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Equations of blowups of ideals of codimension two and three

Susan Morey*

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

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Abstract

Access to the defining equations of blowup algebras is a natural pathway to the study of these algebras. In this paper, a generating set is found for the equations of Rees algebras for certain classes of codimension two ideals. From the equations, the Rees algebras are seen to be Cohen–Macaulay. For related classes of codimension three Gorenstein ideals, information is given regarding the degrees of the generators of the ideal of equations. The results depend on the parity of the dimension of the base ring.

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1. Introduction

In this paper, we are interested in exploring conditions on the syzygies of an ideal I which will enable us to determine the structure of the equations of the Rees algebra of I. The Rees algebra of I, denoted by $\mathscr{R}$, is the graded algebra

$$\mathscr{R} = R \oplus It \oplus I^2t^2 \oplus \cdots .$$

A powerful tool in studying $\mathscr{R}$ is the ideal J of equations of $\mathscr{R}$ defined by

$$0 \to J \to R[T_1, \ldots, T_n] \to \mathscr{R} \to 0$$
$$T_i \to \alpha_i t$$

where $I = (\alpha_1, \ldots, \alpha_n)$. The ideal $J = J_1 + J_2 + \cdots$ is graded with J_1 being the linear relations $f_j = \sum a_{ij}T_i$ where $\sum a_{ij}\alpha_i = 0$. This can be rewritten in the form $J_1 = \boldsymbol{T} \cdot \phi$ where $\boldsymbol{T} = (T_1, \ldots, T_n)$ and $\phi = [a_{ij}]$ is a presentation matrix of I

$$R^m \xrightarrow{\phi} R^n \longrightarrow I \longrightarrow 0.$$

* Current address: Department of Mathematics, University of Texas, Austin, TX 78712, USA.
E-mail: morey@math.utexas.edu

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The symmetric algebra $S(I)$ is defined by J_1 and there is a canonical surjection

$$0 \longrightarrow \mathscr{A} \longrightarrow S(I) \longrightarrow \mathscr{R} \longrightarrow 0$$

where $\mathscr{A} = J/(J_1)$. (See [16] for further details.)

The structure of J is not generally known. The generators of degree one are as above, but where the other generators lie varies. We will study classes of ideals for which the structure of J can be determined. In particular, we will be interested in cases where the generators of J appear in only two degrees. For certain ideals, we will be able to obtain these generators from the presentation matrix of the ideal. Having control over the equations will aid in determining properties of the Rees algebra such as Cohen–Macaulayness and normality.

One class of ideals which is of interest is the class of codimension two ideals whose presentation matrices have content ideals, $I_1(\phi)$, generated by a regular sequence. For such ideals, we will focus on the following question:

Question 1.1. Let R be a regular local ring and let I be an ideal of R that is Cohen–Macaulay of codimension two. If the content ideal of the presentation matrix of I is generated by a regular sequence $(\mathbf{x})$ of length $s \geq 2$ and I is of linear type in codimension $s-1$, then are the equations of $R[It]$ of the form $J = (\mathbf{x} \cdot B(\phi), I_s(B(\phi)))$ and is $R[It]$ Cohen–Macaulay?

Here $B(\phi)$ is the Jacobian dual of ϕ which will be defined later but which is derived directly from ϕ. A main concern is what additional restrictions are required on the regular sequence to make the question feasible. In some cases the question is expected to have an affirmative answer even without R being a regular local ring.

Another class of ideals which shall be considered is that of grade three Gorenstein ideals. The structure here will vary depending on the parity of the dimension of the polynomial ring.

Conjecture 1.2. Let R be a commutative Noetherian local ring and let I be a perfect ideal of R that is Gorenstein of codimension three and is of linear type on the punctured spectrum. If d is odd, then the equations of $R[It]$ are of the form $J = (\mathbf{x} \cdot B(\phi), I_d(B(\phi)))$ and $R[It]$ is not Cohen–Macaulay.

For an even-dimensional ring, the situation appears to be quite different. The Rees algebra is again not expected to be Cohen–Macaulay, unless $n = d + 1$, however the equations will in general have a different form with generators appearing in degree $d - 1$. In the case $n = d + 1$ they will be given by $(\mathbf{x} \cdot B(\phi), \gcd(I_d(B(\phi))))$.

Let $n = v(I)$. If $d = n - 1$, R is a polynomial ring, and ϕ has linear entries in the variables, Question 1.1 is settled in [15]. Our aim is to prove some special cases of the conjecture. The key result will be Proposition 3.1 which asserts:

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local domain and let I be a codimension two perfect ideal minimally generated by n elements, with presentation matrix ϕ. Assume $I = I_{n-1}(\phi)$ is of linear type on the punctured spectrum of R and $I_1(\phi)$ is generated by a regular sequence (x) of length d. If* $\operatorname{height}(I_d(B(\phi)) + \mathfrak{m})/\mathfrak{m}R[T] \geq n - d$ *and $n \geq d + 2$ then $J = (\boldsymbol{x} \cdot B(\phi), I_d(B(\phi)))$.*

Thus in the case of linear entries, finding the equations amounts to finding the height of $I_d(B(\phi))$. In some circumstances (see Theorem 3.5) one can do this. The main application of Proposition 3.1 will be the following theorem.

Theorem 3.6. *Let R be a Cohen–Macaulay local domain and let I be a codimension two perfect ideal minimally generated by $n > d + 1$ elements. Assume $I = I_{n-1}(\phi)$ is of linear type on the punctured spectrum of R and $I_1(\phi)$ is generated by a regular sequence (x) of length d. If* $\operatorname{height}(J(0)_d) = \operatorname{height} J(0)$ *and (x) annihilates $\mathscr{A}_d$, then $(\boldsymbol{x} \cdot B(\phi), I_d(B(\phi)))$ defines $\mathscr{R}$ and $\mathscr{R}$ is Cohen–Macaulay.*

We now present a basic outline of the paper. In Section 2, we will define the Jacobian dual, $B(\phi)$, of a matrix ϕ and examine its structure for certain classes of ideals. In Section 3, we work with ideals of codimension two. The main focus of the section will be a discussion of various techniques leading to the proof of Theorem 3.6. In Section 4, we study codimension three Gorenstein ideals. Their Rees algebras are not always Cohen–Macaulay, as is seen in the examples of this section, but using the complex $\mathscr{D}$ of [8], we are able to shed some light on the degrees of the generators of the defining ideal.

