Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 243 (2007) 101–122

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability radii of positive linear Volterra–Stieltjes equations

Pham Huu Anh Ngoc

Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany

Received 24 November 2006

Available online 24 October 2007

Abstract

We study stability radii of linear Volterra–Stieltjes equations under multi-perturbations and affine perturbations. A lower and upper bound for the complex stability radius with respect to multi-perturbations are given. Furthermore, in some special cases concerning the structure matrices, the complex stability radius can precisely be computed via the associated transfer functions. Then, the class of positive linear Volterra–Stieltjes equations is studied in detail. It is shown that for this class, complex, real and positive stability radius under multi-perturbations or multi-affine perturbations coincide and can be computed by simple formulae expressed in terms of the system matrices. As direct consequences of the obtained results, we get some results on robust stability of positive linear integro-differential equations and of positive linear functional differential equations. To the best of our knowledge, most of the results of this paper are new.

MSC: 93D09; 34D10

Keywords: Linear Volterra–Stieltjes equation; Positive system; Uniformly asymptotic stability; Stability radius

1. Introduction

Motivated by many applications in control engineering, problems of robust stability of dynamical systems have attracted a lot of attention from researchers during the last twenty years. In the study of these problems, the notion of stability radius was proved to be a very effective tool. By definition, the stability radius of a given asymptotically stable system $\dot{x}(t) = Ax(t)$ is

E-mail address: phanhngoc@yahoo.com.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.08.005

the *maximal* $\gamma > 0$ for which all the systems of the form

$$\dot{x}(t) = (A + D\Delta E)x(t), \quad \|\Delta\| < \gamma$$

are asymptotically stable. Here, Δ is unknown disturbance matrix, D and E are given matrices defining the structure of the perturbations. Depending upon whether complex or real disturbances Δ are considered this maximal γ is called complex or real stability radius, respectively. The basic problem in the study of robustness of stability of the system is to characterize and compute these radii in terms of given matrices A, D, E. It is important to note that these two stability radii are in general distinct. The analysis and computation of the complex stability radius for systems under structured perturbations has been done first in [6] in 1986 and extended later in many subsequent papers (see [7] for a survey up till 1990) while the computation of the real stability radius, being a much more difficult problem, was solved ten years later (with a very complicated solution) by a group of international researchers, see [24].

The situation is much simpler for the class of positive systems. It has been shown in [25,26] that if A is a Metzler matrix (that is, the system $\dot{x}(t) = Ax(t)$, $t \geqslant 0$, is positive) and D, E are nonnegative matrices, then the complex and real stability radii coincide and can be computed directly by a simple formula. Then this result has been extended only in recent time to many various classes of positive systems such as positive continuous time-delay systems, see e.g. [27,28], discrete time-delay systems, see e.g. [10,19] and positive linear functional differential equations, see e.g. [22,29].

It is worth noticing that the notion of stability radius can be extended to various perturbation types [7]. Among perturbation types, two of the following perturbation types

$$A \to A + \sum_{i=1}^{N} D_i \Delta_i E_i \quad (\textit{multi-perturbation}),$$

$$A \to A + \sum_{i=1}^{N} \delta_i A_i \quad (\textit{affine-perturbation})$$

are most well known in control theory and include perturbation types studied in the literature.

In this paper, we consider problems of computing stability radii of linear Volterra–Stieltjes equations of the form

$$\dot{x}(t) = Ax(t) + \int_0^t d\big[B(s)\big]x(t-s), \quad t \geqslant 0, \tag{1}$$

under multi-perturbations and affine perturbations of the matrix A and the matrix function $B(\cdot)$. To the best of our knowledge, these problems have never been considered in the literature. The present paper is based on our latest results on positive linear Volterra–Stieltjes equations [15,17,18].

The organization of the paper is as follows. In the next section, we summarize some notations and preliminary results which will be used in the sequel. In Section 3, after giving a brief background on linear Volterra–Stieltjes equations, we deal with the problem of computing stability radii of linear Volterra–Stieltjes equations under multi-perturbations. A lower and upper bound for the complex stability radius with respect to multi-perturbations are given. Furthermore, in some special cases concerning the structure matrices, the complex stability radius can precisely be computed via the associated transfer functions. Then, for the class of positive linear Volterra–Stieltjes equations, it will be shown that the complex, real and positive stability radius under multi-perturbations coincide and a simple formula for their computation is established. Next, we consider the problem of computing stability radii of positive linear Volterra–Stieltjes equations under affine perturbations and an explicit formula for computing the stability radii is given. In the last section, as particular cases of the obtained results, we present some results on robust stability of positive linear integro-differential equations and of positive linear functional differential equations which are encountered frequently in applications. Finally, we give two examples to illustrate the obtained ones.

2. Preliminaries

In this section we shall define some notations and recall some well-known results which will be used in the subsequent sections. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. Let us denote

$$\mathbb{C}^- := \{z \in \mathbb{C}\colon \Re z < 0\}; \qquad \mathbb{C}^+ := \{z \in \mathbb{C}\colon \Re z > 0\}.$$

Then $\mathbb{C} = \mathbb{C}^- \cup \overline{\mathbb{C}^+}$. For integers $l, q \geqslant 1$, $\mathbb{K}^l$ denotes the l-dimensional vector space over $\mathbb{K}$, $(\mathbb{K}^l)^*$ is its dual and $\mathbb{K}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geqslant B$ iff $a_{ij} \geqslant b_{ij}$ for $i = 1, \dots, l$, $j = 1, \dots, q$. We denote by $\mathbb{R}_+^{l\times q}$ the set of all nonnegative matrices $A \geqslant 0$. Similar notations are adopted for vectors. For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. Then, it is easy to see that $|CD| \leqslant |C||D|$. For any matrix $A \in \mathbb{K}^{n\times n}$ the *spectral radius* and *spectral abscissa* of A are denoted by $\rho(A) = \max\{|\lambda|;\ \lambda \in \sigma(A)\}$ and $\mu(A) = \max\{\Re\lambda;\ \lambda \in \sigma(A)\}$, where $\sigma(A) := \{s \in \mathbb{C};\ \det(sI_n - A) = 0\}$ is the spectrum of A. $A \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of A are nonnegative or, equivalently, $tI + A \geqslant 0$ for some $t \geqslant 0$. It is clear that any $A \in \mathbb{R}_+^{n\times n}$ is a Metzler matrix and, moreover, $\rho(A) = \mu(A)$.

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leqslant \|y\|$ whenever $x, y \in \mathbb{K}^n$, $|x| \leqslant |y|$. Every p-norm on $\mathbb{K}^n$, $1 \leqslant p \leqslant \infty$, is monotonic. Throughout the paper, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^q$, that is $\|P\| = \max\{\|Py\|;\ \|y\| = 1\}$. We note that the operator norm is in general not monotonic norm on $\mathbb{K}^{l\times q}$ even if $\mathbb{K}^l$, $\mathbb{K}^q$ are provided with monotonic norms. However, such monotonicity holds for nonnegative matrices. Moreover, we have (see e.g. [26])

$$P \in \mathbb{K}^{l\times q},\ Q \in \mathbb{R}_+^{l\times q}, \quad |P| \leqslant Q \quad \Rightarrow \quad \|P\| \leqslant \big\||P|\big\| \leqslant \|Q\|. \tag{2}$$

We now summarize in the following theorem some existing results on properties of Metzler matrices which will be used in the sequel (see e.g. [1,26]).

Theorem 2.1. *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*:

(i) (*Perron–Frobenius*) *$\mu(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \geqslant 0$, $x \neq 0$ such that $Ax = \mu(A)x$.*

(ii) *Given* $\alpha \in \mathbb{R}$, *there exists a nonzero vector* $x \geqslant 0$ *such that* $Ax \geqslant \alpha x$ *if and only if* $\mu(A) \geqslant \alpha$.
(iii) $(tI_n - A)^{-1}$ *exists and is nonnegative if and only if* $t > \mu(A)$.
(iv) *Given* $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. *Then*

$$|C| \leqslant B \quad \Rightarrow \quad \mu(A + C) \leqslant \mu(A + B). \tag{3}$$

Let $\mathbb{K}^{m\times n}$ be endowed with the norm $\|\cdot\|$ and $C([\alpha,\beta],\mathbb{K}^{m\times n})$ be the Banach space of all continuous functions on $[\alpha,\beta]$ with values in $\mathbb{K}^{m\times n}$ normed by the maximum norm $\|\phi\| = \max_{\theta\in[\alpha,\beta]}\|\phi(\theta)\|$. Let J be an interval of $\mathbb{R}$. For a matrix function $\phi(\cdot): J \to \mathbb{R}^{m\times n}$, we say that $\phi(\cdot)$ is nonnegative and denote it by $\phi(\cdot) \geqslant 0$ if $\phi(\theta) \geqslant 0$ almost everywhere on J. A matrix function $\eta(\cdot): J \to \mathbb{R}^{m\times n}$ is called increasing on J, if

$$\eta(\theta_2) \geqslant \eta(\theta_1) \quad \text{for } \theta_1, \theta_2 \in J,\ \theta_1 < \theta_2.$$

To make the presentation self-contained we present here some basic facts on vector-valued functions of bounded variation and relative knowledge.

