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FOLIATED COBORDISM CLASSES OF CERTAIN FOLIATED S^1-BUNDLES OVER SURFACES

TAKASHI TSUBOI

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INTRODUCTION

IN THIS paper we prove that the Godbillon–Vey invariant is the complete invariant for C^∞-foliated cobordism classes of those foliated S^1-bundles over oriented surfaces whose structural group reduces to a certain discrete group $\mathbf{G}$ (for the definition of $\mathbf{G}$, see §1). It should be noted that these cobordism classes contain Thurston's examples in [25] (see 1.3, 4.5, 5.2) and the Godbillon–Vey invariant for these classes ranges through the whole real numbers (Thurston[25], Brooks[2, 3]).

The Godbillon–Vey invariant is the only invariant known by now for $\mathscr{F}\Omega_{3,1}^\infty$, the group of foliated cobordism classes of oriented 3-manifolds with oriented codimension one foliations. There are several foliations whose Godbillon–Vey invariants are known to be zero (Wallet[28], Herman[9], Nishimori[18], Morita–Tsuboi[17], Mizutani–Morita–Tsuboi[14], Duminy–Sergiescu[30], Tsuchiya[31]). Some of them are known to be cobordant to zero (Mizutani[13], Sergeraert[24], Fukui[6], Oshikiri[19], Mizutani–Morita–Tsuboi[15], Tsuboi[26]). In the construction of their null cobordisms, the theory of the behavior of leaves of these foliations plays an essential role.

In the case of our foliated S^1-bundles, however, the behavior of leaves seems to be quite complicated (Sacksteder[21], Hector[8], Raymond[20]). In order to obtain our main result (Theorem 1.1), we use the homological properties of the discrete group $\mathbf{G}$; in particular, the calculation of $H_2(SL(2, \mathbb{R}); \mathbb{Z}) = K_2(2, \mathbb{R})$ by Sah–Wagoner[23], and the fact that the Godbillon–Vey class coincides with a non-zero multiple of the Euler class for flat $SL(2, \mathbb{R})$-bundles[2, 3].

In §1, we give the precise definition of $\mathbf{G}$ and state our result. In §2, we give several results obtained from those of Sah–Wagoner[23]. We consider, in §3, the relative homology $H_2(\mathbf{G}, R)$, where R is the subgroup consisting of rotations, and we will see that it is isomorphic to a countable direct sum of $\mathbb{R}$'s. In §4 we construct a homomorphism from $H_2(\mathbf{G}, R)$ to $H_3(B\Gamma_1{}^\infty)$, and we complete the proof of our theorem in §5.

§1. DEFINITION OF G AND THE STATEMENT OF THE RESULT

First, we define the group $\mathbf{G}$.

Let G_1 be the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbb{Z}_2$. For a positive integer n, let p_n: $G_n \to G_1$ be the n-fold cyclic covering; then G_n has a topological group structure such that p_n is a homomorphism.

Consider the subgroup R_1 of G_1 which consists of rotations. For $n \in \mathbb{N}$, we put $R_n = p_n{}^{-1}(R_1)$. Then as a topological group, R_n is isomorphic to $\mathbb{R}/\mathbb{Z}$;

$$R_1 \cong R_2 \cong \cdots \cong R_n \cong \cdots \cong \mathbb{R}/\mathbb{Z}.$$

We fix these isomorphisms.

For $n \in \mathbb{N}$, let $G^{(n)}$ be the free product of G_i $(i = 1, \ldots, n)$ with amalgamation

$R_1 = R_2 = \cdots = R_n$;

$$G^{(n)} = G_1 * G_2 * \cdots * G_n / R_1 = R_2 = \cdots = R_n.$$

Then we have a direct system of groups;

$$G^{(1)} \hookrightarrow G^{(2)} \hookrightarrow \cdots \hookrightarrow G^{(n)} \hookrightarrow \cdots.$$

We define $\mathbf{G}$ to be the direct limit group;

$$\mathbf{G} = \xrightarrow{\lim} G^{(n)}.$$

We have the subgroup R of rotations of $\mathbf{G}$ which corresponds to the subgroups of rotations $R_1 = R_2 = \cdots = R_n \subset G^{(n)}$.

G_1 acts on the Poincaré half-plane by the linear fractional transformations, hence on the boundary of it, which is S^1. This action is conjugate to that on S^1 considered as the set of lines of $\mathbb{R}^2$ through the origin, induced by the linear action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$. The group G_n acts on S^1, the n-fold cyclic covering of S^1 and the actions of R_n and R_1 on S^1 are compatible with respect to the isomorphism $R_n \cong R_1$. Therefore $G^{(n)}$ acts on S^1 and so does $\mathbf{G}$. These actions are orientation preserving C^∞-actions (actually real analytic actions), so we have a homomorphism $\mathbf{G} \to \mathrm{Diff}_+{}^\infty(S^1)$, where $\mathrm{Diff}_+{}^\infty(S^1)$ denotes the group of orientation preserving smooth diffeomorphisms of S^1. (This homomorphism is probably injective, but the author has not been able to prove it.)

Now we consider foliated S^1-bundles whose structural group reduces to the discrete group $\mathbf{G}$. For these foliated bundles, there exists a classifying space $B\mathbf{G}$. Naturally, in the second cohomology group of the group $\mathbf{G}$, i.e., in that of the space $B\mathbf{G}$, there are defined the Euler class $e \in H^2(\mathbf{G}, \mathbb{Z})$ and the Godbillon–Vey class $gv \in H^2(\mathbf{G}, \mathbb{R})$ (see [2]). Brooks[2, 3] showed that gv: $H_2(\mathbf{G}, \mathbb{Z}) \to \mathbb{R}$ is surjective.

Our main result is the following theorem.

