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Fibered geometries

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Abstract

Our aim is to initiate the study of fibered geometries, in particular fibered projective planes and fibered generalized polygons. In fact, we apply the theory of fuzzy sets in a particular way on incidence geometry. Combinatorial and geometric questions arise. But also classical objects are recognized by this alternative view: for instance apartments arise naturally in the theory of "contagious values".

1. Introduction

A lot of theories in mathematics have the so-called *fuzzy counterparts*. Basically this means that certain elements of an object get a membership degree (a number in the unit interval $[0,1]$) as an alternative for the classical black-and-white situation of 'belonging to or not belonging to'. Certain rules to deal with this alternative approach have been established (see for instance [9]), and the theory has been applied in many areas. It is our aim to contribute to this theory by introducing a particular kind of "fuzzy geometries", which we will call *fibered geometries*. We motivate our study as follows.

In the theory of fuzzy sets, the notions of (discrete) *fuzzy group* and (discrete) *fuzzy vector space* exist (see [6,2]). In the classical theory, groups and geometries

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have always been very close and interrelated areas. In fact, in finite group theory, the notion of a vector space is crucial, as many (almost simple) groups are understood by their action on a "module" (a finite vector space). Particularly helpful in this respect is in many cases the corresponding projective space, where the action on the points is faithful and primitive. The research of the incidence-geometers is mostly motivated by group-theoretical questions and problems, and conversely, group-theoretical problems arise sometimes from purely geometric problems.

Hence, it seems strange that in the fuzzy algebra theories, the notions of fuzzy group and fuzzy vector space are defined independently. In [4,5], we have shown that these definitions are in fact related, and that the "objects in the middle" are the fuzzy projective spaces. In our search for the most suitable definition of "fuzzy geometry", we encountered the following problem: there are two possibilities to "fuzzify" the points of a geometry. Either one assigns a unique degree of membership to each of them, or one assigns several degrees of membership to all of them.

The first possibility was studied in [4,5], and it turns out that every fuzzy vector space gives rise to a fuzzy projective space, and every fuzzy projective space gives rise to a fuzzy group. The geometric structure involved in fuzzy projective spaces in this sense is not very rich: basically a fuzzy projective space is equivalent with a given flag in the base projective space.

In the present paper, we explore the second possibility, and it turns out that the geometric structures involved are much richer. We will see that a fibered (we have chosen this name to distinguish it from the fuzzy examples in [4,5]) generalized polygon yields closed subconfigurations. Also, apartments arise naturally in this context.

Much more can and has been done. However, we will restrict ourselves to introducing fibered generalized polygons and to proving some first theoretical properties.

The paper is organized as follows. After some definitions from both the fuzzy set theory and the theory of generalized polygons (Section 2), we define fibered points and fibered lines in general incidence geometries (Section 3). In Section 4, we define fibered projective planes and look at fibered versions of some classical (configuration) theorems. In Section 5 we define the crucial notion of *contagious values* and we show a number of properties. In Section 6, we turn to generalized polygons and prove some basic results. Section 7 contains a few final remarks.

2. Preliminaries

We first recall some basic notions from fuzzy set theory.

Definition 2.1 (see Zadeh [9]). A *fuzzy set* λ on a set X is a mapping $\lambda : X \to [0,1] : x \mapsto \lambda(x)$. The number $\lambda(x)$ is called the *degree of membership* of the point x in λ. The *intersection of two fuzzy sets* λ and μ on X is given by the fuzzy set $\lambda \wedge \mu : X \to [0,1] : x \mapsto \lambda(x) \wedge \mu(x)$, where $\wedge$ denotes the minimum operator.

Definition 2.2 (see Kerre [3]). Consider a set X and fuzzy sets μ and λ on X. The *Cartesian product* $\mu \times \lambda$ of the two fuzzy sets is defined as follows:

$$\begin{aligned} \lambda \times \mu : X \times X &\to [0,1] \\ (x,y) &\mapsto \lambda(x) \wedge \mu(y). \end{aligned}$$