A matrix function $\eta(\cdot): [\alpha,\beta] \to \mathbb{K}^{m\times n}$ is said to be of bounded variation if

$$\operatorname{Var}(\eta;\alpha,\beta) := \sup_{P[\alpha,\beta]} \sum_k \big\|\eta(\theta_k) - \eta(\theta_{k-1})\big\| < +\infty,$$

where the supremum is taken over the set of all finite partitions of the interval $[\alpha,\beta]$. The set $BV([\alpha,\beta],\mathbb{K}^{m\times n})$ of all matrix functions $\eta(\cdot)$ of bounded variation on $[\alpha,\beta]$ satisfying $\eta(\alpha) = 0$ is a Banach space endowed with the norm $\|\eta\| = \operatorname{Var}(\eta;\alpha,\beta)$.

Given $\eta(\cdot) \in BV([\alpha,\beta],\mathbb{K}^{m\times n})$ then for any continuous functions $\gamma \in C([\alpha,\beta],\mathbb{K})$ and $\phi \in C([\alpha,\beta],\mathbb{K}^n)$, the integrals

$$\int_\alpha^\beta \gamma(\theta)\,d\big[\eta(\theta)\big] \quad \text{and} \quad \int_\alpha^\beta d\big[\eta(\theta)\big]\,\phi(\theta)$$

exist and are defined respectively as the limits of $S_1(P) := \sum_{k=1}^p \gamma(\zeta_k)(\eta(\theta_k) - \eta(\theta_{k-1}))$ and $S_2(P) := \sum_{k=1}^p (\eta(\theta_k) - \eta(\theta_{k-1}))\phi(\zeta_k)$ as $d(P) := \max_k |\theta_k - \theta_{k-1}| \to 0$, where $P = \{\theta_1 = \alpha \leqslant \theta_2 \leqslant \cdots \leqslant \theta_p = \beta\}$ is any finite partition of the interval $[\alpha,\beta]$ and $\zeta_k \in [\theta_{k-1},\theta_k]$. It is immediate from the definition that

$$\begin{aligned}
\left\|\int_\alpha^\beta \gamma(\theta)\,d\big[\eta(\theta)\big]\right\| &\leqslant \max_{\theta\in[\alpha,\beta]}\big|\gamma(\theta)\big|\|\eta\|,\\
\left\|\int_\alpha^\beta d\big[\eta(\theta)\big]\,\phi(\theta)\right\| &\leqslant \max_{\theta\in[\alpha,\beta]}\big\|\phi(\theta)\big\|\|\eta\|.
\end{aligned} \tag{4}$$

Let $L : C([\alpha,\beta],\mathbb{K}^n) \to \mathbb{K}^n$ be a linear bounded operator. Then, by the Riesz representation theorem, there exists a unique matrix function $\eta(\cdot) \in BV([\alpha,\beta],\mathbb{K}^{n\times n})$ which is *continuous from the right* (or briefly c.f.r.) on (α,β) such that

$$L\phi = \int_{\alpha}^{\beta} d\big[\eta(\theta)\big]\phi(\theta), \quad \forall \phi \in C\big([\alpha,\beta],\mathbb{K}^n\big). \tag{5}$$

Finally, the following spaces will be used frequently in the subsequent sections

$$NBV\big([\alpha,\beta],\mathbb{K}^{l\times q}\big) := \big\{\eta \in BV\big([\alpha,\beta],\mathbb{K}^{l\times q}\big);\ \eta(\alpha)=0,\ \eta \text{ is c.f.r. on } [\alpha,\beta]\big\}. \tag{6}$$

$$NBV\big(\mathbb{R}_+,\mathbb{K}^{l\times q}\big) := \Bigg\{\delta(\cdot)\colon\ \mathbb{R}_+ \to \mathbb{K}^{l\times q}/\delta(\cdot) \text{ is c.f.r. on } \mathbb{R}_+,\ \delta(0)=0, \text{ and } \|\delta\| := \int_0^{+\infty} \big|d\delta(s)\big| < +\infty\Bigg\}. \tag{7}$$

3. Robust stability of linear Volterra–Stieltjes equations

3.1. A brief background on linear Volterra–Stieltjes equations

Consider a linear Volterra–Stieltjes equation of the form (1), where $A \in \mathbb{R}^{n\times n}$ is a given matrix and $B(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a given matrix function of locally bounded variation on $\mathbb{R}_+$. Furthermore, we always assume that $B(\cdot)$ is normalized to be right-continuous on $\mathbb{R}_+$ and vanishes at 0. From theory of Volterra integro-differential equations (see e.g. [3]), it is well known that there exists a unique locally absolutely continuous matrix function $R(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ such that

$$\dot{R}(t) = AR(t) + \int_0^t d\big[B(s)\big]R(t-s), \quad \text{a.e. on } \mathbb{R}_+, \qquad R(0) = I_n. \tag{8}$$

Then $R(\cdot)$ is called the *resolvent* of Eq. (1). Moreover, for a given $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$, the following nonhomogeneous equation

$$\dot{x}(t) = Ax(t) + \int_0^t d\big[B(s)\big]x(t-s) + f(t), \quad t \in \mathbb{R}_+, \tag{9}$$

has a unique locally absolutely continuous solution $x(\cdot)$ satisfying the initial condition $x(0) = x_0$ and it is given by the variation of constants formula

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \in \mathbb{R}_+, \tag{10}$$

see e.g. [3].

Definition 3.1. Let $\sigma \in \mathbb{R}_+$ and $\phi \in C([0,\sigma],\mathbb{R}^n)$. A vector function $x(\cdot):\mathbb{R}_+ \to \mathbb{R}^n$ is called a solution of Eq. (1) through (σ,ϕ) if $x(\cdot)$ is absolutely continuous on any bounded subinterval of $[\sigma,+\infty)$ and satisfies (1) almost everywhere on $[\sigma,+\infty)$, $x(t)=\phi(t)$, $\forall t \in [0,\sigma]$. We denote it by $x(\cdot,\sigma,\phi)$.

Remark 3.2. By the fact mentioned above on solution of the nonhomogeneous equation (9), it is easy to check that for a fixed $\sigma \in \mathbb{R}_+$ and a given $\phi \in C([0,\sigma],\mathbb{R}^n)$, there exists a unique solution of Eq. (1) through (σ,ϕ) and it is given by the formula

$$x(t+\sigma,\sigma,\phi) = R(t)\phi(\sigma) + \int_0^t R(t-u)\left\{\int_u^{u+\sigma} d\big[B(s)\big]\phi(u+\sigma-s)\right\}du, \quad t \in \mathbb{R}_+. \tag{11}$$

Definition 3.3. The zero solution of Eq. (1) is said to be uniformly stable (US) if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\phi \in C\big([0,\sigma],\mathbb{R}^n\big), \quad \|\phi\| < \delta \quad \Rightarrow \quad \big\|x(t,\sigma,\phi)\big\| < \epsilon, \quad \forall t \geqslant \sigma.$$

Definition 3.4. The zero solution of Eq. (1) is said to be uniformly asymptotically stable (UAS) if it is US and if there exists $\delta_0 > 0$ such that $\forall \epsilon > 0$, $\exists T(\epsilon) > 0$:

$$\phi \in C\big([0,\sigma],\mathbb{R}^n\big), \quad \|\phi\| < \delta_0 \quad \Rightarrow \quad \big\|x(t,\sigma,\phi)\big\| < \epsilon, \quad \forall t \geqslant T(\epsilon).$$

If the zero solution of Eq. (1) is US (UAS) then we say that Eq. (1) is US (UAS), respectively. In the rest of this paper, we suppose

$$\int_0^{+\infty} \big|dB(t)\big| < +\infty, \tag{12}$$

that is $B(\cdot) \in NBV(\mathbb{R}_+,\mathbb{R}^{n\times n})$. Then, the Laplace–Stieltjes transform of $B(\cdot)$ is defined by

$$\tilde{B}(z) := \int_0^{+\infty} e^{-zs}\,dB(s), \tag{13}$$

which is well defined for every $z \in \mathbb{C}$, $\Re z \geqslant 0$. Let us define

$$H(z) := zI_n - A - \tilde{B}(z), \tag{14}$$

for appropriate $z \in \mathbb{C}$. Then, $H(\cdot)$ is called the *characteristic matrix* of Eq. (1). Denote by $\sigma(A,B(\cdot))$ the set of all roots of the *characteristic equation* of Eq. (1). That is,

$$\sigma\big(A,B(\cdot)\big) := \big\{z \in \mathbb{C}\colon \det H(z) = 0\big\}. \tag{15}$$

Theorem 3.5. *(See [3,15].) Suppose* (12) *holds true. Then the following statements are equivalent*:

(i) $\sigma(A, B(\cdot)) \subset \mathbb{C}^-$;
(ii) *the resolvent* $R(\cdot)$ *of Eq.* (1) *belongs to* $L^1([0, +\infty), \mathbb{R}^{n\times n})$;
(iii) *Eq.* (1) *is UAS.*

3.2. Stability radii of linear Volterra–Stieltjes equations under multi-perturbations

Assume that Eq. (1) is UAS and A, $B(\cdot)$ are subjected to multi-perturbations of the type

$$A \to A_\Delta := A + \sum_{j=1}^{N} D_{0j}\Delta_j E_{0j},$$

$$B(\cdot) \to B_\delta(\cdot) := B(\cdot) + \sum_{j=1}^{N} D_{1j}\delta_j(\cdot)E_{1j}. \tag{16}$$

In other words, we consider perturbed systems of the form

$$\dot{x}(t) = \left(A + \sum_{j=1}^{N} D_{0j}\Delta_j E_{0j}\right)x(t) + \int_0^t d\left[B(s) + \sum_{j=1}^{N} D_{1j}\delta_j(s)E_{1j}\right]x(t-s), \quad t \geqslant 0. \tag{17}$$

Here $D_{ij} \in \mathbb{C}^{n\times l_{ij}}$, $E_{ij} \in \mathbb{C}^{q_{ij}\times n}$, $i \in I := \{0, 1\}$, $j \in \underline{N} := \{1, 2, \ldots, N\}$ are given matrices determining the *structure* of perturbations and $\Delta_j \in \mathbb{K}^{l_{0j}\times q_{0j}}$, $\delta_j(\cdot) \in NBV(\mathbb{R}_+, \mathbb{K}^{l_{1j}\times q_{1j}})$, $j \in \underline{N}$, are unknown disturbances.