THEOREM 1.1. *Let ξ be a foliated S^1-bundle over a closed oriented surface Σ. Suppose that the structural group of ξ reduces to the discrete group $\mathbf{G}$. Then the foliation of ξ is C^∞-foliated cobordant to zero if and only if $gv(\xi)[\Sigma] = 0$, where $gv(\xi)$ is the Godbillon–Vey class gv pulled back to $H^2(\Sigma, \mathbb{R})$ by the classifying map $\Sigma \to B\mathbf{G}$.*

Since a foliated oriented S^1-bundle is a $\bar{\Gamma}_1$-structure on the total space of the bundle, we have the following homomorphisms:

$$s_n\colon H_2(G_n; \mathbb{Z}) \to H_3(B\bar{\Gamma}_1{}^\infty; \mathbb{Z})$$

and

$$s\colon H_2(\mathbf{G}; \mathbb{Z}) \to H_3(B\bar{\Gamma}_1{}^\infty; \mathbb{Z}).$$

Let GV denote the Godbillon–Vey class in $H^3(B\bar{\Gamma}_1{}^\infty; \mathbb{R})$. Of course $gv = GV \circ s$. Since $\mathscr{F}\Omega_{3,1}^\infty$ is isomorphic to $H_3(B\bar{\Gamma}_1{}^\infty; \mathbb{Z})$ (see e.g. [26]), Theorem 1.1 is equivalent to the following theorem.

THEOREM 1.2. *The restriction of GV to $s(H_2(\mathbf{G}; \mathbb{Z}))$ is an isomorphism onto $\mathbb{R}$; $GV\colon s(H_2(\mathbf{G}\colon \mathbb{Z})) \xrightarrow{\cong} \mathbb{R}$.*

Remark 1.3. Thurston's examples given in [25] are contained in $t_1(H_2(G_1, R_1; \mathbb{Z}))$ (see 4.5, 5.2) which coincides with $s(H_2(\mathbf{G}; \mathbb{Z}))$.

§2. SECOND HOMOLOGY OF G_n

In this section, we study the structure of $H_2(G_n; \mathbb{Z})$.

For a perfect group G, we have the universal central extension $\upsilon: U \to G$, and $\ker \upsilon$ is isomorphic to $H_2(G; \mathbb{Z})$ (see Milnor[12]). This isomorphism is given as follows: Any element x of $H_2(G; \mathbb{Z})$ is represented by a homomorphism $\psi: \pi_1(\Sigma_k, *) \to G$, where Σ_k is a closed oriented surface of genus k. Take generators a_i $(i = 1, \ldots, 2k)$ of $\pi_1(\Sigma_k, *)$ so that $\pi_1(\Sigma_k, *) = \langle a_1, \ldots, a_{2k}: [a_1, a_2] \ldots [a_{2k-1}, a_{2k}] = 1 \rangle$. For each i $(i = 1, \ldots, 2k)$, choose an element $\widetilde{\psi(a_i)}$ of U such that $\upsilon(\widetilde{\psi(a_i)}) = \psi(a_i)$. Then $y = [\widetilde{\psi(a_1)}, \widetilde{\psi(a_2)}] \ldots [\widetilde{\psi(a_{2k-1})}, \widetilde{\psi(a_{2k})}]$ belongs to $\ker \upsilon$. This y is independent of the choices of the homomorphism ψ, the generators of $\pi_1(\Sigma_k, *)$ and the lifts $\widetilde{\psi(a_i)}$, and this is the element of $\ker \upsilon$ which corresponds to $x \in H_2(G, \mathbb{Z})$.

In the case of the perfect group $SL(2, \mathbb{R})$, we have the Steinberg group $St(2, \mathbb{R})$ as the universal central extension of it. Moreover, in [23, (1.18)], Sah and Wagoner proved the following result.

PROPOSITION 2.1 (*Sah–Wagoner* [23]). *$H_2(G_2; \mathbb{Z}) = H_2(SL(2; \mathbb{R}); \mathbb{Z})$ ($= K_2(2, \mathbb{R})$ in the usual notation) is a direct sum of an infinite cyclic group X_2 and a $\mathbb{Q}$-vector space Y_2. As subgroups of $St(2, \mathbb{R})$, by using the Steinberg symbols, the generators of the subgroups X_2, Y_2 of $H_2(G_2; \mathbb{Z})$ are described multiplicatively as follows: X_2 is generated by $c(-1, -1) = h_{12}(-1)^2 = w_{12}(-1)^4$. Y_2 is generated by $c(u, v) = h_{12}(u)h_{12}(v)h_{12}(uv)^{-1}$ where $u, v > 0$.*

Using the following equality of Moore ([16], Lemma 3.2; [23], (1.9))

$$c(t, s)c(s, t)^{-1} = c(t^2, s) = c(s^2, t)^{-1} = h_{12}(t)h_{12}(s)h_{12}(t)^{-1}h_{12}(s)^{-1},$$

and the fact that the homomorphism $\upsilon: St(2, \mathbb{R}) \to SL(2, \mathbb{R})$ maps $h_{12}(u)$ to the element

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},$$

we can see that $c(u, v)$ $(u, v > 0) \in Y_2 \subset H_2(G_2; \mathbb{Z})$ is represented by a 2-cycle

$$\left(\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}\right) - \left(\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}\right),$$

that is, it is represented by a homomorphism $\psi: \pi_1(T^2, *) \cong \mathbb{Z}^2 \to G_2$ such that

$$\psi(1, 0) = \begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}$$

and

$$\psi(0, 1) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}.$$

From Moore's equality, we can also see that $c(u, v)$ is bimultiplicative if both u and v are positive.

Since G_1 is the quotient group of G_2 by the center ($\cong \mathbb{Z}_2$), the composition $St(2, \mathbb{R}) \to G_2 \to G_1$ is the universal central extension of G_1. Moreover, since G_n is a central extension of G_1, we have a homomorphism $St(2,\mathbb{R})\to G_n$ which is the universal central extension of G_n. These facts imply the following proposition.