Definition 2.3 (see Zadeh [10]). For a set X, we denote by $\mathscr{F}(X)$ the set of all fuzzy subsets on X. Let X_1, X_2 and Y be any sets. Suppose f is a mapping from $X_1 \times X_2$ to Y (not necessarily defined everywhere). The *extension principle* states that this mapping can be extended to the following one:

$$f : \mathscr{F}(X_1) \times \mathscr{F}(X_2) \to \mathscr{F}(Y) : (\lambda_1, \lambda_2) \mapsto f(\lambda_1, \lambda_2),$$

where, for $\lambda_i \in \mathscr{F}(X_i)$, $i = 1, 2$, the image $f(\lambda_1, \lambda_2)$ is defined as the following fuzzy set on Y:

$$y \mapsto \begin{cases} \sup_{f(x_1,x_2)=y} \{(\lambda_1 \times \lambda_2)(x_1, x_2) \\ \quad | x_i \in X_i,\ i = 1, 2\} & \text{if } \exists \bar{x} \in X_1 \times X_2 : f(\bar{x}) = y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4. A point-line geometry $\mathscr{P} = (P, B, I)$ is a *generalized n-gon* if the diameter of the incidence graph Γ of $\mathscr{P}$ is equal to n and if the girth of Γ is equal to $2n$. In this paper, we restrict to $n \geqslant 3$ (the case $n = 2$ is a trivial case giving rise to geometries with the property that every line is incident with every point). Usually, we are only interested in *thick* generalized n-gons, i.e., generalized n-gons where every line contains at least three points, and every point is incident with at least three lines. A *generalized polygon*, or briefly a *polygon*, is a generalized n-gon for some $n \geqslant 2$. A thick generalized 3-gon is nothing else than an ordinary projective plane. Generalized polygons are basic geometries in the theory of incidence geometry and were introduced by Tits in [7]. For more information we refer to [8]. We just mention that there is a *principle of duality* for generalized polygons: interchanging the names of points and lines of a generalized n-gon yields again a (not necessarily isomorphic) generalized n-gon. Hence interchanging the names point and line in definitions and statements yields new definitions and statements.

The reason for the restriction to generalized polygons to investigate fibered geometries will become clear later. In fact, it is convenient to start with projective planes.

3. The f-points and f-lines

Let $\mathscr{P} = (P, B, I)$ be any point-line geometry with point set P and line set B. We assume that $\mathscr{P}$ is a partial linear space, i.e., two distinct points in $\mathscr{P}$ are incident with at most one line (and the dual statement of this automatically holds). If p and q are points incident with a common line, then we denote that line by $\langle p, q \rangle$ and we call p and q *collinear*. Dually, if L and M are lines incident with a common point, then we denote that point by $L \cap M$ and we call L and M *concurrent*. The notation suggests that we may view a line as the set of points incident with it. The reason for this nonself-dual notation is historical. We now define fibered points and fibered lines, briefly called f-points and f-lines.

Definition 3.1. Suppose $a \in P$ and $\alpha \in [0,1]$. The *f-point* (a,α) is the following fuzzy set on the point set P of $\mathscr{P}$:

$$(a,\alpha): P \to [0,1]: \begin{cases} a \mapsto \alpha, \\ x \mapsto 0 \quad \text{if } x \in P\backslash\{a\}. \end{cases}$$

Dually, one defines in the same way the *f-line* (L,β) for $L \in B$ and $\beta \in [0,1]$.

Defined in this way, an f-point is just a point a from the geometry $\mathscr{P}$ that is given a nonzero value α, which we will call its *degree of membership*. The point a is called the *base point* of the f-point (a,α). Different f-points may have the same base point. Similarly one defines the *base line* of an f-line.

We use Definition 2.1 to define the intersection f-point of two f-lines (L,α) and (M,β) (provided the lines L and M are concurrent in $\mathscr{P}$). We thus obtain:

Definition 3.2. The f-lines (L,α) and (M,β) intersect in the unique f-point $(L \cap M, \alpha \wedge \beta)$.