2. Jacobian duals

We first collect some notation which will be used throughout the paper. For a detailed discussion, see [17]. If ϕ is a matrix, $I_n(\phi)$ is the ideal generated by the $n \times n$ minors of ϕ. Let $\ell(I)$ denote the analytic spread of an ideal I and let $v(I)$ be the minimal number of generators of I. If M is a finitely generated R-module, the rth Fitting ideal of M, denoted $I_r(M)$, is $I_{n-r}(\psi)$ where ψ presents M and $n = v(M)$. For general notation, see [10].

The linear equations of the Rees algebra, which are the equations of the symmetric algebra $S(I)$, are given by $\boldsymbol{T} \cdot \phi = (f_1, \ldots, f_m)$. These can be reformulated as $\boldsymbol{x} \cdot B(\phi)$ for some (not necessarily unique) matrix $B(\phi)$ where $(x) = (x_1, \ldots, x_s) = I_1(\phi)$. This new matrix provides a source for the higher degree generators of J. Suppose $m \geq s$ and $s \leq d$. Let B_i be any $s \times s$ minor of $B(\phi)$. Then

$$B_i^t(x_1, \ldots, x_s)^t = (f_{i_1}, \ldots, f_{i_s})^t$$

where $i_1, \ldots, i_s$ are the s columns of B which appear in B_i and t denotes transpose. By Cramer's rule $(x_j)\det B_i = \det(B_i^j)$ where B_i^j denotes replacing the jth column of B_i by $(f_{i_1}, \ldots, f_{i_s})^t$. So

$$(x_j)\det B_i \subseteq (f_{i_1}, \ldots, f_{i_s}) \subseteq J.$$

When R is a domain J is prime, and $x_j \notin J$, so $\det B_i \in J$ and thus $I_s(B(\phi)) \subset J$.

Throughout the course of this paper we will be working with ideals whose presentation matrices satisfy certain conditions. At times we will require the presentation matrix to have linear entries in $R = k[x_1, \ldots, x_d]$. In this case, $B(\phi)$ will be unique and will have linear entries in the $\boldsymbol{T}$ variables. In such cases we call $B(\phi)$ the Jacobian dual of ϕ. At other times, we will require $I_1(\phi) = (\boldsymbol{x})$ to be a regular sequence. In this case, $B(\phi)$ may not be unique, but $(I_s(B(\phi)) + (\boldsymbol{x}))/(\boldsymbol{x})R[\boldsymbol{T}]$ will be unique.

When I is of linear type in codimension $s-1$ and $I_1(\phi)$ is generated by a regular sequence of length s we call $L = (\boldsymbol{x} \cdot B(\phi), I_s(B(\phi)))$ the expected form of the equations of $\mathscr{R}$ and we have $L \subseteq J$. We are interested in finding conditions where this inclusion is actually an equality.

We now gather some information on the ideal of expected equations which will prove to be helpful later. When working over a polynomial ring, we shall normally assume that I is a homogeneous ideal so by considering only homogeneous prime ideals, we may treat R as if it were local.

Proposition 2.1. *Let R be a d dimensional commutative Noetherian local ring and let I be an ideal of R with presentation matrix ϕ. Suppose I is of linear type on the punctured spectrum, $v(I) = n \geq d+1$, and $I_1(\phi) = (x_1, \ldots, x_d) = (\boldsymbol{x})$. Then* $\dim S(I) = n$ *and* $\operatorname{height}(\boldsymbol{x} \cdot B(\phi)) = d$.

Proof. We know that

$$\dim S(I) = \sup_{\mathfrak{p} \in \operatorname{Spec}(R)} \{v(I_\mathfrak{p}) + \dim(R/\mathfrak{p})\}$$

by [5, Theorem 2.6] (see also [17]) and I is of linear type on the punctured spectrum of R. If $\mathfrak{p} = \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R, the sum is n. If $\mathfrak{p} \neq \mathfrak{m}$, then $I_\mathfrak{p}$ is of linear type. If $I \not\subseteq \mathfrak{p}$, $v(I_\mathfrak{p}) = 1$ so the sum is $\leq d+1$. If $\mathfrak{p} \neq \mathfrak{m}$ and $I \subseteq \mathfrak{p}$, localize at $\mathfrak{p}$ and set $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then

$$\begin{aligned} v(I_\mathfrak{p}) &= \dim_{k(\mathfrak{p})}(I_\mathfrak{p}/\mathfrak{p}I_\mathfrak{p}) = \dim S(I_\mathfrak{p}/\mathfrak{p}I_\mathfrak{p}) = \dim(S(I_\mathfrak{p}) \otimes k(\mathfrak{p})) \\ &= \dim(\mathscr{R}_\mathfrak{p} \otimes k(\mathfrak{p})) = \ell(I_\mathfrak{p}) \leq \dim(R_\mathfrak{p}), \end{aligned}$$

so the sum is $\leq d$. Then $\dim S(I) = \sup\{n, d, d+1\} = n$ since $n \geq d+1$. Now $\dim(R[\boldsymbol{T}]) = n + d$ and

$$S(I) = \frac{k[\boldsymbol{x}, \boldsymbol{T}]}{(\boldsymbol{x} \cdot B(\phi))},$$

so $\operatorname{height}(\boldsymbol{x} \cdot B(\phi)) = d$. $\square$

Corollary 2.2. *Assume in addition that $R = k[\mathbf{x}]$ is a polynomial ring, I is a homogeneous ideal, and the presentation matrix of I has linear entries. Then $I_d(B(\phi)) \neq 0$.*

Proof. Let $F = \operatorname{coker}(B(\phi))$. Then $S_{k[x]}(I) \cong S_{k[T]}(F)$. By computing the dimensions of each side, we see that

$$n = \dim S_{k[T]}(F) \geq \dim k[\mathbf{T}] + v(F_{(0)}) = n + v(F_{(0)}).$$

Then $v(F_{(0)}) = 0$ and F is a torsion module. This is equivalent to $I_d(B(\phi)) \not\subseteq (0)$ by [2, Lemma 1.4.6], so $I_d(B(\phi)) \neq 0$. $\square$

We actually get more information than this. We can show that, under the above hypotheses, if there are no equations of the Rees algebra coming from degrees 2 through $d-1$, then height $I_d(B(\phi)) \geq 2$. We will need a result proved by Lipman in [9] which is stated here for ease of reference.