PROPOSITION 2.2. *For any positive integer n, $H_2(G_n;\mathbb{Z})$ is a direct sum of an infinite cyclic group X_n and a $\mathbb{Q}$-vector space Y_n. As a subgroup of $St(2,\mathbb{R})$, X_n is generated by $h_{12}(-1)^n = w_{12}(-1)^{2n}$ and $Y_n = Y_2$.*

It follows from this proposition that the projection $H_2(G_n;\mathbb{Z})\to X_n$ coincides with the Euler class (up to sign according to conventions). For the Euler class factors through this projection and, by Milnor[11] and Wood[29], there exists a flat G_n-bundle with the Euler class "one".

Remark. Sah–Wagoner[23] proved that Y_2 is a $\mathbb{Q}$-vector space of dimension equal to the continuum. This implies that there are many non-zero elements of $H_2(G_2;\mathbb{Z})$ which are represented by $(f,g)-(g,f)$'s with f, g belonging to a one parameter subgroup of G_2. In [26], however, the author proved that it is not the case for $H_2(\mathrm{Diff}_K{}^\infty(\mathbb{R});\mathbb{Z})$. Precisely, if f and g belong to a one parameter subgroup of $\mathrm{Diff}_K{}^\infty(\mathbb{R})$ generated by a smooth vectorfield on $\mathbb{R}$, then $(f,g)-(g,f)$ is homologous to zero.

Now we prove the following lemma.

LEMMA 2.3. *Let i_n; $R_n\to G_n$ be the inclusion for $n\in\mathbb{N}$. Then $i_{n*}(H_2(R_n;\mathbb{Z})) = Y_n$.*

Proof of Lemma 2.3 *for* $n=2$. (In this case, this lemma may be implicitely proved in [23].) First, since R_2 is the subgroup consisting of all rotations, $H_2(R_2;\mathbb{Z})$ is divisible. Therefore, $i_{2*}(H_2(R_2;\mathbb{Z}))\subset Y_2$.

We prove that any 2-cycle

$$Z(x,y) = \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1}\end{pmatrix}\right) - \left(\begin{pmatrix} y & 0 \\ 0 & y^{-1}\end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}\right) \quad (x, y > 0)$$

is homologous to a union of 2-cycles of the form $(r_1,r_2)-(r_2,r_1)$ $(r_1,r_2\in R_2)$. By the bimultiplicativity, it suffices to prove it when $1<y<x<\sqrt{2}$.

Let V be the complement of an open tubular neighborhood of the link of Fig. 1 in S^3. Let a, b, c be the element of $\pi_1(V,*)$ corresponding to the meridians and let A, B, C be those corresponding to the longitudes. Then we have

$$\pi_1(V,*) = \langle a,b,c,A,B,C : aA = Aa, bB = Bb, cC = Cc,$$
$$ba^{-1}ca = A, cb^{-1}ab = B, ac^{-1}bc = C\rangle.$$

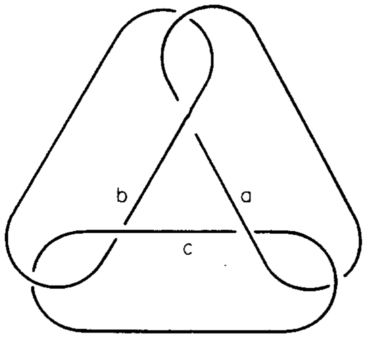

Fig. 1.

We shall define a homomorphism $\psi\colon \pi_1(V, *)\to SL(2, \mathbb{R})$ such that

$$\psi(a)=\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix},\ \psi(Aa^{-1})=\begin{pmatrix} y & 0 \\ 0 & y^{-1}\end{pmatrix},$$

and

$$|\mathrm{trace}\,\psi(b)|<2,\ |\mathrm{trace}\,\psi(c)|<2.$$

Put

$$\psi(b)=\begin{pmatrix} s & t \\ u & v\end{pmatrix}\quad sv-tu=1.$$

A calculation shows that a homomorphism $\psi\colon \pi_1(V, *)\to SL(2,\mathbb{R})$ with given $\psi(a)$, $\psi(Aa^{-1})$, $\psi(b)$ exists if x, y, s, t, u, v satisfy the following equation

$$v^2x^2y^2+sv(x^2+y^2)+s^2=(y^2-x^4)/(1-x^2).$$

Then, we have

$$\psi(c)=\begin{pmatrix} vxy & -ty/x \\ -ux/y & s/(xy)\end{pmatrix}.$$

Since we assumed that $1<y<x<\sqrt{2}$, we can choose s, t, u, v so that

$$|\mathrm{trace}\,\psi(b)|=|s+v|<2$$

and

$$|\mathrm{trace}\,\psi(c)|=|vxy+s/(xy)|<2.$$

(In fact, we can put $s=v=(x^4-y^2)^{1/2}((x^4-1)(y^2+1))^{-1/2}$ and $t=1$.) Thus our cycle $Z(x, y)$ is homologous to

$$-((\psi(b),\psi(B))-(\psi(B),\psi(b)))-((\psi(c),\psi(C))-(\psi(C),\psi(c))).$$

Since $\psi(b)\psi(B)=\psi(B)\psi(b)$, $\psi(c)\psi(C)=\psi(C)\psi(c)$ and $|\mathrm{trace}\,\psi(b)|<2$, $|\mathrm{trace}\,\psi(c)|<2$, $\psi(b)$, $\psi(B)$ and $\psi(c)$, $\psi(C)$ are simultaneously conjugate to pairs of elements of R_2, respectively. Since inner automorphisms act trivially on the homology of a group, $(\psi(b), \psi(B))-(\psi(B), \psi(b))$ and $(\psi(c), \psi(C))-(\psi(C), \psi(c))$ are homologous to 2-cycles of the form $(r_1, r_2)-(r_2, r_1)$ $(r_1, r_2\in R_2)$.