To obtain the f-line spanned by two f-points (a,α) and (b,β), where a and b are distinct collinear points in $\mathscr{P}$, we use the extension principle (referring to Definition 2.3, we put X_1 and X_2 equal to the set of points and we put Y equal to the set of lines; the map f assigns to every pair of distinct points the unique line through these points). With Definition 2.2 we obtain:

$$\begin{aligned} \langle (a,\alpha),(b,\beta)\rangle : B &\to [0,1] \\ \langle a,b\rangle &\mapsto \sup\{(a,\alpha)(x) \wedge (b,\beta)(y) : (x,y) \in P^2, x \neq y \\ &\qquad \text{and } \langle x,y\rangle = \langle a,b\rangle\} \\ L &\mapsto 0 \quad \text{if } L \in B\backslash\{\langle a,b\rangle\}. \end{aligned}$$

Since $(a,\alpha)(x) = 0$ if $x \neq a$ and $(b,\beta)(y) = 0$ if $y \neq b$, this reduces to

$$\begin{aligned} \langle (a,\alpha),(b,\beta)\rangle : B &\to [0,1] \\ \langle a,b\rangle &\mapsto \alpha \wedge \beta \\ L &\mapsto 0 \quad \text{if } L \in B\backslash\{\langle a,b\rangle\}, \end{aligned}$$

yielding the following easy rule:

Definition 3.3. The f-points (a,λ) and (b,β) span the unique f-line $(\langle a,b\rangle, \lambda \wedge \beta)$.

It is actually remarkable to see that the intersection of fuzzy sets (to define the intersection of f-lines) and the extension principle (to define the f-line spanned by two f-points) give rise to mutually dual definitions.

In ordinary partial linear spaces, a line is spanned by any two different points incident with it. Definition 3.3 shows that f-points of which the base points are collinear, do not necessarily span the same f-line: the degrees of membership may be different. Dually, f-lines with concurrent base lines do not necessarily intersect in the same f-point, for the same reason.

4. Fibered projective planes

4.1. Definition

Suppose we have a (thick) projective plane $\mathscr{P}=(P,B,I)$. A *fibered projective plane* $\mathscr{FP}$ consists of a set $\mathscr{FP}$ of f-points of $\mathscr{P}$ and a set $\mathscr{FB}$ of f-lines of $\mathscr{P}$ such that every point and every line of $\mathscr{P}$ is the base point and base line of at least one f-point and f-line, respectively, and such that $(\mathscr{FP},\mathscr{FB})$ satisfies the following fuzzified versions of the axioms of a projective plane:

(F1) every pair of f-points with distinct base points span a unique f-line;
(F2) every pair of f-lines with distinct base lines intersect in a unique f-point.

4.2. Collinear f-points and concurrent f-lines

Definition 4.1. A set of f-points are called *collinear* if each pair of them span the same f-line. Dually, a set of f-lines is called *concurrent* if each pair of them intersect in the same f-point.

Given the definition of the f-line spanned by two f-points, we see that in a set of collinear f-points, all base points are collinear and all degrees of membership are equal, except possibly one degree of membership that can be higher. Dually for a set of concurrent f-lines.

Remark on incidence: There is no logical way to define fuzzy incidence. A necessary requirement is that an f-line which is determined by two f-points is incident with these two f-points. Hence, if an f-point (p,α) would be "f-incident" with an f-line (L,β), then $\alpha\geqslant\beta$. Dually, $\beta\geqslant\alpha$. Consequently $\beta=\alpha$, but then the intersection f-point of two f-lines is not necessarily incident with both these lines, an absurd situation.

However, it might be worthwhile to define fuzzy incidence as a fuzzy set on the set of flags (an "f-flag"). We will not pursue this idea in the present paper.

4.3. Theorem of Desargues

The goal of this section is to show that theoretical theorems can be fuzzified. We give two examples. The reader can certainly think of many more.

For our purpose, a *Desarguesian projective plane* is a projective plane arising from a vector space of dimension 3 over a skew field in the classical way (points are the 1-spaces, lines are the 2-spaces of the vector space, and incidence is symmetrized containment).

We have the following classical result.

Theorem 4.1. *Consider a Desarguesian projective plane* $\mathscr{P}=(P,B,I)$. *We choose three noncollinear points* a_1, a_2, a_3 *and three other noncollinear points* b_1, b_2, b_3, *such that the lines* $\langle a_i,b_i\rangle$ *are concurrent in a point* p, *for* $i\in\{1,2,3\}$ *and* $a_i\neq b_i\neq p\neq a_i$. *Then the intersection points* $\langle a_1,a_2\rangle\cap\langle b_1,b_2\rangle$, $\langle a_1,a_3\rangle\cap\langle b_1,b_3\rangle$ *and* $\langle a_2,a_3\rangle\cap\langle b_2,b_3\rangle$ *are collinear*.