Denote by

$$[\Delta, \delta(\cdot)] := \big((\Delta_1, \ldots, \Delta_N), \big(\delta_1(\cdot), \ldots, \delta_N(\cdot)\big)\big),$$

where $\Delta_j \in \mathbb{K}^{l_{0j}\times q_{0j}}$, $\delta_j(\cdot) \in NBV(\mathbb{R}_+, \mathbb{K}^{l_{1j}\times q_{1j}})$, $j \in \underline{N}$. We shall measure the size of each perturbation $[\Delta, \delta(\cdot)]$ by the norm

$$\big\|[\Delta, \delta(\cdot)]\big\| := \sum_{j=1}^{N}\big(\|\Delta_j\| + \big\|\delta_j(\cdot)\big\|\big). \tag{18}$$

Denote by $\sigma(A_\Delta, B_\delta(\cdot))$ the set of all roots of the characteristic equation of the perturbed equation (17). That is

$$\sigma\big(A_\Delta, B_\delta(\cdot)\big) := \left\{z \in \mathbb{C}\colon \det\left(zI_n - A_\Delta - \int_0^{+\infty} e^{-zs}\, dB_\delta(s)\right) = 0\right\}. \tag{19}$$

Recall that, by Theorem 3.5, a perturbed equation (17) is UAS if and only if $\sigma(A_\Delta, B_\delta(\cdot)) \subset \mathbb{C}^-$. Let us define

$$\mathcal{D}_{\mathbb{C}} := \big\{\big[\Delta, \delta(\cdot)\big]\colon\ \Delta_j \in \mathbb{C}^{l_{0j}\times q_{0j}};\ \delta_j(\cdot) \in NBV\big(\mathbb{R}_+, \mathbb{C}^{l_{1j}\times q_{1j}}\big),\ j \in \underline{N}\big\},$$
$$\mathcal{D}_{\mathbb{R}} := \big\{\big[\Delta, \delta(\cdot)\big]\colon\ \Delta_j \in \mathbb{R}^{l_{0j}\times q_{0j}};\ \delta_j(\cdot) \in NBV\big(\mathbb{R}_+, \mathbb{R}^{l_{1j}\times q_{1j}}\big),\ j \in \underline{N}\big\},$$
$$\mathcal{D}_{+} := \big\{\big[\Delta, \delta(\cdot)\big]\colon\ \Delta_j \in \mathbb{R}_+^{l_{0j}\times q_{0j}};\ \delta_j(\cdot) \in NBV\big(\mathbb{R}_+, \mathbb{R}^{l_{1j}\times q_{1j}}\big),$$
$$\delta_j(\cdot) \text{ is increasing on } \mathbb{R}_+,\ j \in \underline{N}\big\}.$$

Then $\mathcal{D}_{\mathbb{C}}$, $\mathcal{D}_{\mathbb{R}}$, $\mathcal{D}_{+}$ is called respectively the *class of complex, real, nonnegative perturbations*.

In the sequel, we always define $\inf\emptyset = +\infty$, $0^{-1} = +\infty$. To study robustness of stability of the linear Volterra–Stieltjes equation (1), we introduce the following.

Definition 3.6. Let the linear Volterra–Stieltjes equation (1) be UAS. The complex, real and positive stability radius of the equation with respect to multi-perturbations of the form (16), measured by the norm (18), is defined respectively by

$$r_{\mathbb{C}} = \inf\big\{\big\|\big[\Delta, \delta(\cdot)\big]\big\|\colon\ \big[\Delta, \delta(\cdot)\big] \in \mathcal{D}_{\mathbb{C}},\ \sigma\big(A_\Delta, B_\delta(\cdot)\big) \not\subset \mathbb{C}^-\big\},$$
$$r_{\mathbb{R}} = \inf\big\{\big\|\big[\Delta, \delta(\cdot)\big]\big\|\colon\ \big[\Delta, \delta(\cdot)\big] \in \mathcal{D}_{\mathbb{R}},\ \sigma\big(A_\Delta, B_\delta(\cdot)\big) \not\subset \mathbb{C}^-\big\},$$

and

$$r_{+} = \inf\big\{\big\|\big[\Delta, \delta(\cdot)\big]\big\|\colon\ \big[\Delta, \delta(\cdot)\big] \in \mathcal{D}_{+},\ \sigma\big(A_\Delta, B_\delta(\cdot)\big) \not\subset \mathbb{C}^-\big\}.$$

From the definition, it is easy to see that

$$0 < r_{\mathbb{C}} \leqslant r_{\mathbb{R}} \leqslant r_+ \leqslant +\infty. \tag{20}$$

We define the associated transfer functions $G_{(i,j;\,u,v)}: \mathbb{C} \setminus \sigma(A, B(\cdot)) \to \mathbb{C}^{q_{ij}\times l_{uv}}$ of the perturbed system (17) by setting

$$G_{(i,j;\,u,v)}(z) = E_{ij} H(z)^{-1} D_{uv}, \quad i, u \in I;\ j, v \in \underline{N}. \tag{21}$$

Lemma 3.7. *Suppose Eq.* (1) *is UAS. Let $i_0 \in I$, $j_0 \in \underline{N}$ be fixed. Then*:

(i) *For given matrices $D \in \mathbb{C}^{n\times l}$ and $E \in \mathbb{C}^{q\times n}$, we get*

$$\max_{z\in\mathbb{C},\,\Re z\geqslant 0}\left\| E\left(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s)\right)^{-1} D\right\|$$
$$= \max_{z\in\mathbb{C},\,\Re z= 0}\left\| E\left(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s)\right)^{-1} D\right\|. \tag{22}$$

(ii) *If $G_{(i_0,j_0;\,i_0,j_0)}(z_0) \neq 0$, for some $z_0 \in \mathbb{C}$, $\Re z_0 = 0$, then there exists a complex perturbation $[\Delta, \delta(\cdot)] \in \mathcal{D}_{\mathbb{C}}$ such that*

$$\big\|\big[\Delta, \delta(\cdot)\big]\big\| = \frac{1}{\|G_{(i_0,j_0;\,i_0,j_0)}(z_0)\|} \tag{23}$$

and

$$\sigma\big(A_\Delta, B_\delta(\cdot)\big) \not\subset \mathbb{C}^-. \tag{24}$$

Moreover, if $G_{(i_0,j_0;\,i_0,j_0)}(0) \in \mathbb{R}_+^{q_{i_0j_0}\times l_{i_0j_0}}$ *and* $G_{(i_0,j_0;\,i_0,j_0)}(0) \neq 0$, *then there exists a nonnegative perturbation* $[\Delta, \delta(\cdot)] \in \mathcal{D}_+$ *satisfying* (23)–(24) *for* $z_0 = 0$.

Proof. (i) First, we note that $F(z) := \|E(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s))^{-1}D\|$ is well defined on the domain $\overline{\mathbb{C}^+}$ and is a subharmonic function on $\mathbb{C}^+ := \{z \in \mathbb{C},\ \Re z > 0\}$, see e.g. [5, p. 100]. Moreover, it is easy to see that $\lim_{|z|\to+\infty,\,\Re z\geqslant 0} F(z) = 0$. Therefore, by the extended maximum principal of subharmonic functions (see e.g. [11]), we get (i).