Lemma 2.3. *Let S be a local ring and let M be a finitely generated S-module with torsion submodule M^{ι}. Let r be a nonnegative integer. The following conditions are equivalent:*

(i) *The smallest nonzero Fitting ideal of M is $I_r(M)$, and $I_r(M)$ is generated by a single regular element of S.*

(ii) *M is of finite presentation,* $\operatorname{proj}\dim(M) \leq 1$, *and M/M^{ι} is free of rank r.*

Some notation is needed to denote reduction modulo the maximal ideal $\mathfrak{m}$. Let $J(0) = (J + \mathfrak{m})/\mathfrak{m}R[\mathbf{T}]$, let $I_d(B(0)) = (I_d(B(\phi)) + \mathfrak{m})/\mathfrak{m}R[\mathbf{T}]$, and let $B(0)$ be the image of $B(\phi)$ in $k[\mathbf{T}]$ where $k = R/\mathfrak{m}$.

Proposition 2.4. *Let I be an ideal of $R = k[x_1, \ldots, x_d]$ with a minimal $n \times m$ linear presentation matrix ϕ with $m \geq n-1$ and $I_1(\phi) = (x_1, \ldots, x_d)$. Suppose I is of linear type on the punctured spectrum and $v(I) = n \geq d+2$. Assume that the defining ideal of $\mathscr{R}$ has no equations in degree $\leq d-1$ that do not come from degree 1. Then* height $I_d(B(\phi)) \geq 2$.

Proof. We know height $I_d(B(\phi)) \neq 0$ by Corollary 2.2. We need to see that height $I_d(B(\phi)) \neq 1$. First we show that the fiber ring $F(I)$ is a domain. Let $I = (\alpha_1, \ldots, \alpha_n)$ and $\mathfrak{m} = (\mathbf{x})$. Then $\mathscr{R} = k[\mathbf{x}][\alpha_1 t, \ldots, \alpha_n t]$ and $F(I) = \mathscr{R}/\mathfrak{m}\mathscr{R}$ so we have the sequence

$$0 \longrightarrow K \longrightarrow k[\alpha_1 t, \ldots, \alpha_n t] \longrightarrow F(I) \longrightarrow 0$$

with $K \subseteq k[\alpha_1 t, \ldots, \alpha_n t] \cap \mathfrak{m}\mathscr{R}$. Since ϕ has homogeneous entries of the same degree, I is generated by homogeneous forms of the same degree and this intersection vanishes, so $F(I) \cong k[\alpha_1 t, \ldots, \alpha_n t]$ is a domain.

The fiber ring $F(I) = \mathscr{R}/\mathfrak{m}\mathscr{R}$ can be written as $k[\mathbf{T}]/J(0)$, so $J(0)$ is prime. $I_d(B(\phi)) \subseteq k[\mathbf{T}]$, and so $I_d(B(\phi)) \subseteq J(0)$. $J(0)$ has no equations in degree less than d, so any nonzero degree d element of $I_d(B(\phi))$ is prime.

Assume height $I_d(B(\phi)) = 1$. Then $I_d(B(\phi))$ is contained in a height-one prime of $k[\boldsymbol{T}]$ which must be principal since $k[\boldsymbol{T}]$ is factorial. Say $I_d(B(\phi)) \subseteq (c)$ with $\deg(c) \geq 1$ since $(c) \neq k[\boldsymbol{T}]$. Then c divides $\det B_i(\phi)$ which implies $c = u \cdot \det B_i(\phi)$ where u is some unit and $\det B_i(\phi)$ is some $d \times d$ minor of $B(\phi)$. This says that $\det B_1(\phi)$ divides $\det B_i(\phi)$ for each i so $I_d(B(\phi))$ is principal. Then the module $F = \operatorname{coker} B(\phi)$ has projective dimension at most one by Lipman's argument.

The entries of $B(\phi)$ are in $(\boldsymbol{T})$, so we now show that

$$k[\boldsymbol{T}]^m \xrightarrow{B(\phi)} k[\boldsymbol{T}]^d \longrightarrow F \longrightarrow 0$$

is part of a minimal resolution of F. Let $B(\phi) = (v_1 | \ldots | v_m)$ and suppose $v_1 = \sum_{i=2}^m r_i v_i$ for some $r_i \in k[\boldsymbol{T}]$. Since $B(\phi)$ has linear entries, the coefficients r_i can be chosen from k. But then if $\boldsymbol{T} \cdot \phi = (f_1, \ldots, f_m) = \boldsymbol{x} \cdot B(\phi)$, $f_1 = \sum_{i=2}^m r_i f_i$ which contradicts the minimality of ϕ.

Since $\operatorname{proj\,dim} F \leq 1$, $B(\phi)$ must be one to one. Then $m \leq d$. But $m \geq n - 1 \geq d + 1$, which is a contradiction so height $I_d(B(\phi)) \geq 2$. $\square$

In order to use Proposition 2.4, we will need to show that there are no equations in degree $\leq d - 1$ that do not come from degree 1. In the codimension two case, we will achieve this by showing that the approximation complex resolving $S_t(I)$ for $t \leq d - 1$ is not only acyclic, but also the modules have enough depth to force $\operatorname{depth}(S_t(I)) \geq 1$. For codimension three, we verify that the complexes $\mathscr{D}_t$ of [8] resolve $S_t(I)$, bounding the projective dimension. This will ensure depth $S_t(I) \geq 1$ for $t \leq d - 1$ when d is odd and depth $S_t(I) \geq 1$ for $t \leq d - 2$ when d is even.