We now prove the following fuzzified version:

Theorem 4.2. *Suppose we have a fibered projective plane* $\mathscr{FP}$ *with base plane* $\mathscr{P}$ *that is Desarguesian. Choose three f-points* (a_1,α_1), (a_2,α_2) *and* (a_3,α_3) *in* $\mathscr{FP}$ *with noncollinear base points, and three other f-points* (b_1,β_1), (b_2,β_2), (b_3,β_3) *with noncollinear base points, such that the lines* $\langle a_i,b_i\rangle$, *for* $i\in\{1,2,3\}$, *are concurrent in a point* p of $\mathscr{P}$, *with* $a_i\neq b_i\neq p\neq a_i$. *Then the three f-lines* $(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j)$ *and* $(\langle b_i,b_j\rangle,\beta_i\wedge\beta_j)$ (*for* $i\neq j$ *and* $i,j\in\{1,2,3\}$) *intersect in three collinear f-points.*

Proof. Let, for $\{i,j,k\}=\{1,2,3\}$, (x_k,γ_k) be the intersection f-point of the f-lines $(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j)$ and $(\langle b_i,b_j\rangle,\beta_i\wedge\beta_j)$. Then clearly $\gamma_k=\alpha_i\wedge\alpha_j\wedge\beta_i\wedge\beta_j$. It is now clear that the smallest value of $\gamma_1,\gamma_2,\gamma_3$ occurs at least two times (since we use the minimum operator in our definitions). □

Another important theorem in ordinary geometry is the Pappus' theorem. For our purpose, a Pappian plane is a projective plane arising, as above, from a vector space of dimension 3 over a (commutative) field (hence any Pappian plane is Desarguesian).

Theorem 4.3. *Let* a_1, b_1 *and* c_1 *be three distinct points on a line* L_1 *and* a_2, b_2 *and* c_2 *three distinct points on a line* $L_2\neq L_1$, *all in a Pappian plane* $\mathscr{P}$. *Define three points* a_3, b_3 *and* c_3 *as follows*: $a_3=\langle b_1,c_2\rangle\cap\langle b_2,c_1\rangle$, $b_3=\langle a_1,c_2\rangle\cap\langle a_2,c_1\rangle$, $c_3=\langle a_1,b_2\rangle\cap\langle a_2,b_1\rangle$. *If no three of* a_1, b_1, a_2, b_2 *are collinear and* c_1, c_2 *are arbitrary, then the points* a_3, b_3 *and* c_3 *are collinear*.

The fibered version of this theorem is as follows:

Theorem 4.4. *Suppose we have a fibered projective plane* $\mathscr{FP}$ *with base plane* $\mathscr{P}$ (*a Pappian plane*). *Choose two different lines* L_1 *and* L_2 *in* $\mathscr{P}$. *Now choose two triples of f-points* (a_i,α_i), (b_i,β_i) *and* (c_i,γ_i) *with* a_i,b_i,c_iIL_i, *for* $i=1,2$, *and such that no three of the base points* a_1, b_1, a_2, b_2 *are collinear. Then the three intersection f-points* $(a_3,\alpha_3)=\langle(b_1,\beta_1),(c_2,\gamma_2)\rangle\cap\langle(b_2,\beta_2),(c_1,\gamma_1)\rangle$, $(b_3,\beta_3)=\langle(a_1,\alpha_1),(c_2,\gamma_2)\rangle\cap\langle(a_2,\alpha_2),(c_1,\gamma_1)\rangle$, $(c_3,\gamma_3)=\langle(a_1,\alpha_1),(b_2,\beta_2)\rangle\cap\langle(a_2,\alpha_2),(b_1,\beta_1)\rangle$ *are collinear*.

This theorem can be proved in the same way as Theorem 4.2.

4.4. Construction

One might wonder if fibered projective planes really exist. They do, and in fact, there is a kind of universal construction that goes as follows.