Proof of Lemma 2.3 *for general n.* First we note that, for $n\in N$, the homomorphism $(p_n|R_n)_*\colon H_2(R_n;\mathbb{Z})\to H_2(R_1;\mathbb{Z})$ is induced by the chain map which maps a base (f, g) $(f, g\in R_n)$ of $C_2(R_n)$ to $(p_n(f), p_n(g))$. Since, under the isomorphism $R_n\cong R_1\cong\mathbb{R}/\mathbb{Z}$, $(p_n|R_n)\colon \mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is the multiplication by n, the homomorphism $(p_n|R_n)_*\colon H_2(R_n;\mathbb{Z})\to H_2(R_1;\mathbb{Z})$ is surjective.

Now the commutative diagram

$$\begin{array}{ccc} R_n & \xrightarrow{i_n} & G_n \\ \downarrow{\scriptstyle p_n|R_n} & & \downarrow{\scriptstyle p_n} \\ R_1 & \xrightarrow[i_1]{} & G_1 \end{array}$$

gives the following commutative diagram:

$$\begin{array}{ccc} H_2(R_n;\mathbb{Z}) & \xrightarrow{i_{n*}} & H_2(G_n;\mathbb{Z}) \\ {\scriptstyle (p_n|R_n)_*}\downarrow & & \downarrow{\scriptstyle p_{n*}} \\ H_2(R_1;\mathbb{Z}) & \xrightarrow[i_{1*}]{} & H_2(G_1;\mathbb{Z}). \end{array}$$

We also note that p_{n*}: $H_2(G_n;\mathbb{Z})\to H_2(G_1;\mathbb{Z})$ coincides with the inclusion $X_n\bigoplus Y_n\to X_1\bigoplus Y_1$ of subgroups of $St(2,\mathbb{R})$.

If $n=2$, the above diagram implies that $i_{1*}(H_2(R_1:\mathbb{Z}))=Y_1$, because we know that $i_{2*}(H_2(R_2;\mathbb{Z}))=Y_2$. Then, if $n\geq 3$, the above diagram again implies that $i_{n*}(H_2(R_n;\mathbb{Z}))=Y_n$.

§3. SECOND HOMOLOGY OF (G, *R*)

In this section we show that the second homology $H_2(\mathbf{G}, R;\mathbb{Z})$ of the pair $(\mathbf{G}, R)$ is a countable direct sum of $\mathbb{R}$'s.

Let H be a subgroup of a group G, and consider the classifying spaces BH and BG. We may assume that these are obtained as realizations of semi-simplicial complexes associated to these groups, and that BH is a subcomplex of BG. Let $H_i(G,H;\mathbb{Z})$ denote the relative homology group $H_i(BG, BH;\mathbb{Z})$ $(i\geq 0)$.

Considering BR_n as a subcomplex of BG_n, we can construct the classifying space $BG^{(n)}$ as the space obtained from the disjoint union of BG_i $(i=1,\dots,n)$ by identifying the subcomplexes BR_i $(i=1,\dots,n)$;

$$BG^{(n)}=BG_1\cup\dots\cup BG_n/BR_1=\cdots=BR_n.$$

Then the classifying space $B\mathbf{G}$ is obtained as the direct limit $\xrightarrow{\lim} BG^{(n)}$.

The homology exact sequence for (BG_n, BR_n) is as follows:

$$H_2(R_n;\mathbb{Z})\xrightarrow{i_{n*}}H_2(G_n;\mathbb{Z})\xrightarrow{k_{n*}}H_2(G_n,R_n;\mathbb{Z})\to H_1(R_n;\mathbb{Z})\to H_1(G_n;\mathbb{Z}).$$

Here, $H_2(G_n;\mathbb{Z})=X_n\bigoplus Y_n$, $X_n\cong\mathbb{Z}$ by Proposition 2.2, $i_{n*}(H_2(R_n;\mathbb{Z}))=Y_n$ by Lemma 2.3, R_n is commutative and G_n is perfect, i.e. $H_1(G_n;\mathbb{Z})=0$. Hence we have an exact sequence

$$0\to X_n\to H_2(G_n,R_n;\mathbb{Z})\to R_n\to 0.$$

Now we prove the following lemma.

LEMMA 3.1. *There exists an isomorphism q_n: $H_2(G_n,R_n;\mathbb{Z})\to\mathbb{R}$ such that the following diagram commutes.*

$$\begin{array}{ccccccccc} 0 & \to & X_n & \to & H_2(G_n,R_n;\mathbb{Z}) & \to & R_n & \to & 0 \\ & & \downarrow\cong & & \downarrow q_n & & \downarrow\cong & & \\ 0 & \to & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \to & 0, \end{array}$$

where the first row is the above exact sequence and $R_n\cong\mathbb{R}/\mathbb{Z}$ is the fixed isomorphism.

Proof. Let $\tilde{\omega}_n$: $\tilde{G}_n\to G_n$ be the topological universal covering, and put $\tilde{R}_n=\tilde{\omega}_n^{-1}(R_n)$. Note that $\tilde{G}_n\to G_n$ is a central extension.