(ii) By the definition of operator norm, there exists a vector $u_0 \in \mathbb{C}^{l_{i_0j_0}}$, $\|u_0\| = 1$, such that $\|G_{(i_0,j_0;\,i_0,j_0)}(z_0)\| = \|G_{(i_0,j_0;\,i_0,j_0)}(z_0)u_0\|$. Then, by the Hahn–Banach theorem, there exists a linear form $y^* \in (\mathbb{C}^{q_{ij}})^*$ of dual norm $\|y^*\| = 1$ such that

$$y^* G_{(i_0,j_0;\ i_0,j_0)}(z_0)u_0 = \big\|G_{(i_0,j_0;\,i_0,j_0)}(z_0)u_0\big\|.$$

Define

$$\Delta_0 := \big\|G_{(i_0,j_0;\,i_0,j_0)}(z_0)\big\|^{-1} u_0 y^* \in \mathbb{C}^{l_{i_0j_0}\times q_{i_0j_0}}.$$

It is easy to see that $\|\Delta_0\| = \|G_{(i_0,j_0;\,i_0,j_0)}(z_0)\|^{-1}$. Set

$$x_0 := \left(z_0 I_n - A - \int_0^{+\infty} e^{-z_0 s}\,dB(s)\right)^{-1} D_{i_0j_0}u_0.$$

Then, we get $\Delta_0 E_{i_0j_0}x_0 = u_0$, which implies that $x_0 \neq 0$. Thus, $x_0 = (z_0 I_n - A - \int_0^{+\infty} e^{-z_0 s}\,dB(s))^{-1} D_{i_0j_0}\Delta_0 E_{i_0j_0}x_0$. It follows that

$$\left(z_0 I_n - (A + D_{i_0j_0}\Delta_0 E_{i_0j_0}) - \int_0^{+\infty} e^{-z_0 s}\,dB(s)\right)x_0$$
$$= \left(z_0 I_n - A - \int_0^{+\infty} e^{-z_0 s}\,d\big[B(s) + D_{i_0j_0}\delta_0(s)E_{i_0j_0}\big]\right)x_0 = 0,$$

where $\delta_0(s) := (\int_0^s e^{(z_0-1)\tau}\,d\tau)\Delta$, $s \in [0, +\infty)$. We consider two separate cases as follows:

- If $i_0 = 0$, then we set $\Delta := (\Delta_1, \dots, \Delta_N)$ where $\Delta_j = \Delta_0$, if $j = j_0$ otherwise $\Delta_j = 0$ and $\delta(\cdot) := (\delta_1(\cdot), \dots, \delta_N(\cdot)) = (0, \dots, 0)$. Then it is easy to see that $[\Delta, \delta(\cdot)] \in \mathcal{D}_\mathbb{C}$ satisfies (23)–(24).
- If $i_0 = 1$, then we set $\Delta := (\Delta_1, \dots, \Delta_N) = (0, \dots, 0)$ and $\delta(\cdot) = (\delta_1(\cdot), \dots, \delta_N(\cdot))$, where $\delta_j(\cdot) = \delta_0(\cdot)$, if $j = j_0$ otherwise $\delta_j(\cdot) = 0$. Then, $[\Delta, \delta(\cdot)] \in \mathcal{D}_\mathbb{C}$ satisfies (23)–(24).

Moreover, if $G_{(i_0,j_0;\,i_0,j_0)}(z_0) \in \mathbb{R}_+^{q_{i_0j_0}\times l_{i_0j_0}}$ for $z_0 = 0$ then we have

$$\left\|G_{(i_0,j_0;\,i_0,j_0)}(0)\right\| = \max_{u\in\mathbb{R}_+^{l_{i_0j_0}},\,\|u\|=1} \left\|G_{(i_0,j_0;\,i_0,j_0)}(0)u\right\|,$$

see e.g. [12]. Thus we can choose $u_0 \in \mathbb{R}_+^{l_{i_0j_0}}$ so that $\|u_0\| = 1$, $\|G_{(i_0,j_0;\,i_0,j_0)}(0)u_0\| = \|G_{(i_0,j_0;\,i_0,j_0)}(0)\|$. Since $G_{(i_0,j_0;\,i_0,j_0)}(0)u_0 \geqslant 0$ there exists by a theorem of Krein and Rutman [14] a positive linear form $y^* \in (\mathbb{C}^{q_{i_0j_0}})^*$ of dual norm $\|y^*\| = 1$ such that $y^*G_{(i_0,j_0;\,i_0,j_0)}(0)u_0 = \|G_{(i_0,j_0;\,i_0,j_0)}(0)u_0\|$. Hence the perturbation $[\Delta, \delta(\cdot)]$ constructed as above belongs to $\mathcal{D}_+$. This completes our proof. □

Using the above lemma we obtain the following estimates for the complex stability radius.

Theorem 3.8. *Let Eq.* (1) *be UAS. Assume that A, $B(\cdot)$ are subjected to multi-perturbations of the form* (16)*. Then*

$$\frac{1}{\max_{i,u\in I;\,j,v\in\underline{N}}\{\max_{z\in\mathbb{C},\,\Re z=0}\|G_{(i,j;\,u,v)}(z)\|\}} \leqslant r_{\mathbb{C}} \leqslant \frac{1}{\max_{i\in I;\,j\in\underline{N}}\{\max_{z\in\mathbb{C},\,\Re z=0}\|G_{(i,j;\,i,j)}(z)\|\}}. \tag{25}$$

In particular, if $D_{ij} = D$ for all $i \in I$, $j \in \underline{N}$ or $E_{ij} = E$ for all $i \in I$, $j \in \underline{N}$, then

$$r_{\mathbb{C}} = \frac{1}{\max_{i\in I;\,j\in\underline{N}}\{\max_{z\in\mathbb{C},\,\Re z=0}\|G_{(i,j;\,i,j)}(z)\|\}}. \tag{26}$$

Proof. Obviously, (26) is immediate from (25), if $D_{ij} = D$ for all $i \in I$, $j \in \underline{N}$ or $E_{ij} = E$ for all $i \in I$, $j \in \underline{N}$. Let $[\Delta, \delta(\cdot)] \in \mathcal{D}_{\mathbb{C}}$ be a destabilizing disturbance, that is $\sigma(A_\Delta, B_\delta(\cdot)) \not\subset \mathbb{C}^-$. It follows that there exist a nonzero $x_0 \in \mathbb{C}^n$ and $z_0 \in \mathbb{C}$, $\Re z_0 \geqslant 0$ such that

$$\left(z_0I_n - \left(A + \sum_{j=1}^N D_{0j}\Delta_jE_{0j}\right) - \int_0^{+\infty} e^{-z_0s}\,d\left[B(s) + \sum_{j=1}^N D_{1j}\delta_j(s)E_{1j}\right]\right)x_0 = 0.$$

This implies

$$H(z_0)x_0 = \left(\sum_{j=1}^N D_{0j}\Delta_jE_{0j} + \int_0^{+\infty} e^{-z_0s}\,d\left[\sum_{j=1}^N D_{1j}\delta_j(s)E_{1j}\right]\right)x_0,$$

where $H(\cdot)$ is given by (14). Since Eq. (1) is UAS, it follows that

$$x_0 = H(z_0)^{-1}\left(\sum_{j=1}^N D_{0j}\Delta_jE_{0j} + \sum_{j=1}^N D_{1j}\int_0^{+\infty} e^{-z_0s}\,d\big[\delta_j(s)\big]\,E_{1j}\right)x_0. \tag{27}$$

Let $i_0 \in I,\ j_0 \in \underline{N}$ be indexes such that $\|E_{i_0 j_0} x_0\| = \max\{\|E_{ij} x_0\|\colon\ i \in I,\ j \in \underline{N}\}$. From (27), it follows that $E_{i_0 j_0} x_0 \neq 0$. Multiplying Eq. (27) with $E_{i_0 j_0}$ from the left and taking norms, we deduce that

$$\sum_{j=1}^{N} \left(\left\| E_{i_0 j_0} H(z_0)^{-1} D_{0j} \right\| \|\Delta_j\| + \left\| E_{i_0 j_0} H(z_0)^{-1} D_{1j} \right\| \left\| \int_0^{+\infty} e^{-z_0 s}\, d\delta_j(s) \right\| \right) \geqslant 1. \tag{28}$$

Since $\| \int_0^{+\infty} e^{z_0 \theta}\, d[\delta_j(\theta)] \| \leqslant \|\delta_j(\cdot)\|$, $j \in \underline{N}$, (28) yields

$$\Big(\max_{i,u \in I;\ j,v \in \underline{N}} \left\| G_{(i,j;\,u,v)}(z_0) \right\| \Big) \sum_{j=1}^{N} \big(\|\Delta_j\| + \left\| \delta_j(\cdot) \right\| \big) \geqslant 1.$$

Thus,

$$\Big(\max_{i,u \in I;\ j,v \in \underline{N}} \Big\{ \max_{z \in \mathbb{C},\, \Re z \geqslant 0} \left\| G_{(i,j;\,u,v)}(z) \right\| \Big\} \Big) \left\| \left[\Delta, \delta(\cdot) \right] \right\| \geqslant 1,$$

or equivalently,

$$\left\| \left[\Delta, \delta(\cdot) \right] \right\| \geqslant \frac{1}{\max_{i,u \in I;\ j,v \in \underline{N}} \{ \max_{z \in \mathbb{C},\, \Re z = 0} \| G_{(i,j;\,u,v)}(z) \| \}},$$

by Lemma 3.7(i). By the definition of the complex radius $r_{\mathbb{C}}$, we get

$$r_{\mathbb{C}} \geqslant \frac{1}{\max_{i,u \in I;\ j,v \in \underline{N}} \{ \max_{z \in \mathbb{C},\, \Re z = 0} \| G_{(i,j;\,u,v)}(z) \| \}}.$$

