasons, it is convenient to study the related spaces ($s \in \mathbb{R}$)

$$H^s_0(\mathbb{R}_+) = \operatorname{clos} C_0^\infty(\mathbb{R}_+) \quad \text{in } H^s(\mathbb{R}_+), \tag{A.6}$$

$$\tilde{H}^s(\mathbb{R}_+) = \left[H^{-s}(\mathbb{R}_+)\right]', \tag{A.7}$$

$$\tilde{H}_s(\mathbb{R}_+) = r_+ H^s_+ \subset H^s(\mathbb{R}_+) \tag{A.8}$$

with the norm induced by H^s_+, which yields that the embedding is continuous. The first space is very important for approximation arguments, proof technique, the study of boundary value problems, etc. It is well known that in general the elements of $H_0^{1/2}(\mathbb{R}_+)$ are not extendable by zero to elements in $H^{1/2}$, in contrast to the cases where $|s-k|<1/2$, $k\in\mathbb{N}_0$; see [16]. All the spaces in (A.6)–(A.8) can be seen as subspaces of $(r_+\mathscr{S})'$ [1, 10, 26]. So we find, e.g.,

$$\tilde{H}_{1/2}(\mathbb{R}_+)=\tilde{H}^{1/2}(\mathbb{R}_+)\subset H_0^{1/2}(\mathbb{R}_+)=H^{1/2}(\mathbb{R}_+) \tag{A.9}$$

as a dense but nonclosed linear manifold. The same holds for $s=-1/2$ (by duality, e.g.):

$$r_+H_+^{-1/2}\subset H^{-1/2}(\mathbb{R}_+) \tag{A.10}$$

is a proper dense manifold. The *tilde spaces* have been successfully used in the study of boundary and transmission problems, (see, for instance, [5, 9, 18, 30, 31]), in particular for mixed boundary value problems and screen and wedge problems. The two definitions (A.7) and (A.8) are equivalent for $s\geq -1/2$, but not for $s<-1/2$. For example, the δ distribution belongs to $\tilde{H}^s(\mathbb{R}_+)$, and not to $\tilde{H}_s(\mathbb{R}_+)$ [30] (the notation is rectified by an early definition in [9]). According to the number of papers that now use (A.7), we decided not to write $\tilde{H}^s(\mathbb{R}_+)$ for (A.8), and we note that

$$\tilde{H}^s(\mathbb{R}_+)=\tilde{H}_s(\mathbb{R}_+)\dot{+}\operatorname{span}\{D^j\delta\colon j=0,1,\ldots,k-1\} \tag{A.11}$$

can be identified, where

$$k=\min\{l\in\mathbb{N}\colon s+l<1/2\}.$$

The following results are mainly collected from [7] and [21], but see also [2, 19, 24] and other references in particular cases. Consider the WHOs (or PDOs) defined in (1.2). In the *elliptic case* A acts bijectively and both Φ and Φ^{-1} (with r and s exchanged) satisfy the conditions (1.3).

THEOREM A.1 (Lifting theorem). *W is equivalent to a lifted WHO*

$$W_0=r_+A_0|_{[L_+^2]^n}\colon [L_+^2]^n\to L^2(\mathbb{R}_+)^n, \tag{A.12}$$

where $A_0=\mathscr{F}^{-1}\Phi_0\cdot\mathscr{F}$, $\Phi_0\in L^\infty(\mathbb{R})^{n\times n}$. An equivalence relation is given by

$$W=(r_+\Lambda_-^{-s}\ell^{(0)})\,W_0\ell_0(r_+\Lambda_+^r), \tag{A.13}$$

where $\ell^{(0)}$ is any extension from $L^2(\mathbb{R}_+)^n$ into $L^2(\mathbb{R})^n$ (even element wise [7]) and ℓ_0r_+ can be dropped. Further $\Lambda_+^r=\operatorname{diag}(\Lambda_+^{r_1},\ldots,\Lambda_+^{r_n})$, etc. [see definition (2.6)] and the operators in parentheses are invertible in the corresponding spaces. Conversely

$$W_0=(r_+\Lambda_-^s\ell^{(s)})W\Lambda_+^{-r}, \tag{A.14}$$

where $\ell^{(s)}$ *denotes an arbitrary extension into* H^s [*and we cannot introduce* $\ell_0 r_+$ *between* W *and* Λ_+^{-r} *if some* $r_j \leq -1/2$ *because of* (A.11)].