Let $\mathscr{P}=(P,B,I)$ be a projective plane. A *closed configuration* S is a subset of $P\cup B$ which is closed under taking intersection points of any pair of lines in S and joining lines of any pair of distinct points of S. Let $P'\subseteq P$ and $B'\subseteq B$ be such that the unique closed configuration containing $P'\cup B'$ is $P\cup B$. For each element x of $P'\cup B'$, we choose arbitrarily a nonempty subset Σ_x of $[0,1]$, and we define a fibered projective plane $\mathscr{FP}$ as follows. For each $x\in P'\cup B'$ and for each $\alpha\in\Sigma_x$, the element (x,α) belongs to $\mathscr{FP}$. This is step 1 of the construction. We now describe step i, $i>1$.

For any pair $((x,\alpha),(y,\beta))$ of f-points—with different base points x,y—that we already obtained, the f-line $(\langle x,y\rangle,\alpha\wedge\beta)$ spanned by it also belongs by definition to $\mathscr{FP}$. Dually, for any pair of f-lines—with different base lines—that we already obtained, the intersection f-point belongs to $\mathscr{FP}$.

The set of all f-points and of all f-lines constructed this way in a finite number of steps is readily verified to constitute a fibered projective plane.

We see that f-points can be considered as an ordinary projective plane (its *base plane* $\mathscr{P}$) where to every point and line, a set of values from $[0,1]$ are assigned.

Definition 4.2. The values assigned to the points and lines of $\mathscr{P}$ in step 1 to create the first f-points and f-lines are called the *initial values* of the respective base elements.

Sometimes we will refer to the values of a certain base point (line) that are different from its initial value, as *values that have been caught in the construction process*, or we say that the base point (line) has caught that value in the construction process. If there exists an f-point or f-line (a,x), $a\in P\cup B$, $a\in[0,1]$, with x not an initial value of a, then we say that *a has got* (*caught*) *the value x*.

It is clear that every fibered projective plane can be constructed as above. Indeed, one can always take for each element all its corresponding values as initial values.

5. Contagious values and fibers

5.1. Contagious values

As an example (and an application) of the potential of a theory of fibered geometries, we will show how apartments in generalized n-gons arise in a natural way. It is convenient to treat the case of projective planes separately.

We will look at a special class of fibered projective planes, namely those that can be constructed (referring to the construction above) using only, but all, points in step 1, together with one initial value for every point. We call these fibered projective planes *mono-point-generated*. Fibered planes like that must be looked at as a “dynamic” geometry: in each step, points and lines catch more values. One can consider the values as the gradation of a kind of disease; by identifying an ordinary projective plane with its point-line incidence graph, the construction process says that an element p catches the disease if at least two neighbors p', p'' already have got it, and the degree p gets is the lowest among the degrees of p' and p''. We stay in the medical sector with the following definition.

Recall that from now on we only consider mono-point-generated fibered projective planes.

Definition 5.1. An initial value x is *contagious* if in the course of the construction process as described in Section 4.4, all base points with a higher initial value $z \geqslant x$ catch the value x. The highest initial value that occurs (if it exists) is of course always contagious, therefore we call it *trivially contagious*.

Definition 5.2. An *f-triangle* in a fibered projective plane $\mathscr{FP}$ with base plane $\mathscr{P}$ consists of three f-points, called the *f-vertices*, and three f-lines, called the *f-edges*, of which the base points and base lines form a triangle, called the *base triangle* in $\mathscr{P}$, such that the f-edges intersect in the respective f-vertices, and such that the f-vertices span the respective f-edges. We denote such an f-triangle by the set of its f-vertices (because, clearly, an f-triangle is completely determined by its three f-vertices).

Lemma 5.1. *In an f-triangle, all elements have the same degree of membership.*

Proof. If not all degrees of membership are equal, then there is a point p and a line L of the base triangle with distinct degrees of membership α, β, respectively. Without loss of generality, we may assume that $\alpha < \beta$. But now (L, β) can never be the f-line spanned by (p, α) and the other f-point of the f-triangle whose base point is incident with L. □

The common degree of membership of all elements of an f-triangle will be called the *degree of membership of the f-triangle*.

Lemma 5.2. *If in the fibered projective plane $\mathscr{FP}$ there exists an f-triangle with degree of membership x, then x is contagious.*

Proof. Suppose in $\mathscr{FP}$ there exists an f-triangle $\mathscr{FT}$, with base triangle $\mathscr{T}$ and with degree of membership x. To be contagious, every base point with an initial value $z \geqslant x$, has to end up with the value x. We take such an arbitrary f-point (p, z) in $\mathscr{FP}$ with z the initial value of p. There are two possibilities: either p is incident with an edge of $\mathscr{T}$, or p is not.