It remains to prove that

$$r_{\mathbb{C}} \leqslant \frac{1}{\max_{i \in I;\ j \in \underline{N}} \{ \max_{z \in \mathbb{C},\, \Re z = 0} \| G_{(i,j;\,i,j)}(z) \| \}}.$$

However, this inequality directly follows from Lemma 3.7(ii) and the definition of $r_{\mathbb{C}}$. This completes our proof. □

As noted in the Introduction, the problem of computation of the real stability radius is much more difficult. It has been solved firstly for ordinary linear differential systems of the form $\dot{x}(t) = Ax(t)$, $t \geqslant 0$, where the system matrix A is subjected to single perturbations and then extended to linear time-invariant time-delay differential systems in only recent time, see [13,24]. However, the obtained formulae for the real stability radii in these papers are very complicated. We note that, by definition, $r_{\mathbb{C}} \leqslant r_{\mathbb{R}}$, so $r_{\mathbb{C}}$ can be accepted as the lower bound for $r_{\mathbb{R}}$. Unfortunately, as shown in many previous papers (see e.g. [7]) these two stability radii can be arbitrarily distinct. Therefore, it is an interesting problem to find classes of systems of practical interest for which these two stability radii coincide. Motivated by the results of [8,9,19–22,27–29] and basing on our new results on positive linear Volterra–Stieltjes equations [15], in the rest of this subsection, we show that for the class of positive linear Volterra–Stieltjes equations, the positive, real and complex stability radius under multi-perturbations coincide and can be computed by a simple formula.

Definition 3.9. Equation (1) is called positive, if for every $\sigma \geqslant 0$ and every $\phi \in C([0,\sigma],\mathbb{R}^n)$, $\phi(\cdot) \geqslant 0$, the corresponding solution $x(\cdot,\sigma,\phi)$ is also nonnegative, that is $x(t,\sigma,\phi) \geqslant 0$, $\forall t \geqslant \sigma$.

Theorem 3.10. *(See [15].) Equation* (1) *is positive if and only if $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $B(\cdot)$ is an increasing matrix function on $\mathbb{R}_+$.*

The following theorem offers an explicit criterion for uniformly asymptotic stability of positive linear Volterra–Stieltjes equations.

Theorem 3.11. *(See [15].) Suppose that Eq.* (1) *is positive. Then, Eq.* (1) *is UAS if and only if $\mu(A + \int_0^{+\infty} dB(t)) < 0$.*

To prove the main results of this paper, we need the following technical lemma.

Lemma 3.12. *Suppose Eq.* (1) *is UAS, positive and $D \in \mathbb{R}_+^{n\times l}$, $E \in \mathbb{R}_+^{q\times n}$. Let $H(\cdot)$ be the characteristic matrix of Eq.* (1) *defined by* (14). *Then*:

(i)
$$\left|H(z)^{-1}x\right| \leqslant H(0)^{-1}|x|, \quad \forall x \in \mathbb{C}^n, \tag{29}$$

for every $z \in \mathbb{C}$, $\Re z \geqslant 0$. In particular, $H(0)^{-1}$ is a nonnegative matrix.

(ii)
$$\max_{z\in\mathbb{C},\,\Re z\geqslant 0}\left\| E\left(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s)\right)^{-1} D\right\| = \left\| E\left(-A - \int_0^{+\infty} dB(s)\right)^{-1} D\right\|. \tag{30}$$

Proof. (i) Since Eq. (1) is positive, A is a Metzler matrix and $B(\cdot)$ is increasing on $\mathbb{R}_+$. For every $z \in \mathbb{C}$, $\Re z \geqslant 0$, by Theorem 2.1(iv), we get

$$\mu\left(A + \int_0^{+\infty} e^{-zs}\,dB(s)\right) \leqslant \mu\left(A + \int_0^{+\infty} e^{-\Re zs}\,dB(s)\right) \leqslant \mu\left(A + \int_0^{+\infty} dB(s)\right).$$

On the other hand, because Eq. (1) is UAS, we get $\mu(A + \int_0^{+\infty} dB(s)) < 0$, by Theorem 3.11. Therefore, $\mu(A + \int_0^{+\infty} e^{-zs}\,dB(s)) < 0$, for every $z \in \mathbb{C}$, $\Re z \geqslant 0$. For a fixed $z \in \mathbb{C}$, $\Re z \geqslant 0$, we can represent the following

$$\left(zI_n - \left(A + \int_0^{+\infty} e^{-zs}\,dB(s)\right)\right)^{-1} x = \int_0^{+\infty} e^{-z\theta} e^{\theta(A+\int_0^{+\infty} e^{-zs}\,dB(s))} x\,d\theta, \quad x \in \mathbb{C}^n, \tag{31}$$

see e.g. [16]. As A is a Metzler matrix, there exists a real number $\alpha_0 > 0$ such that $(A + \alpha_0 I_n) \geqslant 0$. Since $(A + \alpha_0 I_n) \geqslant 0$ and $B(\cdot)$ is increasing on $\mathbb{R}_+$, it follows that

$$\begin{aligned} e^{\alpha_0\theta}\left|e^{\theta(A+\int_0^{+\infty} e^{-zs}\,dB(s))}\right| &= \left|e^{\alpha_0\theta I_n} e^{\theta(A+\int_0^{+\infty} e^{-zs}\,dB(s))}\right| = \left|e^{\theta((A+\alpha_0 I_n)+\int_0^{+\infty} e^{-zs}\,dB(s))}\right| \\ &\leqslant e^{\theta((\alpha_0 I_n+A)+\int_0^{+\infty} dB(s))} = e^{\alpha_0\theta} e^{\theta(A+\int_0^{+\infty} dB(s))}, \quad \theta \geqslant 0. \end{aligned}$$

This implies that

$$\left|e^{\theta(A+\int_0^{+\infty} e^{-zs}\,dB(s))}\right| \leqslant e^{\theta(A+\int_0^{+\infty} dB(s))}, \quad \theta \geqslant 0,\ z \in \mathbb{C},\ \Re z \geqslant 0. \tag{32}$$

Taking (31), (32) into account, we get

$$\begin{aligned}\left|\left(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s)\right)^{-1} x\right| &\leqslant \int_0^{+\infty} e^{\theta(A+\int_0^{+\infty} dB(s))}\,d\theta\,|x| \\ &= \left(-A - \int_0^{+\infty} dB(s)\right)^{-1} |x|,\end{aligned}$$

for every $z \in \mathbb{C}$, $\Re z \geqslant 0$.

(ii) Since D, E are the nonnegative matrices, it follows from (i) that

$$\left|E\left(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s)\right)^{-1} Dx\right| \leqslant E\left(-A - \int_0^{+\infty} dB(s)\right)^{-1} D|x|, \quad x \in \mathbb{C}^n,$$

for every $z \in \mathbb{C}$, $\Re z \geqslant 0$. By monotonicity property of the vector norm and the definition of operator norm, we get

$$\left\|E\left(zI_n - A - \int_0^{+\infty} e^{-zs}\,dB(s)\,dt\right)^{-1} D\right\| \leqslant \left\|E\left(-A - \int_0^{+\infty} dB(s)\right)^{-1} D\right\|,$$

for every $z \in \mathbb{C}$, $\Re z \geqslant 0$. This completes our proof. □

We are now in the position to prove the main result of this paper.

Theorem 3.13. *Let Eq.* (1) *be positive and UAS. Assume that A, $B(\cdot)$ are subjected to multi-perturbations of the form* (16) *where $D_{ij} \in \mathbb{R}_+^{n\times l_{ij}}$, $E_{ij} \in \mathbb{R}_+^{q_{ij}\times n}$, $i \in I$, $j \in \underline{N}$. If $D_{ij} = D$ for all $i \in I$, $j \in \underline{N}$ or $E_{ij} = E$ for all $i \in I$, $j \in \underline{N}$ then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max_{i\in I,\, j\in \underline{N}} \|G_{(i,j;\,i,j)}(0)\|}.$$

Proof. By (26) and Lemma 3.12(ii), we get

$$r_{\mathbb{C}} = \frac{1}{\max_{i\in I,\, j\in \underline{N}} \|G_{(i,j;\,i,j)}(0)\|}. \tag{33}$$

Since Eq. (1) is UAS, positive and $D_{ij} \in \mathbb{R}_+^{n\times l_{ij}}$, $E_{ij} \in \mathbb{R}_+^{q_{ij}\times n}$, $i \in I$, $j \in \underline{N}$, it follows from Lemma 3.12(i) that $G_{(i,j;i,j)}(0) \in \mathbb{R}_+^{q_{ij}\times l_{ij}}$, for all $i \in I$, $j \in \underline{N}$. By the definition of r_+ and Lemma 3.7(ii), we get

$$r_+ \leqslant \frac{1}{\max_{i\in I,\, j\in \underline{N}} \|G_{(i,j;\,i,j)}(0)\|}. \tag{34}$$

Finally, it follows from (33), (34) and the inequalities $r_{\mathbb{C}} \leqslant r_{\mathbb{R}} \leqslant r_+$ that

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max_{i\in I,\, j\in \underline{N}} \|G_{(i,j;\,i,j)}(0)\|}. \qquad \square$$

3.3. Stability radii of positive linear Volterra–Stieltjes equations under affine perturbations

We now deal with the problem of computing stability radius of positive linear Volterra–Stieltjes equations of the form (1) under affine perturbations. To do this, we assume that Eq. (1) is UAS and A, $B(\cdot)$ are subjected to *affine perturbations* of the form

$$A \rightsquigarrow A + \sum_{i=1}^{N} \alpha_i A_i, \qquad B(\cdot) \rightsquigarrow B(\cdot) + \sum_{i=1}^{N} \beta_i B_i(\cdot), \tag{35}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i(\cdot) \in NBV(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $i \in \underline{N}$ are given and $\alpha_i, \beta_i \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), $i \in \underline{N}$, are unknown scalars.