The Fourier symbol of W_0 *is given by*

$$\Phi_0 = \lambda_-^s \Phi \lambda_+^{-r} = (\lambda_-^{s_j} \Phi_{jl} \lambda_+^{-r_l})_{j,\, l=1,\, \ldots,\, n}. \tag{A.15}$$

REMARK A.2. The operator in (A.12) can be identified with

$$\tilde{W}_0 = r_+ A_0 \ell_0 \colon L^2(\mathbb{R}_+)^n \to L^2(\mathbb{R}_+)^n \tag{A.16}$$

by restriction and zero extension, since

$$\tilde{W}_0 = W_0 \ell_0, \qquad W_0 = \tilde{W}_0 r_+. \tag{A.17}$$

Similarly we can write

$$\tilde{W} = r_+ A \ell_0 = W \ell_0 \colon \tilde{H}_r(\mathbb{R}_+) \to H^s(\mathbb{R}_+) \tag{A.18}$$

if $r_j \geq -1/2$ for $j = 1, \ldots, n$. For $r_j < -1/2$ the two operators $\tilde{W}$ and W cannot be related in this way according to (A.11) and the present notation of ℓ_0 and r_+. However, particularly for the orders $\pm 1/2$ and operators $A = I + B$, where B is smoothing, the notation of $\tilde{W}$ gives a more direct understanding of compatibility conditions and normalization.

Let us assume for the rest of the section [cf. (6.1)–(6.5)] that

$$\Phi_0 \in \mathscr{G}C^\nu(\ddot{\mathbb{R}})^{n\times n} \quad \text{for some } \nu \in]0, 1[. \tag{A.19}$$

THEOREM A.3. *The following assertions are equivalent*:

(i) W_0 *is normally solvable*;

(ii) W_0 *it is generalized invertible*;

(iii) W_0 *is a Fredholm operator*;

(iv) W *has one of these properties*;

(v) $\det(\mu \Phi_0(-\infty) + (1-\mu)\Phi_0(+\infty)) \neq 0$, $\mu \in]0, 1[$; (A.20)

(vi) Φ_0 *admits a generalized factorization with respect to* $L^2(\mathbb{R})^n$, *i.e.*,

$$\Phi_0 = \Phi_{0-} \operatorname{diag}(z^{\kappa_j}) \Phi_{0+}, \tag{A.21}$$

where $z(\xi) = \lambda_-(\xi)/\lambda_+(\xi) = (\xi - k_0)/(\xi + k_0)$, $\xi \in \mathbb{R}$ [*see also* (2.1)], $\kappa_j \in \mathbb{Z}$, $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$,

$$\Phi_{0\pm}, \Phi_{0\pm}^{-1} \in L_\pm^2(\mathbb{R}, \lambda_\pm^{-1})^{n\times n} = \mathscr{F}\Lambda_\pm^{-1}\ell_0 L^2(\mathbb{R}_\pm)^{n\times n} \tag{A.22}$$

and

$$r_+ A_{0+}^{-1} \ell_0 r_+ A_{0-}^{-1} \ell_0 \in \mathscr{L}(L^2(\mathbb{R}_+)^n) \tag{A.23}$$

for $A_{0\pm} = \mathscr{F}^{-1}\Phi_{0\pm} \cdot \mathscr{F}$, *which are unbounded operators on* $L^2(\mathbb{R})^n$, *in general, such that the composed operator* (A.23) *is bounded*.

COROLLARY A.4 [2]. *The symbol factorization* (A.21) *yields an operator factorization*

$$A_0 = A_{0-}CA_{0+}\colon L^2(\mathbb{R})^n \leftarrow Z \leftarrow Z \leftarrow L^2(\mathbb{R})^n, \tag{A.24}$$

which can be seen as a composition of bounded operators with the help of an intermediate space Z *defined by*

$$\begin{aligned} Z &= \operatorname{im} A_{0+} \subset S', \\ \|f\|_Z &= \|A_{0+}^{-1} f\|_{L^2(\mathbb{R})^n}. \end{aligned} \tag{A.25}$$

A generalized inverse of W_0 *is then given by*

$$W_0^- = A_{0+}^{-1} P C^{-1} P A_{0-}^{-1} \ell_0\colon L^2(\mathbb{R}_+)^n \to [L_+^2]^n, \tag{A.26}$$

where P *is the continuous extension of* $\ell_0 r_+$ *from* $Z \cap L^2(\mathbb{R}_+)^n$ *onto* Z *and* $C = \mathcal{F}^{-1} \operatorname{diag}(z^{\kappa_j}) \cdot \mathcal{F} \in \mathcal{GL}(Z)$, *for arbitrary integers* κ_j.