First we define a homomorphism r_n: $H_2(G_n, R_n; \mathbb{Z}) \to \tilde{R}_n$ as follows. Since BR_n is connected, any element x of $H_2(G_n, R_n; \mathbb{Z})$ is represented by a homomorphism ψ: $\pi_1(\Sigma_k - \mathring{D}^2, *) \to G_n$ such that $\psi(\partial(\Sigma_k - \mathring{D}^2)) \in R_n$ where $\Sigma_k - \mathring{D}^2$ is a closed oriented surface of genus k taken off a small disk in it, and $* \in \partial(\Sigma_k - \mathring{D}^2)$. Choose generators a_i $(i = 1, \dots, 2k)$ of $\pi_1(\Sigma_k - \mathring{D}^2, *)$ so that $[\sigma(\Sigma_k - \mathring{D}^2)] = [a_1, a_2] \dots [a_{2k-1}, a_{2k}]$. For each i, choose an element $\widetilde{\psi(a_i)}$ of $\tilde{G}_n$ such that $\tilde{\omega}_n(\widetilde{\psi(a_i)}) = \psi(a_i)$; then $y = [\widetilde{\psi(a_1)}, \widetilde{\psi(a_2)}] \dots [\widetilde{\psi(a_{2k-1})}, \widetilde{\psi(a_{2k})}]$ is an element of $\tilde{R}_n$. This y is independent of the choices of the homomorphism ψ, generators of $\pi_1(\Sigma_k - \mathring{D}^2, *)$ and lifts $\psi(a_i)$. We put $r_n(x) = y$. It is easy to see that r_n is a well-defined homomorphism.

If x comes from $H_2(G_n; \mathbb{Z})$, i.e. $x \in k_{n*}(H_2(G_n; \mathbb{Z}))$, then $r_n(x) \in \ker \tilde{\omega}_n$ and $r_n(x)$ is given by Milnor's algorithm for computing the Euler class (Milnor[11], Wood[29]). Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & X_n & \to & H_2(G_n, R_n; \mathbb{Z}) & \to & R_n & \to & 0 \\ & & \downarrow r'_n & & \downarrow r_n & & \| & & \\ 0 & \to & \mathbb{Z} & \longrightarrow & \tilde{R}_n & \longrightarrow & R_n & \to & 0, \end{array}$$

where r'_n is an isomorphism ([11, 29]). Consequently, r_n is an isomorphism.

Identifying the second row with $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, we obtain the desired isomorphism q_n.

The homomorphism p_n: $G_n \to G_1$ induces a homomorphism p_{n*}: $H_2(G_n, R_n; \mathbb{Z}) \to H_2(G_1, R_1; \mathbb{Z})$. Since, in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X_n & \to & H_2(G_n, R_n; \mathbb{Z}) & \to & R_n & \to & 0 \\ & & \downarrow p_{n*} & & \downarrow p_{n*} & & \downarrow p_n & & \\ 0 & \to & X_1 & \to & H_2(G_1, R_1; \mathbb{Z}) & \to & R_1 & \to & 0, \end{array}$$

p_{n*}: $X_n \to X_1$ is the multiplication by n under the identifications $X_n \cong \mathbb{Z}$, $X_1 \cong \mathbb{Z}$ and p_n: $R_n \to R_1$ is also the multiplication by n under the identifications $R_n \cong \mathbb{R}/\mathbb{Z}$, $R_1 \cong \mathbb{R}/\mathbb{Z}$, we have the following formula.

LEMMA 3.2. $q_n = (1/n) q_1 p_{n*}$.

Now we consider the exact sequence of the pair $(B\mathbf{G}, BR)$:

$$H_2(R; \mathbb{Z}) \to H_2(\mathbf{G}; \mathbb{Z}) \xrightarrow{k} H_2(\mathbf{G}, R; \mathbb{Z}).$$

By excision, we have

$$H_2(\mathbf{G}, R; \mathbb{Z}) \cong \bigoplus_{n=1}^{\infty} H_2(G_n, R_n; \mathbb{Z}).$$

We write by j_n the projection $H_2(\mathbf{G}, R; \mathbb{Z}) \to H_2(G_n, R_n; \mathbb{Z})$.

We remark here that we can write down the Euler class and the Godbillon–Vey class by q_n and j_n by using Milnor's algorithm ([11, 29]) and a result of Brooks ([2, 3]).

$$e = \Sigma\, q_n j_n k$$

and

$$gv = C\, \Sigma\, n^2 q_n j_n k,$$

where C is a non-zero constant. We will explain the latter later.

§4. HOMOMORPHISM $H_2(G, R; \mathbb{Z}) \to H_3(B\bar{\Gamma}_1^\infty; \mathbb{Z})$

In this section, we study the image of s_n: $H_2(G_n; \mathbb{Z}) \to H_3(B\bar{\Gamma}_1^\infty; \mathbb{Z})$, and we will define a homomorphism t_n: $H_2(G_n, R_n; \mathbb{Z}) \to H_3(B\bar{\Gamma}_1^\infty; \mathbb{Z})$ such that $t_n k_n = s_n$. To compare t_n and $t_1 p_{n*}$ we use the following lemma.

LEMMA 4.1. *Let $(M, \mathscr{H})$ be a $\bar{\Gamma}_1^\infty$-structure on a closed oriented 3-manifold. Let p: $M^{(n)} \to M$ be an n-fold branched covering whose branching locus is a compact codimension 2 submanifold of M. Let $(M^{(n)}, \mathscr{H}^{(n)})$ be the induced $\bar{\Gamma}_1^\infty$-structure. Then, $(M^{(n)}, \mathscr{H}^{(n)})$ is $\bar{\Gamma}_1^\infty$-cobordant to a disjoint union of n copies of $(M, \mathscr{H})$.*

Proof. Let $\tau: n(M, \mathscr{H}) \to (M, \mathscr{H})$ be the trivial n-fold covering. Let X be the space obtained from the mapping cylinders of p and τ by identifying the images of p and τ, X has the induced $\bar{\Gamma}_1^\infty$-structure $\mathscr{X}$. Consider the classifying map $B\mathscr{X}$: $X \to B\bar{\Gamma}_1^\infty$; then we have

$$\partial B\mathscr{X} = B\mathscr{H}^{(n)} \cup (-nB\mathscr{H}),$$

where $B\mathscr{H}^{(n)}$ and $B\mathscr{H}$ are the classifying maps for $(M^{(n)}, \mathscr{H}^{(n)})$ and $(M, \mathscr{H})$, respectively. Since $\Omega_3(B\bar{\Gamma}_1^\infty) = H_3(B\bar{\Gamma}_1^\infty; \mathbb{Z})$[4], we have proved Lemma 4.1.