In the first case p is incident with an edge E of $\mathscr{T}$ and the line L passing through p and the third vertex of $\mathscr{T}$, not on E. There exists an f-line with degree of membership x on each of these base lines: the f-line with base line L is spanned by an f-vertex with degree of membership x and the f-point (p, z) and thus has degree of membership $x \wedge z = x$. This means that p will catch the value $x \wedge x = x$.

If p is not incident with any of the three edges of $\mathscr{T}$, then it is collinear with each of the three vertices of $\mathscr{T}$. On each of these lines there exists an f-line with degree of membership x: spanned by (p,z) and the respective f-vertices of $\mathscr{FT}$, the degrees of membership will be $x \wedge z = x$. Again, p catches the value x. □

A triple $\{(p,x),(q,y),(r,z)\}$ of f-points will be called a *weak f-triangle* if the base points p,q,r are not collinear in $\mathscr{P}$.

Lemma 5.3. *There exists a weak f-triangle* $\{(p,x),(q,y),(r,z)\}$ *(*x,y,z *not necessary distinct) in the fibered projective plane* $\mathscr{FP}$ *with base plane* $\mathscr{P}$ *if and only if one of the following holds.*

(1) *There exists a weak f-triangle* $\{(p',x),(q',y),(r',z)\}$ *in* $\mathscr{FP}$ *such that the degrees of membership* x, y, z *are the initial values of the base points* p', q', r', *respectively*;

(2) *The values* x,y,z *are existing initial values, all points of* $\mathscr{P}$ *with these initial values are incident with a unique common line* L, *and there exists at least one base point with initial value* $w \geqslant \min\{x,y,z\}$ *that is not incident with* L.

Proof. $\Leftarrow$ If (1) is satisfied, then the assertion is trivial.

Suppose now all f-points with initial values x, y or z have collinear base points (spanning the unique line L of $\mathscr{P}$). Since L is unique, we can find f-points (p_1,x), (p_2,y) and (p_3,z), with $p_i I L$, for $i=1,2,3$, such that $\{p_1,p_2,p_3\}$ has size at least 2. Suppose that $x \leqslant y \leqslant z$. Suppose further that there exists a base point s with initial value $w \geqslant x$ such that s is not incident with L. Consider now the triangle $\{s,p_2,p_3\}$ in $\mathscr{P}$. Since $x \leqslant w$, we have the f-line $(\langle s,p_1\rangle,x)$. If $p_2 \neq p_1$, then it is clear that the f-lines $(\langle s,p_2\rangle, y \wedge w)$ and $(\langle s,p_1\rangle,x)$ intersect in the f-point (s,x). Similarly if $p_3 \neq p_1$. We thus find the weak f-triangle $\{(s,x),(p_2,y),(p_3,z)\}$.

$\Rightarrow$ Suppose there exists an f-triangle $\{(p,x),(q,y),(r,z)\}$ in $\mathscr{FP}$. Suppose all base points with initial values x, y, z and higher than $\min\{x,y,z\}$ are collinear (spanning the line L in $\mathscr{P}$; the line L is unique since otherwise $x=y=z$ is the highest degree of membership and it is given as initial value to only one point: clearly a contradiction). Thus the initial values of all f-points with base points not on L are strictly smaller than $\min\{x,y,z\}$. Suppose in step i, $i>1$ and i minimal, of the construction process, a point s not on L catches the value $w \geqslant \min\{x,y,z\}$. Then at least one line different from L must have caught the value w in a step j, with $j<i$. This implies that in a previous step, some point not on L, with value w or bigger must exist, contradicting the minimality of i. □

Theorem 5.4. *An initial value* x *in a fibered projective plane* $\mathscr{FP}$ *with base plane* $\mathscr{P}$ *is contagious if and only if one of the following holds.*

(1) *The value* x *is the highest value that occurs*;

(2) *For any point* p *having initial value* x, *there exist two f-points* (p_1,y) *and* (p_2,z), $y,z \geqslant x$, *such that* $\{(p,x),(p_1,y),(p_2,z)\}$ *is a weak f-triangle in* $\mathscr{FP}$; *hence there exists an f-triangle containing* (p,x).