In other words, we consider perturbed equations of the form

$$\dot{x}(t) = \left(A + \sum_{i=1}^{N} \alpha_i A_i\right) x(t) + \int_0^t d\left[B(s) + \sum_{i=1}^{N} \beta_i B_i(s)\right] x(t-s), \quad t \geqslant 0. \tag{36}$$

Suppose that A is a Metzler matrix and $B(\cdot)$ is increasing on $\mathbb{R}_+$ under which the linear Volterra–Stieltjes equation (1) is positive. Furthermore, we assume that $A_i \in \mathbb{R}_+^{n\times n}$ and $B_i(\cdot)$ is increasing on $\mathbb{R}_+$ for every $i \in \underline{N}$. Then, we define the complex and the real stability radius of the linear Volterra–Stieltjes equation (1) under affine parameter perturbations (35) by setting, for $\mathbb{K} = \mathbb{C}$ and, respectively, $\mathbb{K} = \mathbb{R}$,

$$\begin{aligned} r_{\mathbb{K}}^a = \inf\Bigg\{ &\max\Big(\max_{i\in \underline{N}} |\alpha_i|; \max_{i\in \underline{N}} |\beta_i|\Big)\colon\ \alpha_i, \beta_i \in \mathbb{K}, \\ &\left(A + \sum_{i=1}^{N} \alpha_i A_i,\ B(\cdot) + \sum_{i=1}^{N} \beta_i B_i(\cdot)\right) \not\subset \mathbb{C}^- \Bigg\}. \end{aligned} \tag{37}$$

Similarly, the positive stability radius r_+^a is obtained by restricting, in the above definition, the disturbances $(\alpha, \beta) := ((\alpha_i)_{i\in \underline{N}}, (\beta_j)_{j\in \underline{N}})$ to be nonnegative.

It is clear that

$$0 < r_{\mathbb{C}}^a \leqslant r_{\mathbb{R}}^a \leqslant r_+^a. \tag{38}$$

The following theorem proves that the equalities in (38) hold true and gives an explicit formula for stability radii $r^a_{\mathbb{K}}$, r^a_+ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$).

Theorem 3.14. *Suppose that the linear Volterra–Stieltjes equation* (1) *is UAS and* A, $B(\cdot)$ *are subjected to affine perturbations of the form* (35). *If the stability radii of the equation are given by* (37) *then*

$$r^a_{\mathbb{C}} = r^a_{\mathbb{R}} = r^a_+ = \frac{1}{\mu[(-A - \int_0^{+\infty} dB(s))^{-1}(\sum_{i=1}^N A_i + \sum_{i=1}^N \int_0^{+\infty} dB_i(s))]}. \tag{39}$$

Proof. We first prove that

$$r^a_+ = \frac{1}{\mu[(-A - \int_0^{+\infty} dB(s))^{-1}(\sum_{i=1}^N A_i + \sum_{i=1}^N \int_0^{+\infty} dB_i(s))]}.$$

Let $(\alpha, \beta) = ((\alpha_i)_{i \in \underline{N}}, (\beta_i)_{i \in \underline{N}})$ be an arbitrary nonnegative destabilizing perturbation, that is

$$\sigma\left(A + \sum_{i=1}^N \alpha_i A_i,\, B(\cdot) + \sum_{i=1}^N \beta_i B_i(\cdot)\right) \not\subset \mathbb{C}^-.$$

Then, there exist a complex number z, $\Re z \geqslant 0$, and a nonzero vector $x \in \mathbb{C}^n$ such that

$$\left(\left(A + \sum_{i=1}^N \alpha_i A_i\right) + \int_0^{+\infty} e^{-zs}\, d\left[B(s) + \sum_{i=1}^N \beta_i B_i(s)\right]\right)x = zx.$$

Because Eq. (1) is UAS, this implies that

$$\left(zI_n - A - \int_0^{+\infty} e^{-zs}\, dB(s)\right)^{-1} \left(\sum_{i=1}^N \alpha_i A_i + \sum_{i=1}^N \beta_i \int_0^{+\infty} e^{-zs}\, dB_i(s)\right)x = x.$$

Since Eq. (1) is positive, $A_i \in \mathbb{R}_+^{n \times n}$ and $B_i(\cdot)$ is increasing on $\mathbb{R}_+$ for every $i \in \underline{N}$. Using (29), we obtain the following estimates

$$\begin{aligned}
|x| &= \left|\left(zI_n - A - \int_0^{+\infty} e^{-zs}\, dB(s)\right)^{-1} \left(\sum_{i=1}^N \alpha_i A_i + \sum_{i=1}^N \beta_i \int_0^{+\infty} e^{-zs}\, dB_i(s)\right)x\right| \\
&\leqslant \left(-A - \int_0^{+\infty} dB(s)\right)^{-1} \left|\left(\sum_{i=1}^N \alpha_i A_i + \sum_{i=1}^N \beta_i \int_0^{+\infty} e^{-zs}\, dB_i(s)\right)x\right| \\
&\leqslant \left(-A - \int_0^{+\infty} dB(s)\right)^{-1} \left(\sum_{i=1}^N \alpha_i A_i + \sum_{i=1}^N \beta_i \int_0^{+\infty} dB_i(s)\right)|x|
\end{aligned}$$

$$\leqslant \gamma\left[\left(-A-\int_0^{+\infty} dB(s)\right)^{-1}\left(\sum_{i=1}^N A_i+\sum_{i=1}^N \int_0^{+\infty} dB_i(s)\right)\right]|x|,$$

where $\gamma := \max\{\max_{i\in \underline{N}} \alpha_i, \max_{i\in \underline{N}} \beta_i\}$. Because

$$B := \left[\left(-A-\int_0^{+\infty} dB(s)\right)^{-1}\left(\sum_{i=1}^N A_i+\sum_{i=1}^N \int_0^{+\infty} dB_i(s)\right)\right]$$

is a nonnegative matrix, it follows from Theorem 2.1(ii) that

$$\mu(B) \geqslant \frac{1}{\gamma} > 0.$$

Hence,

$$\gamma \geqslant \frac{1}{\mu(B)}.$$

Since this holds for arbitrary destabilizing nonnegative perturbation (α, β), we conclude that

$$r_+^a \geqslant \frac{1}{\mu(B)}.$$

We shall prove that the converse inequality holds true. In fact, by Theorem 2.1(i) (Perron–Frobenius), there exists a nonzero vector $y \in \mathbb{R}_+^n$ such that $By = \mu(B)y$. This implies that

$$\left(\left(A_0+\sum_{i=1}^N \frac{1}{\mu(B)} A_i\right)+\int_0^{+\infty} d\left[B(s)+\sum_{i=1}^N \frac{1}{\mu(B)} B_i(s)\right]\right)y = 0.$$

It means that the nonnegative perturbation (α^*, β^*) defined by $\alpha_i^* = 1/\mu(B)$, $\beta_i^* = 1/\mu(B)$, $i \in \underline{N}$, is destabilizing. By the definition of r_+^a, we have

$$r_+^a \leqslant \frac{1}{\mu(B)}.$$

Thus, we obtain

$$r_+^a = \frac{1}{\mu(B)} = \frac{1}{\mu[(-A-\int_0^{+\infty} dB(s))^{-1}(\sum_{i=1}^N A_i+\sum_{i=1}^N \int_0^{+\infty} dB_i(s))]}.$$

We are now ready to show that $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_+^a$. Let $(\alpha, \beta) = ((\alpha_i)_{i\in \underline{N}}, (\beta_j)_{j\in \underline{N}})$ be an arbitrary complex destabilizing perturbation. By a similar argument as the above, we get

$$\left[\left(-A-\int_0^{+\infty} dB(s)\right)^{-1}\left(\sum_{i=1}^N |\alpha_i| A_i+\sum_{i=1}^N |\beta_i| \int_0^{+\infty} dB_i(s)\right)\right]|x_0| \geqslant |x_0|, \tag{40}$$

for some $x_0 \in \mathbb{C}^n$, $x_0 \neq 0$. By Theorem 2.1(ii),

$$\mu\left[\left(-A - \int_0^{+\infty} dB(s)\right)^{-1}\left(\sum_{i=1}^{N} |\alpha_i| A_i + \sum_{i=1}^{N} |\beta_i| \int_0^{+\infty} dB_i(s)\right)\right] \geqslant 1.$$