3. Ideals of codimension two

The focus of this section is Question 1.1. The key ingredient will be the following proposition.

Proposition 3.1. *Let R be a Cohen–Macaulay local domain of dimension d and let I be a codimension two perfect ideal of R minimally generated by n elements, with presentation matrix ϕ. Assume $I = I_{n-1}(\phi)$ is of linear type on the punctured spectrum of R and $I_1(\phi)$ is generated by a regular sequence $(\boldsymbol{x})$ of length d. If* $\operatorname{height}(I_d(B(\phi)) + \mathfrak{m})/\mathfrak{m}R[\boldsymbol{T}] \geq n - d$ *and $n \geq d + 2$ then $L = (\boldsymbol{x} \cdot B(\phi), I_d(B(\phi)))$ defines $\mathscr{R}$.*

Proof. Let J be the defining ideal of $\mathscr{R}$. We know $L \subseteq J$ and J is prime of height $n - 1$, so we need to show that L has the same height and is prime.

Let P be a prime minimal over L and $\mathfrak{p} = P \cap R$. Suppose $\mathfrak{m}$ is not contained in $\mathfrak{p}$. $I_{\mathfrak{p}}$ is of linear type so the relations of $\mathscr{R}_{\mathfrak{p}}$ are equal to the relations of $S(I)_{\mathfrak{p}}$ which are $(\boldsymbol{x} \cdot B(\phi))_{\mathfrak{p}}$. So we have the sequence

$$0 \longrightarrow (\boldsymbol{x} \cdot B(\phi))_{\mathfrak{p}} \longrightarrow R[\boldsymbol{T}]_{\mathfrak{p}} \longrightarrow \mathscr{R}_{\mathfrak{p}} \longrightarrow 0$$

and $\text{height}(\boldsymbol{x} \cdot B(\phi))_{\mathfrak{p}} = \text{height}(\mathfrak{p}) + n - (\text{height}(\mathfrak{p}) + 1) = n - 1$. Since $(\boldsymbol{x} \cdot B(\phi))_{\mathfrak{p}} \subseteq L_{\mathfrak{p}} \subseteq P_{\mathfrak{p}}$, $\text{height}\, P \geq n - 1$.

Now suppose $\mathfrak{m} = \mathfrak{p}$. We have

$$(I_d(B(\phi)), \mathfrak{m}) \subseteq (L, \mathfrak{m}) \subseteq P.$$

Further $\text{height}(I_d(B(\phi)), \mathfrak{m}) \geq \text{height}\, I_d(B(0)) + d$ since $(\boldsymbol{x})$ is a regular sequence of length d and the radical of $(\boldsymbol{x})$ is $\mathfrak{m}$. So height $P \geq \text{height}\, I_d(B(0)) + d$. Thus $\text{height}\, P \geq n$ by the hypothesis.

We now have that if P is minimal over L the height of P is at least $n - 1$. Thus L has height $n - 1$. In addition, if $L \subset P$ and $\mathfrak{m} = P \cap R$, then $\text{height}\, L \neq \text{height}\, P$.

Claim: *L is prime.*

We know that $\text{height}\, L = n - 1$. But this is the height for the generic case and the generic case is Cohen–Macaulay by [6, Example 3.4]. So by a specialization argument L is Cohen–Macaulay.

Since $R[\boldsymbol{T}]/L$ is Cohen–Macaulay, all associated primes of L must have height $n - 1$. In particular, $m \nsubseteq P$ for every associated prime P of $R[\boldsymbol{T}]/L$. Thus there is an element a of $\mathfrak{m}$ not contained in any associated prime of $R[\boldsymbol{T}]/L$. In other words, there is an element a in $\mathfrak{m}$ which is regular on $R[\boldsymbol{T}]/L$.

To check primality of L, it suffices to check primality of $(L)_a$. If I_a is of linear type, then as before $(\boldsymbol{x} \cdot B(\phi))_a$ is prime, has the correct height, and is contained in $(L)_a$, so $(L)_a$ is prime. So we must show that I_a is of linear type.

Consider the sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow S(I) \longrightarrow \mathscr{R} \longrightarrow 0.$$

Let $\mathfrak{p} \in \text{Spec}(R_a)$. Then a is not contained in $\mathfrak{p}$ so $(\mathscr{A}_a)_{\mathfrak{p}} = \mathscr{A}_{\mathfrak{p}}$. But $\mathfrak{p} \neq \mathfrak{m}$ since $a \in \mathfrak{m}$ so $\mathscr{A}_{\mathfrak{p}} = 0$, $\text{Supp}(\mathscr{A}_a) = \emptyset$ and $\mathscr{A}_a = 0$. This says that I_a is of linear type. But then L is prime. Thus $J = L$. □

Corollary 3.2. *Under the same conditions, $\mathscr{R}$ is Cohen–Macaulay.*

Proof. We have shown in the proof of the theorem that L defines $\mathscr{R}$ and is Cohen–Macaulay. This gives us that $\mathscr{R} \cong R[\boldsymbol{T}]/L$ is Cohen–Macaulay. □

Remark 3.3. We would like to generalize the proposition to ideals whose presentation matrices are generated by regular sequences of length $s \leq d$. To do this, an appropriate generalization of linear type on the punctured spectrum is needed. On examining the proof, one sees that the important feature is that I_P is of linear type for any P which does not contain $I_1(\phi)$. Thus if we assume I is of linear type on the open set $D(I_1(\phi))$ we can generalize Proposition 3.1, but the statement is not as straightforward.