Proof. Let p be a base point with initial value x.

$\Leftarrow$ If x is the highest occurring value, then x is trivially contagious. Suppose now that there exists a weak f-triangle $\{(p,x),(p_1,y),(p_2,z)\}$, with $y,z\geqslant x$. All f-vertices and all f-edges of this f-triangle will get the degree of membership x in the course of the construction process, yielding the f-triangle $\{(p,x),(p_1,x),(p_2,x)\}$. By Lemma 5.2 we know that x is contagious.

$\Rightarrow$ Suppose x is (not trivially) contagious and let z be an occurring value with $z\geqslant x$. First, suppose that z is the only value that is greater than x, occurring at only one base point p', and that p is the only base point with initial value x. Then the f-line $(\langle p,p'\rangle,x)$ is the only f-line with degree of membership x, and all other lines in $\mathscr{P}$ will only catch values strictly smaller than x. It is now clear that p' will never catch the value x, meaning x is not contagious, a contradiction.

Next suppose that besides the f-point (p',z) with $z\geqslant x$, there exists another f-point (q,y) with $y\geqslant x$, such that p, p' and q are collinear (spanning a line L), and suppose the initial values of all points not on L are strictly smaller than x. Similarly as in the previous paragraph, we see that neither p' nor q will catch the value x is $z>x$ or $y>x$, nor (if existing) any other point of L with value $z'>x$, a contradiction. If $x=y=z=z'$, then x is the highest occurring value. □

Corollary 5.5. *If a value x is contagious, but not trivially contagious, then every value $y<x$ is contagious.*

Proof. Suppose the base point p has the initial (not trivially) contagious value x. The preceding theorem then tells us that there exists a weak f-triangle $\{(p,x),(p_1,z_1),(p_2,z_2)\}$ such that $z_1,z_2\geqslant x$. Suppose the point p' has initial value $y<x$. It is clear that at least one of the sets $\{p',p,p_1\}$, $\{p',p,p_2\}$, $\{p',p_1,p_2\}$ contains three noncollinear points. The assertion now follows from Theorem 5.4(2). □

5.2. Fibers

In this section, we discuss the fibers of a fibered projective plane. They are closely related to closed configurations, which we have defined above.

Let $\mathscr{P}=(P,B,I)$ be a projective plane and $P'\subseteq P$, $B'\subseteq B$. The intersection of all closed configurations containing $P'\cup B'$ will be called the *closure of* $X:=P'\cup B'$, and is denoted by $\langle X\rangle$. This definition extends in an obvious way to the class of linear spaces.

The closed configuration $\langle X\rangle$ can be constructed as follows (see [1]). Put $X_0=X$. For every odd natural number i, let X_i be the union of X_{i-1} and the set of all lines spanned by any pair of two points of X_{i-1}. Let X_{i+1} be the union of X_i and the set of all intersection points of the lines in X_i. Then the union of all X_i, $i\geqslant 0$, is the closure of X.

Definition 5.3. Consider a fibered projective plane $\mathscr{FP}$ with base plane $\mathscr{P}$, and let x be any initial value. We denote by P_x the set of all base points with initial value x, and by $P_{\geqslant x}$ (respectively $P_{>x}$) the set of base points with initial value at least x_i (respectively strictly greater than x_i).

Lemma 5.6. *A value x is contagious if and only if all points and lines in $\langle P_{\geqslant x}\rangle$ get the value x.*

Proof. $\Rightarrow$ Since x is contagious, all points in $P_{\geqslant x}$ catch the value x in the construction process. From the explicit construction of $\langle P_{\geqslant x}\rangle$ above it follows directly that every element of $\langle P_{\geqslant x}\rangle$ also catches the value x.

$\Leftarrow$ Trivial since $P_{\geqslant x} \subseteq \langle P_{\geqslant x}\rangle$. □

Definition 5.4. Let $\mathscr{FP}$ be a fibered projective plane with base plane $\mathscr{P}$ and consider all f-points and f-lines with the value x. The set of all the base points and base lines of these f-points and f-lines, is called the *fiber of value* x. We denote this fiber by F_x. Clearly, every fiber is a closed configuration.