REMARK A.5. In the scalar case ($n = 1$), the intermediate space is a fractional Sobolev space, namely,

$$Z = H^\delta, \qquad |\delta| < \tfrac{1}{2}, \tag{A.27}$$

where $\delta = \operatorname{Re} w - \kappa$, putting $\kappa = [\operatorname{Re} w + 1/2]$ (see Lemma 2.1). In the matrix case, our assumptions (A.19) and (A.20) admit not only

$$Z = H^\delta, \qquad \delta = (\delta_1, \dots, \delta_n), \qquad |\delta_j| < \tfrac{1}{2}, \tag{A.28}$$

but also certain manifolds in such spaces and, moreover, Fourier images of weighted L^2 spaces with logarithmic weights; see [3].

Define in the scalar case the *2-index* of $\Phi_0 \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$, i.e., the index in L^2 of the closed curve formed by the graph of Φ_0 and the straight line segment connecting the points $\Phi_0(+\infty)$ and $\Phi_0(-\infty)$, by

$$\operatorname{ind}_2 \Phi_0 = \left[\sigma + \tfrac{1}{2}\right] \quad \text{if } \sigma + \tfrac{1}{2} \notin \mathbb{Z}, \tag{A.29}$$

where the brackets denote the integer part of a real number and

$$\sigma = \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Phi_0(\xi) \tag{A.30}$$

is the *fractional real winding number* of Φ_0 [cf. (2.4)].

COROLLARY A.6. *Let* $W_0\colon L_+^2 \to L^2(\mathbb{R}_+)$ *be a WHO with Fourier symbol* $\Phi_0 \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$. *Then*

$$\begin{aligned} \alpha(W_0) &= \dim \ker W_0 = \begin{cases} -\left[\sigma + \tfrac{1}{2}\right], & \text{if } \sigma < 0, \\ 0, & \text{if } \sigma \geq 0, \end{cases} \\ \beta(W_0) &= \dim L^2(\mathbb{R}_+) \big/ \overline{\operatorname{im} W_0} = \begin{cases} -\left[-\sigma + \tfrac{1}{2}\right], & \text{if } \sigma > 0, \\ 0, & \text{if } \sigma \leq 0. \end{cases} \end{aligned} \tag{A.31}$$

Analogously, the 2-index of the lifted Fourier symbol of $W_s\colon H^s_+ \to H^s(\mathbb{R}_+)$ reads as

$$\operatorname{ind}_2 \Phi_{s,0} = \operatorname{ind}_2(\lambda^s_- \Phi \lambda^{-s}_+) = \left[s + \sigma + \tfrac{1}{2}\right] \tag{A.32}$$

if $s + \sigma + 1/2 \notin \mathbb{Z}$, and the defect numbers of W_s are given by

$$\begin{aligned} \alpha(W_s) &= \dim\ker W_s = \begin{cases} -\left[s+\sigma+\frac{1}{2}\right], & \text{if } s < -\sigma, \\ 0, & \text{if } s \geq -\sigma, \end{cases} \\ \beta(W_s) &= \dim H^s(\mathbb{R}_+)/\overline{\operatorname{im} W_s} = \begin{cases} -\left[-\left(s+\sigma-\frac{1}{2}\right)\right], & \text{if } s > -\sigma, \\ 0, & \text{if } s \leq -\sigma. \end{cases} \end{aligned} \tag{A.33}$$

COROLLARY A.7. *Under the assumptions* (A.14), (A.15), (A.19), *and* (*A*.20), *a generalized inverse of W defined in* (1.2) *reads as*

$$W^- = \Lambda^{-r}_+ A^{-1}_{0+} P C^{-1} P A^{-1}_{0-} \Lambda^s_- \ell^{(s)}, \tag{A.34}$$

and the Fredholm index is given by

$$\operatorname{Ind} W = \operatorname{Ind} W_0 = -\operatorname{ind}_2 \det \Phi_0 = -\sum_{j=1}^{n} \kappa_j, \tag{A.35}$$

where ind_2 *denotes the 2-index of the graph of* $\det \Phi_0$ *closed by a straight line between the values of* $\det \Phi_0(+\infty)$ *and* $\det \Phi_0(-\infty)$.

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