Since $C := (-A - \int_0^{+\infty} dB(s))^{-1}(\sum_{i=1}^{N} |\alpha_i| A_i + \sum_{i=1}^{N} |\beta_i| \int_0^{+\infty} dB_i(s))$ is a nonnegative matrix, using Theorem 2.1(i) again, we have $Cx_1 = \mu(C)x_1$, for some nonzero vector $x_1 \in \mathbb{R}^n_+$. This gives

$$\left(\left(A_0 + \sum_{i=1}^{N} \frac{|\alpha_i|}{\mu(C)} A_i\right) + \int_0^{+\infty} d\left[B(s) + \sum_{i=1}^{N} \frac{|\beta_i|}{\mu(C)} B_i(s)\right]\right)x_1 = 0,$$

which means that

$$\big(|\alpha|, |\beta|\big) := \left(\left(\frac{|\alpha_i|}{\mu(C)}\right)_{i\in\underline{N}}, \left(\frac{|\beta_i|}{\mu(C)}\right)_{i\in\underline{N}}\right)$$

is a nonnegative destabilizing perturbation. Hence, it follows from the definition of r_+^a that

$$\max\left(\max_{i\in\underline{N}}\left(\frac{|\alpha_i|}{\mu(C)}\right), \max_{i\in\underline{N}}\left(\frac{|\beta_i|}{\mu(C)}\right)\right) \geqslant r_+^a,$$

or

$$\max\Big(\max_{i\in\underline{N}} |\alpha_i|, \max_{i\in\underline{N}} |\beta_i|\Big) \geqslant \mu(C) r_+^a \geqslant r_+^a,$$

which implies that $r_{\mathbb{C}}^a \geqslant r_+^a$. In combining with the inequalities $r_{\mathbb{C}}^a \leqslant r_{\mathbb{R}}^a \leqslant r_+^a$, it implies that $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_+^a$. In addition, from the above arguments, we observe that $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_+^a = +\infty$ if and only if $\mu(B) = 0$. This completes our proof. □

4. Particular cases and examples

In this section, we consider particular cases of Theorems 3.13 and 3.14 which are most frequent in applications. Then, we illustrate the obtained results by two simple examples.

4.1. Robust stability of positive linear integro-differential equations

We now consider a linear Volterra integro-differential equation of convolution type

$$\dot{x}(t) = Ax(t) + \int_0^t C(s)x(t-s)\,ds, \quad t \in \mathbb{R}_+, \tag{41}$$

where $A \in \mathbb{R}^{n\times n}$ and $C(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ are given.

Note that Eq. (41) can be rewritten in the form (1) with $B(t) = \int_0^t C(s)\,ds$, $t \in \mathbb{R}_+$. By $C(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, (12) holds true. Then, it follows from Theorem 3.10 that Eq. (41) is positive if and only if $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $C(\cdot) \geqslant 0$.

The following theorems are straightforward from Theorems 3.13 and 3.14, respectively.

Theorem 4.1. *Suppose Eq.* (41) *is positive, UAS and $E \in \mathbb{R}_+^{q\times n}$, $D_{ij} \in \mathbb{R}_+^{n\times l_{ij}}$ ($i \in I$, $j \in \underline{N}$) are given. Then a perturbed equation of the form*

$$\dot{x}(t) = \left(A + \sum_{j=1}^{N} D_{0j}\Delta_j E\right)x(t) + \int_0^t \left(C(s) + \sum_{j=1}^{N} D_{1j}\delta_j(s)E\right)x(t-s)\,ds, \quad t \geqslant 0,$$

where $\Delta_j \in \mathbb{R}^{l_{0j}\times q}$, $\delta_j(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{l_{1j}\times q})$, $j \in \underline{N}$, is still UAS if

$$\sum_{j=1}^{N}\left(\|\Delta_j\| + \int_0^{+\infty} \|\delta_j(s)\|\,ds\right) < \frac{1}{\max_{i\in I,\, j\in \underline{N}}\{\|E(-A - \int_0^{+\infty} C(s)\,ds)^{-1} D_{ij}\|\}}.$$

Theorem 4.2. *Suppose Eq.* (41) *is positive, UAS. Let $A_i \in \mathbb{R}_+^{n\times n}$, $C_i(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $C_i(\cdot) \geqslant 0$, $i \in \underline{N}$, be given. Then a perturbed equation of the form*

$$\dot{x}(t) = \left(A + \sum_{i=1}^{N} \alpha_i A_i\right)x(t) + \int_0^t \left(C(s) + \sum_{i=1}^{N} \beta_i C_i(s)\right)x(t-s)\,ds, \quad t \geqslant 0,$$

where $\alpha_i, \beta_i \in \mathbb{R}$ ($i \in \underline{N}$) is still UAS if

$$\max\left(\max_{i\in \underline{N}} |\alpha_i|, \max_{i\in \underline{N}} |\beta_i|\right) < \frac{1}{\mu[(-A - \int_0^{+\infty} C(s)\,ds)^{-1}(\sum_{i=1}^{N} A_i + \sum_{i=1}^{N}\int_0^{+\infty} C_i(s)\,ds)]}.$$

4.2. Robust stability of positive linear functional differential equations

Let h be a given positive number and let us consider a linear functional differential equation of the form

$$\dot{x}(t) = Ax(t) + \int_0^h d\big[\eta(s)\big]x(t-s), \quad t \geqslant h, \tag{42}$$

where $A \in \mathbb{R}^{n\times n}$ and $\eta(\cdot) \in NBV([0,h], \mathbb{R}^{n\times n})$ are given.

It is well known that for an initial function $\phi \in C([0,h], \mathbb{R}^n)$, Eq. (42) has a unique solution $x(\cdot, \phi)$ satisfying the initial condition

$$x(t) = \phi(t), \quad t \in [0,h], \tag{43}$$

see e.g. [2,4].

Definition 4.3. Equation (42) is called positive if for every initial function $\phi \in C([0,h],\mathbb{R}^n)$ being nonnegative, the corresponding solution $x(\cdot,\phi)$ is also nonnegative.

It is important to note that if the matrix function $B(\cdot)$ in Eq. (1) is defined by

$$B(s) := \begin{cases} \eta(s) & \text{if } s \in [0,h), \\ \eta(h) & \text{if } s \in [h,+\infty), \end{cases} \tag{44}$$

then Eq. (1) coincides with Eq. (42) on the interval $[h,+\infty)$. Therefore, we get the following.

Theorem 4.4. *(See [15].) Equation* (42) *is positive if and only if $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $\eta(\cdot)$ is an increasing matrix function on* $[0,h]$.

Before stating the next results, we recall the notion of exponential stability of the linear functional differential equations of the form (42).

Definition 4.5. Equation (42) is said to be exponentially stable if there exist $M \geqslant 1$ and $\alpha > 0$ such that for every initial function $\phi \in C([0,h],\mathbb{R}^n)$ the corresponding solution $x(\cdot,\phi)$ satisfies

$$\left\| x(t,\phi) \right\| \leqslant Me^{-\alpha t}\|\phi\|, \quad \forall t \geqslant 0.$$

Furthermore, it is well known that Eq. (42) is exponentially stable if and only if its characteristic equation has no zeros in the closed right half plane. That is $\det(zI_n - A - \int_0^h e^{-zt}\,d\eta(t)) \neq 0$, $\forall z \in \mathbb{C}$, $\Re z \geqslant 0$, see e.g. [2,4,23].

As in the above subsection, the following theorems directly follow from Theorems 3.13 and 3.14, respectively.

Theorem 4.6. *Suppose Eq.* (42) *is exponentially stable, positive and $E \in \mathbb{R}_+^{q\times n}$, $D_{ij} \in \mathbb{R}_+^{n\times l_{ij}}$ ($i \in I$, $j \in \underline{N}$) are given. Then a perturbed equation of the form*

$$\dot{x}(t) = \left(A + \sum_{j=1}^{N} D_{0j}\Delta_j E\right)x(t) + \int_0^h d\left[\eta(s) + \sum_{j=1}^{N} D_{1j}\delta_j(s)E)\right]x(t-s), \quad t \geqslant h,$$

where $\Delta_j \in \mathbb{R}^{l_{0j}\times q}$, $\delta_j(\cdot) \in NBV([0,h],\mathbb{R}^{l_{1j}\times q})$, $j \in \underline{N}$, is still exponentially stable if

$$\sum_{j=1}^{N}\left(\|\Delta_j\| + \|\delta_j\|\right) < \frac{1}{\max_{i\in I,\, j\in \underline{N}}\{\|E(-A-\eta(h))^{-1}D_{ij}\|\}}.$$

Theorem 4.7. *Suppose Eq.* (42) *is exponentially stable, positive. Let $A_i \in \mathbb{R}_+^{n\times n}$, $i \in \underline{N}$, be given matrices and let $\eta_i(\cdot) \in NBV([0,h],\mathbb{R}^{n\times n})$ be a given increasing matrix function for every $i \in \underline{N}$. Then a perturbed equation of the form*

$$\dot{x}(t) = \left(A + \sum_{i=1}^{N}\alpha_i A_i\right)x(t) + \int_0^h d\left[\eta(s) + \sum_{i=1}^{N}\beta_i\eta_i(s)\right]x(t-s), \quad t \geqslant 0,$$

where $\alpha_i, \beta_i \in \mathbb{R}$ $(i \in \underline{N})$ is still exponentially stable if

$$\max\Big(\max_{i\in\underline{N}} |\alpha_i|, \max_{i\in\underline{N}} |\beta_i|\Big) < \frac{1}{\mu[(-A-\eta(h))^{-1}(\sum_{i=1}^N A_i + \sum_{i=1}^N \eta_i(h))]}.$$

Remark 4.8. Theorems 4.6, 4.7 which are the main results of [22], have just been found by ourselves in only recent time.