As a colollary to this lemma, we have

Corollary 4.2. $s_n = ns_1 p_{n*}$.

For, if an element x of $H_2(G_n; \mathbb{Z})$ corresponds to a homomorphism ψ: $\pi_1(\Sigma, *) \to G_n$ with Σ being a closed oriented surface, $p_{n*}(x)$ corresponds to $p_n \circ \psi$: $\pi_1(\Sigma, *) \to G_1$. It is easy to see that there is an n-fold covering map between the corresponding foliated S^1-bundles which respects the $\bar{\Gamma}_1^\infty$-structures.

For the homomorphism s_n, we have the following lemma. This is a special case of our theorem for foliated S^1-bundles whose structural groups reduce to G_n. Note that in this case the Godbillon–Vey class is proportional to the Euler class[2, 3].

LEMMA 4.3. $\ker s_n = Y_n$.

Proof. Every generator of Y_n is represented by a foliated S^1-bundle over a torus whose structural group reduces to R_n (Lemma 2.3) and this foliation can be defined by a non-vanishing closed 1-form. Hence this foliation is C^∞-foliated cobordant to zero, that is, $Y_n \subset \ker s_n$. On the other hand, since for these bundles, the Godbillon–Vey class is a non–zero multiple of the Euler class and the Euler class is nothing but the projection $H_2(G_n; \mathbb{Z}) \to X_n$, we have $Y_n \supset \ker s_n$.

The above lemma implies that there exists a well defined homomorphism $\bar{s}_n$: $\mathrm{Im}\, k_n \to H_3(B\bar{\Gamma}_1^\infty; \mathbb{Z})$ such that $\bar{s}_n k_n = s_n$. By Corollary 4.2, we also have $\bar{s}_n = n\bar{s}_1 p_{n*}$. We shall define a homomorphism t_n: $H_2(G_n, R_n; \mathbb{Z}) \to H_3(B\bar{\Gamma}_1^\infty; \mathbb{Z})$ so that $t_n k_n = s_n$, $t_n = nt_1 p_{n*}$, $s = \sum_n t_n j_n k$ and $GVt_n = Cn^2 q_n j_n k$ with C being a non-zero constant.

In order to define t_n, we need a $\bar{\Gamma}_1^\infty$-structure $\mathscr{H}^{(n)}$ on D^2, the 2-disk.

Let η: $[0, 1/2] \to \mathbb{R}$ be a C^∞-function such that $\eta(x) = 0, x \in [0, 1/3]$ and $\eta(x) > 0$, $x \in (1/3, 1/2]$. Let f: $(1/3, 1] \to \mathbb{R}$ be a C^∞-function such that $f(x) = 0$, $x \in [2/3, 1]$ and $(f'(x))^{-1} = \eta(x)$, $x \in (1/3, 1/2]$.

First we define a C^∞-foliation $\mathscr{H}$ of $[0, 1] \times \mathbb{R}$, invariant under translations in the direction of $\mathbb{R}$. For $\theta \in \mathbb{R}$, put

$$L_\theta = \{(x, y) \in [0, 1] \times \mathbb{R};\ y = f(x) + \theta,\ x \in (1/3, 1]\},$$

and for $r \in [0, 1/3]$, put

$$L'_r = \{(x, y) \in [0, 1] \times \mathbb{R};\ x = r,\ y \in \mathbb{R}\}.$$

Then the foliation $\mathscr{H}$ is defined by

$$\mathscr{H} = \{L_\theta;\ \theta \in \mathbb{R}\} \cup \{L'_r;\ r \in [0, 1/3]\}.$$

Consider the map p'_n: $[0, 1] \times \mathbb{R} \to D^2 = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 \leq 1\}$ defined by $p'_n(r, \theta) = (r \cos(2\pi\theta/n),\ r \sin(2\pi\theta/n))$. Since $\mathscr{H}$ is invariant under translations, we have a $\Gamma_1{}^\infty$-structure $\mathscr{H}^{(n)}$ on D^2 satisfying $p'^*_n \mathscr{H}^{(n)} = \mathscr{H}$. Note that $\mathscr{H}^{(n)}$ is invariant under rotations.

For a rotation α, let $\sigma_\alpha(\mathscr{H}^{(n)})$ denote the "suspended Γ_1^∞-structure"; more precisely, $\sigma_\alpha(\mathscr{H}^{(n)})$ is the $\Gamma_1{}^\infty$-structure of $D^2 \times S^1 = D^2 \times [0, 1]/(z, 0) \sim (\alpha(z), 1)$ which is induced from the product $\Gamma_1{}^\infty$-structure $(D^2 \times [0, 1], \mathscr{H}^{(n)} \times [0, 1])$. $\sigma_\alpha(\mathscr{H}^{(n)})$ is well defined since α preserves $\mathscr{H}^{(n)}$. This $\Gamma_1{}^\infty$-structure is not defined as foliation along $\{0\} \times S^1$. If one prefers to remain always in the category of foliations, one can put a Reeb component along $\{0\} \times S^1$ without changing the following argument.

Let p''_n: $D^2 \to D^2$ be the n-fold branched covering map defined by p''_n $(r \cos\theta, r \sin\theta) = (r \cos n\theta,\ r \sin n\theta)$. Then we have $p''^*_n \mathscr{H}^{(1)} = \mathscr{H}^{(n)}$ and p''_n induces an n-fold branched covering map p_n: $D^2 \times S^1 \to D^2 \times S^1$ such that $\sigma_\alpha(\mathscr{H}^{(n)}) = p^*_n(\sigma_{n\alpha}(\mathscr{H}^{(1)}))$.