Using Proposition 3.1, we can prove special cases of Question 1.1. First we will need the following lemma.

Lemma 3.4. *Let R be a Cohen–Macaulay local domain and let I be an ideal of R. If I is strongly Cohen–Macaulay, of linear type in codimension $s - 1$, and has height 2, then the approximation complex $\mathbb{Z}_t$ is acyclic and $\operatorname{depth} S_t(I) \geq 1$ for $t \leq s - 1$.*

Proof. Consider the approximation complex

$$0 \to Z_n \otimes B[-n] \to Z_{n-1} \otimes B[-n+1] \to \cdots \to Z_1 \otimes B[-1] \to B \to S(I) \to 0. \tag{1}$$

Here B is a polynomial ring in $n = v(I)$ variables and Z_i is the ith Koszul cycles. See [17, pp. 49–53]. In degree t, this becomes

$$0 \longrightarrow Z_t \xrightarrow{\psi_t} Z_{t-1} \otimes B_1 \xrightarrow{\psi_{t-1}} \cdots \longrightarrow Z_1 \otimes B_{t-1} \longrightarrow B_t \longrightarrow S_t(I) \longrightarrow 0. \tag{2}$$

Since I is strongly Cohen–Macaulay, $\operatorname{depth} Z_i \geq \min\{d, d - g + 2\}$ where $g = 2$ is the height of the ideal (see [17, pp. 66]). $I_\mathfrak{p}$ is of linear type for height $\mathfrak{p} < s$ and is strongly Cohen–Macaulay, so by [3, Theorems 5.1 and 12.9] $I_\mathfrak{p}$ is generated by a d-sequence and the localized approximation complex is acyclic. Then for each $i > 0$ for which $H_i(\mathbb{Z}_t) \neq 0$, the minimal elements of the support of $H_i(\mathbb{Z}_t)$ (which are the minimal associated primes) have height at least s. Let $\mathfrak{p}_0$ be a minimal element of the set of all primes which are minimal elements of the support of $H_i(\mathbb{Z}_t)$ for some $i > 0$. Localize (2) at $\mathfrak{p}_0$. As $\operatorname{depth} H_i(\mathbb{Z}_t)_{\mathfrak{p}_0} = 0$ and $\operatorname{depth} Z_i \otimes B_{i-1} = \operatorname{height} \mathfrak{p}_0 \geq s$ for all $i > 0$ so by the acyclicity lemma [11] (see also [17]) the localized complex is acyclic. But then $H_i(\mathbb{Z}_t)_{\mathfrak{p}_0} = 0$ for all $i > 0$ which can only occur if $H_i(\mathbb{Z}_t) = 0$ for all $i > 0$, so the sequence is acyclic.

Let $C_k = \operatorname{im} \psi_{t-k} = \ker \psi_{t-k-1}$. Then (2) is made up of a series of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Z_t \arrow[r, "\psi_t"] & Z_{t-1} \otimes B_1 \arrow[r, "\psi_{t-1}"] \arrow[dr] & Z_{t-2} \otimes B_2 \arrow[r] & \cdots \arrow[r, "\psi_1"] \arrow[dr] & Z_0 \otimes B_t \arrow[r] & S_t(I) \arrow[r] & 0 \\
& & & C_1 \arrow[u] & & C_{t-1} \arrow[u] & &
\end{tikzcd}$$

By chasing the depths through these short exact sequences, we get depth $C_k \geq d - k \geq 2$ since $k \leq d - 2$. In particular, $\operatorname{depth} C_{t-1} \geq d - t + 1 \geq 2$, so depth $S_t(I) \geq 1$ for $t \leq s - 1$. $\square$

Theorem 3.5. *Let I be a Cohen–Macaulay codimension two ideal of $R = k[x_1, \ldots, x_d]$, minimally generated by n elements, with presentation matrix ϕ. Assume $\phi: R^{n-1} \to R^n$ has linear entries with $I_1(\phi) = (x_1, \ldots, x_d) = (\mathbf{x})$ and $d = n - 2$. Assume $I = I_{n-1}(\phi)$ is of linear type on the punctured spectrum. Then $L = (\mathbf{x} \cdot B(\phi), I_d(B(\phi)))$ defines $\mathscr{R}$ (and $\mathscr{R}$ is Cohen–Macaulay).*

Proof. When ϕ has homogeneous entries of the same degree, both I and J are homogeneous ideals, so we may treat R as if it were local. By Proposition 3.1 it suffices to show that

$$\operatorname{height} I_d(B(0)) \geq 2 = n - d.$$

Since ϕ has linear entries in $(\boldsymbol{x})$, $B(\phi)$ has linear entries in $(\boldsymbol{T})$. Thus $I_d(B(\phi))$ remains unchanged under evaluation at $(\boldsymbol{x})$.

Notice that I is strongly Cohen–Macaulay (it is in the linkage class of a complete intersection, see [4, Example 2.1]), of linear type on the punctured spectrum, and has height 2. So by Lemma 3.4, we have depth $S_t(I) \geq 1$ for $t \leq d - 1$. Then the maximal ideal is not associated to $S_t(I)$ so if we localize $S_t(I)$ at any of its associated primes, $S_t(I)_{\mathfrak{p}} = R_{\mathfrak{p}}[I_{\mathfrak{p}}t]_t$. Thus $(\mathscr{A}_t)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in Ass(\mathscr{A}_t)$ and $\mathscr{A}_t = 0$. Since $S(I)$ has no torsion up to degree $d - 1$, there are no equations of the Rees algebra of degree $\leq d - 1$ that do not come from degree one. Apply Proposition 2.4 to get height $I_d(B(\phi)) \geq 2$. Thus by Proposition 3.1 and its corollary we have the result. □

We would now like to apply Proposition 3.1 to a more general setting. When I is of linear type on the punctured spectrum, we know that $\mathscr{A}_d$ is a module of finite length and so is annihilated by a power of $\mathfrak{m}$ and thus by a power of $I_1(\phi)$. If we assume that $\mathscr{A}_d$ is annihilated by $I_1(\phi)$ and that height$(J(0)_d) =$ height $J(0)$ then we are able to show height $I_d(B(0)) = n - d$ and thus are able to find the equations of the Rees algebra.