4.3. Examples

Example 4.9. Consider a linear positive Volterra differential equation in $\mathbb{R}^2$ given by

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-\tau)x(\tau)\,d\tau, \quad x(t) \in \mathbb{R}^2,\ t \geqslant 0, \tag{45}$$

where

$$A = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} e^{-t} & 0 \\ \frac{1}{(t+1)^2} & e^{-t} \end{pmatrix}, \quad t \geqslant 0.$$

By Theorem 3.11, it is easy to see that Eq. (45) is UAS. We now consider a perturbed equation of the form

$$\dot{x}(t) = A_\Delta x(t) + \int_0^t B_\delta(\tau)x(t-\tau)\,d\tau, \quad x(t) \in \mathbb{R}^2,\ t \geqslant 0, \tag{46}$$

where

$$A_\Delta = \begin{pmatrix} -3+2a_1 & 1+2a_2 \\ 0 & -3 \end{pmatrix}, \qquad B_\delta(t) = \begin{pmatrix} e^{-t} & 0 \\ \frac{1}{(t+1)^2} + \delta_1(t) & e^{-t} + \delta_2(t) \end{pmatrix}, \quad t \geqslant 0,$$

where $a_1, a_2 \in \mathbb{R}$ and $\delta_1(\cdot), \delta_2(\cdot) \in L^1([0,+\infty),\mathbb{R}) \cap C([0,+\infty),\mathbb{R})$ are unknown.

It is important to note that we can rewrite A_Δ and $B_\delta(\cdot)$ in the following form

$$A_\Delta = A + D_1 \Delta E, \qquad B_\delta(\cdot) = B(\cdot) + D_2\delta(\cdot)E,$$

where

$$D_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad E = I_2,$$

and

$$\Delta = (a_1, a_2), \qquad \delta(\cdot) = \big(\delta_1(\cdot), \delta_2(\cdot)\big).$$

Assume that $\mathbb{R}^2$ is endowed with 2-norm, by Theorem 4.2, the system (46) is still UAS if

$$\sqrt{a_1^2 + a_2^2} + \int_0^{+\infty} \sqrt{\delta_1^2(t) + \delta_2^2(t)}\, dt < \frac{3\sqrt{5}}{10}.$$

Example 4.10. Consider a positive linear functional differential equation

$$\dot{x}(t) = -x(t) + \int_0^1 e^{-s} x(t-s)\, ds, \quad t \geqslant 0,\ x(t) \in \mathbb{R}. \tag{47}$$

Equation (47) can be represented of the form (42), with $\eta(s) = 1 - e^{-s}$, $s \in [0, 1]$. By Theorem 3.11, it is easy to see that (47) is exponentially stable. Assume that the system (47) is perturbed as follows

$$\dot{x}(t) = (-1 + \delta_0)x(t) + \int_0^1 \big(e^{-s} + 2006\Delta_1(s) + 2007\Delta_2(s)\big)x(t-s)\, ds, \tag{48}$$

where $\delta_0 \in \mathbb{R}$ is an unknown parameter scalar and $\Delta_1(\theta)$, $\Delta_2(\theta)$ are unknown integrable functions on $[0, 1]$. This perturbed system can be rewritten in the form

$$\dot{x}(t) = (-1 + \delta_0)x(t) + \int_0^1 d\big[\eta(\theta) + 2006\delta_1(s) + 2007\delta_2(s)\big]\, x(t-s), \tag{49}$$

where

$$\delta_1(s) = \int_0^s \Delta_1(\tau)\, d\tau, \qquad \delta_2(s) = \int_0^s \Delta_2(\tau)\, d\tau, \quad s \in [0, 1].$$

By Theorem 4.7, we conclude that the perturbed system (48) is exponentially stable for all $\delta_0 \in \mathbb{R}$, $\Delta_1(\cdot), \Delta_2(\cdot) \in L_1([0, 1], \mathbb{R})$ satisfying

$$|\delta_0| + \int_0^1 \big|\Delta_1(\theta)\big|\, d\theta + \int_0^1 \big|\Delta_2(\theta)\big|\, d\theta < \frac{1}{2007e}.$$

Acknowledgments

The author would like to thank Professor Achim Ilchmann (Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany) for his constant support, advice and encouragement to this work.

References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in Mathematical Sciences, Academic Press, New York, 1979.

[2] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, H.O. Walther, Delay Equations, Functional-, Complex- and Nonlinear Analysis, Springer-Verlag, New York, 1995.

[3] G. Gripenberg, S.O. Londen, O. Staffans, Volterra Integral and Functional Equations, Cambridge Univ. Press, 1990.

[4] J. Hale, Theory of Functional Differential Equations, Academic Press, New York, 1977.

[5] E. Hille, R.S. Phillips, Functional Analysis and Semigroups, Amer. Math. Soc., Providence, RI, 1957.

[6] D. Hinrichsen, A.J. Pritchard, Stability radius for structured perturbations and the algebraic Riccati equation, Systems Control Lett. 8 (1986) 105–113.

[7] D. Hinrichsen, A.J. Pritchard, Real and complex stability radii: A survey, in: D. Hinrichsen, B. Mårtensson (Eds.), Control of Uncertain Systems, in: Progress in System and Control Theory, vol. 6, Birkhäuser, Basel, 1990, pp. 119–162.

[8] D. Hinrichsen, N.K. Son, μ-analysis and robust stability of positive linear systems, Appl. Math. Comp. Sci. 8 (1998) 253–268.

[9] D. Hinrichsen, N.K. Son, Stability radii of positive discrete-time systems under affine parameter perturbations, Internat. J. Robust Nonlinear Control 8 (1998) 1169–1188.

[10] D. Hinrichsen, N.K. Son, P.H.A. Ngoc, Stability radii of positive higher order difference systems, Systems Control Lett. 49 (2003) 377–388.

[11] D. Hinrichsen, A.J. Pritchard, Mathematical Systems Theory, I: Modelling, State Space Analysis, Stability and Robustness, Springer-Verlag, Berlin, 2005.

[12] R.A. Horn, Ch.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1993.

[13] G. Hu, Edward J. Davison, Real stability radii of linear time-invariant time-delay systems, Systems Control Lett. 50 (2003) 209–219.

[14] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Transl. 10 (1948) 199–325.

[15] S. Murakami, T. Naito, J.S. Shin, P.H.A. Ngoc, Characterizations of positive linear Volterra–Stieltjes equations, submitted for publication, 2006.

[16] R. Nagel (Ed.), One-Parameter Semigroups of Positive Operators, Springer-Verlag, Berlin, 1986.

[17] T. Naito, J.S. Shin, S. Murakami, P.H.A. Ngoc, On stability and robust stability of positive linear Volterra equations, SIAM J. Control Optim., 2007, in press.

[18] T. Naito, J.S. Shin, S. Murakami, P.H.A. Ngoc, Characterizations of positive linear Volterra integro-differential systems, Integral Equations Operator Theory 58 (2007) 255–272.

[19] P.H.A. Ngoc, N.K. Son, Stability radii of positive linear difference equations under affine parameter perturbations, Appl. Math. Comput. 134 (2003) 577–594.

[20] P.H.A. Ngoc, N.K. Son, Stability radii of linear systems under multi-perturbations, Numer. Funct. Anal. Optim. 25 (2004) 221–238.

[21] P.H.A. Ngoc, Strong stability radii of positive linear time-delay systems, Internat. J. Robust Nonlinear Control 15 (2005) 459–472.

[22] P.H.A. Ngoc, N.K. Son, Stability radii of positive linear functional differential equations under multi perturbations, SIAM J. Control Optim. 43 (2005) 2278–2295.

[23] P.H.A. Ngoc, T. Naito, J.S. Shin, Characterizations of positive linear functional differential equations, Funkcial. Ekvac. 50 (2007) 1–17.

[24] L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young, J.C. Doyle, A formula for computation of the real stability radius, Automatica 31 (1995) 879–890.

[25] B. Shafai, J. Chen, M. Kothandaraman, Explicit formulas for stability radii of nonnegative and Metzlerian matrices, IEEE Trans. Automat. Control 42 (1997) 265–269.

[26] N.K. Son, D. Hinrichsen, Robust stability of positive continuous-time systems, Numer. Funct. Anal. Optim. 17 (1996) 649–659.

[27] N.K. Son, P.H.A. Ngoc, Stability radius of linear delay systems, Proceedings of the American Control Conference, San Diego, California, June 1999, pp. 815–816.

[28] N.K. Son, P.H.A. Ngoc, Robust stability of positive linear time delay systems under affine parameter perturbations, Acta Math. Vietnam. 24 (1999) 353–372.

[29] N.K. Son, P.H.A. Ngoc, Robust stability of linear functional differential equations, Adv. Stud. Contemp. Math. 3 (2001) 43–59.