Now we define the homomorphism t_n: $H_2(G_n, R_n; \mathbb{Z}) \to H_3(B\Gamma_1{}^\infty; \mathbb{Z})$. Any element $x \in H_2(G_n, R_n; \mathbb{Z})$ is represented by a continuous map b: $(N, \partial N) \to (BG_n, BR_n)$, where N is a compact oriented 2-manifold with ∂N consisting of m circles (m can be zero). Then the boundary of the foliated S^1-bundle $\mathscr{F}_b$ associated to b consists of foliated S^1-bundles over m (oriented) circles with the holonomies being rotations $\alpha_1, \ldots, \alpha_m$. We attach $(D^2 \times S^1, \sigma_{\alpha_i}(\mathscr{H}^{(n)}))$ $(i = 1, \ldots, m)$ to the boundary of $\mathscr{F}_b$; then $\sigma_{\alpha_i}(\mathscr{H}^{(n)})$'s and $\mathscr{F}_b$ define a $\Gamma_1{}^\infty$-structure $\bar{\mathscr{F}}_b$ on the resulted closed oriented 3-manifold.

By the following lemma, we define $t_n(x)$ to be the class of $\bar{\mathscr{F}}_b$ in $H_3(B\Gamma_1{}^\infty; \mathbb{Z})$

LEMMA 4.4. *The class of* $\bar{\mathscr{F}}_b$ *in* $H_3(B\Gamma_1{}^\infty; \mathbb{Z})$ *is independent of the choice of b.*

Proof. Let b': $(N', \partial N') \to (BG_n, BR_n)$ represent also x; then there exist a compact oriented 3-manifold W and a continuous map β: $W \to BG_n$ such that $\partial W = N \cup M \cup (-N')$, M being a 2-manifold, $N \cap M = \partial N$, $N' \cap M = \partial N'$ and $\beta(M) \subset BR_n$ [4]. Since the $\Gamma_1{}^\infty$-structure of the foliated S^1-bundle corresponding to $\beta|\partial W$ is homologous to zero, the difference between $\bar{\mathscr{F}}_b$ and $\bar{\mathscr{F}}_{b'}$ is homologous to $\bar{\mathscr{F}}_{\beta|M}$.

On the other hand, since $\beta(M) \subset BR_n$, $\beta|M$ defines a "foliated disk bundle" over M. More precisely, since $\mathscr{H}^{(n)}$ is invariant under R_n, there exists a $\Gamma_1{}^\infty$-structure $\mathscr{G}$ on the 2-disk bundle over M associated to $\beta|M$, which is transverse to the fibers with the restriction to a fiber isomorphic to $\mathscr{H}^{(n)}$ and with the holonomy given by $\beta|M$. It is easy to see that the boundary of $\mathscr{G}$ is precisely $\bar{\mathscr{F}}_{\beta|M}$. Thus we have proved Lemma 4.4.

Remark 4.5. The above construction of $\bar{\mathscr{F}}_b$ is a generalization of Thurston's examples given in [25] where he used a continuous map b from a 2-disk with small open disks deleted to (BG_1, BR_1) to represent a class of $H_2(G_1, R_1)$ and constructed $\bar{\mathscr{F}}_b$. Thus his examples are in the image of t_1.

Remark 4.5′. To obtain $\bar{\mathscr{F}}_b$, it is not necessary to use the same $\mathscr{H}^{(n)}$ for every boundary component. In fact we can use any $\Gamma_1{}^\infty$-structure on the 2-disk with a non-foliation point at the center which is invariant under rotations and transverse to the boundary circle. Then we modify the proof of Lemma (4.4) as follows: the difference between $\mathscr{F}_b$ and $\mathscr{F}_{b'}$ is homologous to $\mathscr{F}_{\beta|M}$ as before. The $\Gamma_1{}^\infty$-structure $\mathscr{F}_{\beta|M}$ is a foliation almost without holonomy off the non-foliation points and trivial in a neighborhood of the non-foliation points. (The non-foliation points correspond to the center of the 2-disk.) We can see that the holonomy of any compact leaf is contained in a one parameter group of germs of diffeomorphisms generated by a germ of a C$^\infty$-vectorfield. Moreover, since $\beta(M) \subset BR_n$, the Novikov transformations of components bounded by compact leaves are contained in the group of translations of $\mathbb{R}$. Thus, by a result of Mizutani–Morita–Tsuboi [15], $\mathscr{F}_{\beta|M}$ is $\Gamma_1{}^\infty$-cobordant to zero. (However, for simplicity, we use the fixed $\mathscr{H}^{(n)}$ in the rest of this paper.)

By Lemma 4.4, t_n is well defined. It is easy to see that t_n is a homomorphism and $t_n k_n = s_n$. Note that we can always represent an element $x \in H_2(G_n, R_n)$ by a map b: $(\Sigma_k - \mathring{D}^2, \partial(\Sigma_k - \mathring{D}^2)) \to (BG_n, BR_n)$, that is, a homomorphism ψ: $\pi_1(\Sigma_k - D^2, *) \to G_n$ with $\alpha = \psi(\partial(\Sigma_k - \mathring{D}^2)) = \partial x \in R_n$. Then we write $\mathscr{F}_b$ as $\mathscr{F}_\psi \cup \sigma_{\partial x}(\mathscr{H}^{(n)})$.