Theorem 3.6. *Let R be a Cohen–Macaulay local domain and let I be a codimension two perfect ideal of R minimally generated by $n > d + 1$ elements. Assume $I = I_{n-1}(\phi)$ is of linear type on the punctured spectrum of R and $I_1(\phi)$ is a regular sequence $(\boldsymbol{x})$ of length d. If* height$(J(0)_d) =$ height $J(0)$ *and* $(\boldsymbol{x})$ *annihilates* $\mathscr{A}_d$, *then* $(\boldsymbol{x} \cdot B(\phi), I_d(B(\phi)))$ *defines* $\mathscr{R}$ *(and* $\mathscr{R}$ *is Cohen–Macaulay).*

Proof. We know $(\boldsymbol{x})$ annihilates $\mathscr{A}_d = J_d / J_1 B_{d-1}$ where B_{d-1} is the degree $d - 1$ component of the polynomial ring $R[T_1, \dots, T_n]$ and $I_d(B(0)) \hookrightarrow J_d / J_1 B_{d-1}$. Let $h \in J_d$. Then $x_j h \in J_1 B_{d-1}$ so $x_j h = \sum g_{ij} f_i$ where $J_1 = (f_1, \dots, f_n)$ and $g_{ij} \in B_{d-1}$. Combine these equations to get

$$\boldsymbol{x} \cdot h = \boldsymbol{x} \cdot B(\phi) \cdot G$$

where $G = [g_{ij}]$ and $(f_1, \dots, f_n) = \boldsymbol{x} \cdot B(\phi)$. Rewrite this as

$$\boldsymbol{x}(hE_d - B(\phi) \cdot G) = 0$$

where E_d is the $d \times d$ identity matrix. Each row is a syzygy of the regular sequence $(x_1, \dots, x_d)$ so the entries of $(hE_d - B(\phi) \cdot G)$ must all be in $(\boldsymbol{x})R[\boldsymbol{T}]$. Let "(0)" denote reduction modulo $\mathfrak{m}$. Then $h(0) \in J(0)_d$ and $h(0)E_d = B(0) \cdot G(0)$. Let P be any prime in $k[\boldsymbol{T}]$ containing $I_d(B(0))$. Let $^-$ denote reduction modulo P. Rank $\overline{B(0)} \leq d - 1$ and

$\det(B(0) \cdot G(0)) = h(0)^d$ so $h(0) \in P$. Thus $I_d(B(0))$ and $(J(0)_d)$ have the same radical and in particular the same height. From the sequence

$$0 \longrightarrow J(0) \longrightarrow k[\boldsymbol{T}] \longrightarrow F(I) \longrightarrow 0$$

we see that height $J(0) = n - \ell$. Since I is of linear type on the punctured spectrum, satisfies sliding depth, and $v(I) \geq d + 1$, we have $\ell(I) = d$ by [17, Corollary 5.3.16]. Thus height $I_d(B(0)) = n - d$ and the result follows from Proposition 3.1 and its corollary. $\square$

Using the previous theorem, Bernd Ulrich is able to generalize Theorem 3.5 to hold for all $n \geq d + 2$ [14]. By working through the reduction number, he is able to show that conditions on the height of $(J(0)_d)$ and on the annihilator of $\mathscr{A}_d$ are satisfied under the hypotheses of Theorem 3.5.

One way to achieve the condition on the height of $J(0)$ is through the reduction number. We know that when there are no new equations in degrees 2 through $d - 1$, the reduction number is at least $d - 1$. If we assume that the reduction number is equal to $d - 1$ then we have height$(J(0)_d) =$ height $J(0)$. Combining the following with Proposition 3.6 will give us a result similar to [13, Theorem 5.5].

Proposition 3.7. *Let R be a Cohen–Macaulay local ring and let I be an ideal of R which is of linear type on the punctured spectrum, has sliding depth, and for which $v(I) > d$. Suppose that there are no equations of $\mathscr{R}$ of degree less than d which do not come from degree one. Then if the reduction number of I is $d - 1$,* height$(J(0)_d) =$ height$(J(0))$.

Proof. Since I is of linear type on the punctured spectrum, satisfies sliding depth, and $v(I) \geq d + 1$, we have $\ell(I) = d$ by [17, Corollary, 5.3.16]. Consider the fiber ring $F(I) = k[\boldsymbol{T}]/J(0)$. Let $k[z_1, \ldots, z_d] \hookrightarrow F(I)$ be a Noether normalization. Use a change of variables of the T's to choose a set of generators $a_1, \ldots, a_{v(I)}$ for $k[\boldsymbol{T}]$ with $a_i = z_i$ for $1 \leq i \leq d$. We know that the generators of $F(I)$ over $k[z_1, \ldots, z_d]$ have degree at most the reduction number of I which is $d - 1$, so by choosing the monomial order $a_1 < a_2 < \cdots < a_{v(I)}$, we have $a_{d+1}^d, \ldots, a_{v(I)}^d$ in the initial ideal of $J(0)$. In particular these elements are in the initial ideal of $J(0)_d$ since there are no equations of $\mathscr{R}$ of degree less than d which do not come from degree one. So height$(J(0)_d) \geq v(I) - d$. But we know

$$\text{height}(J(0)) = v(I) - \dim F(I) = v(I) - \ell(I) = v(I) - d$$

so height$(J(0)_d) =$ height$(J(0))$. $\square$

Corollary 3.8. *Let R be a Cohen–Macaulay local ring and let I be a codimension two ideal of R which is of linear type on the punctured spectrum, is strongly Cohen–Macaulay, and is minimally generated by $v(I) > d$ elements. Then if $\mathscr{R}$ is Cohen–Macaulay,* height$(J(0)_d) =$ height $J(0)$.

Proof. We may assume that the residue field is infinite. By [7, Theorem 2.3] we have the reduction number of I less than or equal to $d-1$. But since the reduction number is not zero and by Lemma 3.4 there are no new equations in degree less than d, we have $\mathrm{rd}(I) \geq d-1$. So $\mathrm{rd}(I) = d-1$ and the result follows from the proposition. □

If the hypothesis in Theorem 3.5 requiring the presentation matrix to have linear entries is removed, the theorem may fail. We will give an example of a codimension two ideal which is of linear type on the punctured spectrum (and $d = n-2$) whose Rees algebra is not Cohen–Macaulay. First we need a method of verifying linear type on the punctured spectrum.

Lemma 3.9. *Let R be a ring and I an ideal minimally generated by $v(I)$ elements with presentation matrix ϕ. Suppose R has dimension d, I has height g, and $k = v(I) - g$. If $I_k(\phi)$ has height at least d, then I is a complete intersection on the punctured spectrum.*

Proof. Let P be a prime of height $d-1$ which contains I. If I_P is of linear type for all such P, then I is of linear type on the punctured spectrum. Suppose one $k \times k$ minor of ϕ_P is invertible. Then ϕ_P can be transformed (using elementary operations) into

$$\begin{pmatrix} I_k & 0 \\ 0 & \phi'_P \end{pmatrix}.$$

Since ϕ_P presents I_P, I_P has at most g generators (consider the sequence

$$R^{m-k} \xrightarrow{\phi'_P} R^g \longrightarrow I_P \longrightarrow 0$$

where ϕ is a $v(I) \times m$ matrix). But I_P has at least g generators. So $v(I_P) = g$ and I_p is a complete intersection and thus of linear type by [3, Corollary 3.7]. To insure that for any P of height $d-1$ at least one $k \times k$ minor of ϕ_P is invertible, it suffices to have $\mathrm{height}(I_k(\phi)) \geq d$. □

Example 3.10. This example shows that in the setting of Theorem 3.5 if the entries of ϕ are not homogeneous of the same degree then the Rees algebra may not be Cohen–Macaulay.

Let $R = k[x, y, z]$ be a polynomial ring. Consider the matrix ϕ defining the ideal I

$$\begin{pmatrix} y^2 & 0 & -y^2 & 0 \\ 0 & x^2 & z^2 & xz \\ y^2 & -yz & 0 & z^2 \\ x^2 & 0 & z^2 & 0 \\ 0 & z & x & y \end{pmatrix}.$$

Using *Macaulay* [1] one finds the equations of the Rees algebra $\mathscr{R}$ of this ideal. The ideal of 3×3 minors has codimension $3 = d$, so by Lemma 3.9 I is of linear type on the punctured spectrum. However, in degree three, $(J(0)_3)$ is generated by

$-T_5^3, T_1T_5^2, -T_2T_5^2 - T_4T_5^2$ and so has height 1. In this example, $d = 3$, so by Corollary 3.8 since height$(J(0)_3) \neq 2$, $\mathscr{R}$ is not Cohen–Macaulay. □

4. Pfaffians

The next case where some structure is known is that of codimension three Gorenstein ideals. We will examine the equations of the Rees algebras of these ideals. As the examples at the end of this section show, the Rees algebras will not always be Cohen–Macaulay, however we are still able to discover the degrees of the generators of the equations.

For codimension three Gorenstein ideals we must consider the cases where d is even and where d is odd separately. The reason for these two cases arises from the lengths of the complexes $\mathscr{D}^t(I)$ of [8] which will resolve $S_t(I)$ for appropriate t.

Remark 4.1. Let R be a Cohen–Macaulay ring and let I be a perfect grade three Gorenstein ideal of R which is of linear type on the punctured spectrum. Then $\mathscr{D}^t(\phi)$ is a resolution of $S_t(I)$ for $t \leq d-1$ when d is odd and for $t \leq d-2$ when d is even. Thus depth $S_t(I) \geq 1$ for t in the above range.

Proof. By [8, Proposition 4.13] $H_0(\mathscr{D}^t) = S_t(I)$ and by [8, Observation 4.3(g)] the length of $\mathscr{D}^t(\phi)$ is at most $d-1$. Thus it suffices to check exactness on the punctured spectrum. For $Q \in V(I)$, Q not maximal, form the complex $\mathscr{D}^t(\phi_Q)$ over the ring R_Q. Since I is of linear type on the punctured spectrum, we have $\nu(I_P) \leq$ depth R_P for every $P \in Spec(R_Q)$, $I_Q \subseteq P$. Combining [8, Observations 6.23 and 6.17(c)(i)] (which can be done by the definitions of SPC_r and WPC_r given on p. 52 of [8]) we have that $\mathscr{D}^t(\phi_Q)$ is acyclic (although not necessarily minimal) for t in the given range. By observing that $(\mathscr{D}^t(\phi))_Q = \mathscr{D}^t(\phi_Q)$ we have the first part of the Remark.

Since the modules of $\mathscr{D}^t(\phi)$ are free, we have a bound on the projective dimension of $S_t(I)$ given by the length of $\mathscr{D}^t(\phi)$. Combine this with the Auslander–Buchsbaum formula applied to $S_t(I)$ to see that depth $S_t(I) \geq 1$ *for* $t \leq d-1$, d odd, or $t \leq d-2$, d even. □

Thus for d odd there are no new equations between degree one and degree $d-1$ and $L = (\boldsymbol{x} \cdot B(\phi), I_d(B(\phi)))$ is a candidate for the equations of $\mathscr{R}$ when $I_1(\phi)$ is generated by a regular sequence of length d. If $n = d+2$ and ϕ has linear entries in a polynomial ring, then by Proposition 2.4 height $I_d(B(\phi)) \geq 2$ and arguing as in the proof of Proposition 3.1 we see that the height of L is $n-1$. Thus if L is prime, we have all the generators for the equations. We are no longer able to use the specialization argument to see that L is prime, however.