LEMMA 4.6. *$t_n = n t_1 p_{n*}$.*

Proof. Let $x \in H_2(G_n, R_n; \mathbb{Z})$ be represented by ψ: $\pi_1(\Sigma_k - \mathring{D}^2, *) \to G_n$ such that $\alpha = \psi(\partial(\Sigma_k - \mathring{D}^2)) \in R_n$. Then $t_n(x)$ is given by $\mathscr{F}_\psi \cup \sigma_\alpha(\mathscr{H}^{(n)})$ by definition. On the other hand, $p_{n*}(x)$ is represented by $p_n \circ \psi$. Since $p_n \circ \psi(\partial(\Sigma_k - \mathring{D}^2)) = n\alpha \in R_1$, $t_1 p_{n*}(x)$ is represented by $\mathscr{F}_{p_n \circ \psi} \cup \sigma_{n\alpha}(\mathscr{H}^{(1)})$. Now we have an n-fold branched covering map $\bar{p}_n$: $D^2 \times S^1 \to D^2 \times S^1$ such that $\sigma_\alpha(\mathscr{H}^{(n)}) = \bar{p}_n^*(\sigma_{n\alpha}(\mathscr{H}^{(1)}))$. Moreover $\bar{p}_n$ is compatible with the n-fold covering map $\mathscr{F}_\psi \to \mathscr{F}_{p_n \circ \psi}$ induced by p_n along the boundary. Thus we have an induced branched covering map from $\mathscr{F}_\psi \cup \sigma_\alpha(\mathscr{H}^{(n)})$ to $\mathscr{F}_{p_n \circ \psi} \cup \sigma_{n\alpha}(\mathscr{H}^{(1)})$. By Lemma 4.1, we have $t_n = n t_1 p_{n*}$.

LEMMA 4.7. *$s = \sum_n t_n j_n k$.*

Proof. For $x \in H_2(\mathrm{G}; \mathbb{Z})$, put $\Lambda = \{\mathrm{n};\ j_n k(x) \neq 0\}$. Let M' be the two-sphere S^2 with $\# \Lambda$ small open disks deleted. We define a foliated S^1-bundle ζ over M' such that the total holonomies along the boundaries are $\partial(j_n k(x)) \in R_n \cong \mathbb{R}/\mathbb{Z}$, $n \in \Lambda$. Consider a $\Gamma_1{}^\infty$-structure $\mathscr{H}'$ on $S^1 \times M' \cup (\# \Lambda)\ D^2 \times S^1$ given as $\mathscr{F}_\zeta \cup \bigcup_{n \in \Lambda} (\sigma_{\partial(j_n k(x))}(\mathscr{H}^{(n)}))$. We can see that the difference between $s(x)$ and $\sum t_n j_n k(x)$ is represented by $\mathscr{H}'$. Since $\mathscr{H}$ belongs to the special type of $\Gamma_1{}^\infty$-structures that we described in remark 4.5′, we see that $\mathscr{H}'$ is $\Gamma_1{}^\infty$-cobordant to zero. This proves Lemma 4.7.

§5. COMPLETION OF THE PROOF OF THE MAIN THEOREM

First we prove the following lemma which explains how the Godbillon–Vey invariant varies continuously.

LEMMA 5.1. *$GVt_1 = Cq_1$, where C is a non-zero constant.*

Proof. Let $x \in H_2(G_1, R_1; \mathbb{Z})$ be represented by ψ. We consider Godbillon-Vey forms associated to the $\Gamma_1{}^\infty$-structure $\mathscr{F}_\psi \cup \sigma_{\partial x}(\mathscr{H}^{(1)})$. By a result of Mizutani–Morita–Tsuboi [14], since the holonomy of the compact leaf with nontrivial holonomy in $\sigma_{\partial x}(\mathscr{H}^{(1)})$ is contained in a one parameter group generated by a C^∞-vectorfield, we can take a Godbillon–Vey form for $\mathscr{F}_\psi \cup \sigma_{\partial x}(\mathscr{H}^{(1)})$ which vanishes on $\sigma_{\partial x}(\mathscr{H}^{(1)})$. Then, by a result of Brooks [2, 3], recalling that q_1 is computed by Milnor's algorithm (Lemma 3.1), we have $GVt_1 = Cq_1$.

The following lemma is an immediate consequence of Lemmas 3.1 and 5.1.

LEMMA 5.2. *Let x be an element of $H_2(G_1, R_1; \mathbb{Z})$. If $GVt_1(x) = 0$, then $t_1(x) = 0$.*

By Remark 4.5 and Lemma 5.2, Thurston's examples in [25] are C^∞-foliated cobordant to zero if their Godbillon–Vey invariants are zero.

Now we are ready to prove Theorem 1.2, our main theorem.

Proof of Theorem 1.2. Let x be an element of $H_2(\mathbf{G}; \mathbb{Z})$. By Lemma 4.7, we have

$$s(x) = \sum_n t_n j_n k(x).$$

Then by Lemmas 4.6 and 4.4, we have

$$s(x) = \sum_n n t_1 p_{n*} j_n k(x) = t_1\Big(\sum_n n p_{n*} j_n k(x)\Big).$$

Now Lemma 5.2 implies that $s(x) = 0$ provided that $GV \circ s(x) = 0$. Since $GV \circ s$ is surjective onto $\mathbb{R}$ (Brooks [2, 3]), we have proved Theorem 1.2.

Finally, we give the formula for gv: $H_2(\mathbf{G}; \mathbb{Z}) \to \mathbb{R}$ which we gave at the end of §3.

PROPOSITION 5.3. $gv = c \sum n^2 q_n j_n k$

Proof. By the above formula and Lemmas 5.1 and 3.2, we have

$$\begin{aligned} gv(x) &= GV \circ s(x) \\ &= GVt_1\Big(\sum n p_{n*} j_n k(x)\Big) \\ &= Cq_1\Big(\sum n p_{n*} j_n k(x)\Big) \\ &= C \sum n^2 q_n j_n k(x). \end{aligned}$$

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo 113
Japan