For d even the equations have a different form. By Remark 4.1 we see that there are no generators of the equations in degrees 2 through $d-2$. There may be, however, generators in degree $d-1$. When $n=d+1$ and R is a polynomial ring, this is always the case for an ideal with a linear presentation matrix.

Remark 4.2. Let I be a strongly Cohen–Macaulay grade g ideal of $R=k[x_1,\dots,x_d]$ which is of linear type on the punctured spectrum and whose presentation matrix has linear entries with $I_1(\phi)=(x)$. If $n=d+1$ then $J=(\boldsymbol{x}\cdot B(\phi),f)$ where f is of degree $n-g+1$ and satisfies $f^r=\gcd(I_d(B(\phi)))$ for some r. Thus f can be found from the matrix $B(\phi)$.

Proof. By [12, Theorem 4.10(d)] we have $J=(\boldsymbol{x}\cdot B(\phi),f)$ where f is of degree $n-g+1$. It remains to find f from $B(\phi)$. We know that $I_d(B(\phi))\neq 0$ by Corollary 2.2. Since $n-d=1$ and $I_d(B(\phi))\subseteq J(0)$ which has height $n-d$, we have height $I_d(B(\phi))=1$. Suppose $I_d(B(\phi))$ is principal. Then by Lemma 2.3 the module $F=\operatorname{coker} B(\phi)$ has projective dimension at most one. But then $n\leq d$ which is a contradiction. Thus there is a gcd, denoted by h, of $I_d(B(\phi))$ of degree less than d. Using the isomorphism $S_{k[x]}(I)\cong S_{k[T]}(F)$ and the primary decomposition

$$0=\mathscr{A}\cap\mathfrak{m}S(I)$$

we see that $\mathscr{A}=(f)=\operatorname{Ann}(F)$. Since $I_d(B(\phi))=I_0(F)$, we have $f^d\in I_d(B(\phi))$. So $f^d=hv$ for some polynomial v and, since $(f)=\mathscr{A}$ is prime, by unique factorization $f^r=h$ for some r. □

Corollary 4.3. *Let I be a grade three ideal of Pfaffians of $R=k[x_1,\dots,x_d]$ which is of linear type on the punctured spectrum and whose presentation matrix has linear entries with $I_1(\phi)=(x)$. If $n=d+1$ then $(\boldsymbol{x}\cdot B(\phi),\gcd I_d(B(\phi)))=J$ and in particular there is a generator of degree $d-1$.*

Proof. Since the degree of f is $d-1$ we must have $t=1$ above. □

We need another computational test to determine if the Rees algebra of a given ideal is Cohen–Macaulay. The following test can be used to show that the Rees algebra is not Cohen–Macaulay.

Remark 4.4 (Bruns and Herzog [2, Corollary 4.1.10]). If the Rees algebra of an ideal I is Cohen–Macaulay, then all the coefficients h_i of the Hilbert–Poincaré function

$$\frac{h_0+h_1t+\cdots+h_rt^r}{(1-t)^{d+1}}$$

are positive.

Example 4.5. The second set of counterexamples shows that for a grade three ideal of Pfaffians which is of linear type on the punctured spectrum and has a linear presentation matrix the Rees algebra need not be Cohen–Macaulay either in the case $n = d + 2$ or the case $n = d + 3$.

Let $R = k[x_1, \dots, x_4]$ and let ϕ be the matrix

$$\begin{pmatrix} 0 & -x_1 & -x_3 & x_2 & -x_1 & x_4 & -x_3 \\ x_1 & 0 & -x_3 & x_2 & x_1 & -x_4 & -x_1 \\ x_3 & x_3 & 0 & 0 & -x_3 & x_1 & -x_4 \\ -x_2 & -x_2 & 0 & 0 & -x_4 & x_2 & 0 \\ x_1 & -x_1 & x_3 & x_4 & 0 & -x_3 & x_1 \\ -x_4 & x_4 & -x_1 & -x_2 & x_3 & 0 & -x_2 \\ x_3 & x_1 & x_4 & 0 & -x_1 & x_2 & 0 \end{pmatrix}$$

or let $R = k[x_1, \dots, x_5]$ and let ϕ be

$$\begin{pmatrix} 0 & -x_1 & -x_3 & x_2 & -x_5 & x_4 & -x_3 \\ x_1 & 0 & -x_3 & x_2 & x_1 & x_5 & -x_1 \\ x_3 & x_3 & 0 & 0 & -x_3 & x_1 & -x_4 \\ -x_2 & -x_2 & 0 & 0 & -x_4 & x_2 & 0 \\ x_5 & -x_1 & x_3 & x_4 & 0 & -x_3 & x_1 \\ -x_4 & -x_5 & -x_1 & -x_2 & x_3 & 0 & -x_2 \\ x_3 & x_1 & x_4 & 0 & -x_1 & x_2 & 0 \end{pmatrix}.$$

Let I be the ideal of Pfaffians of ϕ. To see that the ideal is of linear type on the punctured spectrum, compute the ideal of 4×4 minors of ϕ. The codimension of this ideal is d in each case, so the ideal of Pfaffians is of linear type on the punctured spectrum. (See Lemma 3.9 for details.)

Now compute the Hilbert function of the ideal of equations of $\mathscr{R}$ (this was done using *Macaulay* [1]). For the first matrix, the Hilbert function is

$$\frac{1 + 6t + 14t^2 + 11t^3 - 19t^4 + 5t^5 - t^6}{(1-t)^5},$$

and for the second matrix it is

$$\frac{1 + 6t + 14t^2 + 15t^3 + 6t^4 - 15t^5 + 20t^6 - 15t^7 + 6t^8 - t^9}{(1-t)^6}.$$

So by Remark 4.4, $\mathscr{R}$ cannot be Cohen–Macaulay. For the second matrix, the equations of $\mathscr{R}$ are of the expected form. This can be seen by computing both L and J and showing that $L:J = (1)$. However, for the first matrix there are generators of J in degree $d - 1$.

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