Restating the previous lemma, we obtain:

Lemma 5.7. *The value x is contagious if and only if $\langle P_{\geqslant x}\rangle \subseteq F_x$.*

In particular, if x is not contagious, then $\langle P_{\geqslant x}\rangle \neq F_x$. We now make the previous lemma more precise.

Lemma 5.8. *The value x is contagious if and only if $\langle P_{\geqslant x}\rangle = F_x$.*

Proof. We only have to show that, if x is contagious, then $\langle P_{\geqslant x}\rangle = F_x$. This is of course true if x is trivially contagious. So suppose that x is contagious, but not trivially contagious. By Lemma 5.7, $\langle P_{\geqslant x}\rangle \subseteq F_x$. Since there exists a triangle in $P_{\geqslant x}$, the construction process now guarantees that every f-point and every f-line having the degree of membership x as a noninitial value, is constructed as the intersection point of two f-lines with degree of membership x, or as the f-line through two f-points with the degree of membership x, thus $\langle P_{\geqslant x}\rangle = F_x$. □

Theorem 5.9. *Let $J \subseteq [0,1]$ be the set of initial values of a fibered projective plane $\mathscr{FP}$ with base plane $\mathscr{P}$. Let $J = J_1 \cup J_2$, with J_1 the set of nontrivially contagious values, and $J_2 = J \backslash J_1$.*

(1) *If x and y are two distinct elements of J_2, then the fibers F_x and F_y are not contained in one another, but they are both contained in every fiber F_z with $z \in J_1$.*

(2) *The set J_1 is nonempty. If $x > y$ and both belong to J_1, then $F_x \subseteq F_y$.*

Proof. (1) Theorem 5.4 implies that all the base points with an initial value in J_2 are incident with a line L, which we may assume to be unique (otherwise $|J_2| \in \{0,1\}$ and (1) is trivial). Let $x, y \in J_2$. As in the proof of Theorem 5.4, we see that the base point(s) with the initial value x will never catch the value y, and vice versa. If one of these values, e.g. x, is the highest value (if existing), then we know that $F_x = \langle P_x\rangle$, since x is (trivially) contagious (see Lemma 5.8), while F_y will consist of P_y and L. Hence the fibers F_x and F_y are not contained in one another. If $z \in J_1$, then $(F_x \cup F_y) \subseteq F_z$ by the definition of contagious values.

(2) Let $\{p_1, p_2, p_3\}$ be a triangle in the base plane $\mathscr{P}$, and let x_i be the initial value of p_i, $i=1,2,3$. Then by Theorem 5.4, the smallest among x_1, x_2, x_3 belongs to J_1, hence $J_1 \neq \emptyset$. Now let $x, y \in J_1$, with $x > y$. Clearly, we have $P_{\geqslant x} \subseteq P_{\geqslant y}$. Hence $\langle P_{\geqslant x}\rangle \subseteq \langle P_{\geqslant y}\rangle$. The assertion now follows from Lemma 5.8. □

6. Fibered generalized polygons

6.1. Definition

From now on, we suppose that $\mathscr{P}=(P,B,I)$ is a thick generalized n-gon (see Definition 2.4), $n \geqslant 3$.

There are essentially two ways to generalize the definition of a fibered projective plane (a generalized 3-gon) to a fibered generalized n-gon.

First, one could require that the intersection f-point of two f-lines (of which the base lines intersect in the base generalized n-gon) belongs to the fibered generalized n-gon, and dually. However, this is not the most natural way, since it only uses the fact that generalized n-gons are partial linear spaces. Moreover, it turns out that the theory of fibered projective planes as given above, cannot be extended in full generality to fibered generalized n-gons. But certainly, interesting combinatorial questions would arise. For instance, when is a value contagious in that case? The answer heavily depends on the particular structure of the generalized polygon in question. In general, one can only formulate some sufficient conditions (for generalized quadrangles, for example, an initial value x is contagious if the set P_x, defined as above, is the set of points of an ovoid, or the set of points collinear with a given point, or the set of points incident with one of the four lines of an ordinary quadrangle).

Secondly, we could view the intersection point p of two lines L_1 and L_2 in a projective plane as the unique element incident with L_1 and at minimal distance (measured in the incidence graph) from L_2, and we denote $p=\mathrm{proj}_{L_1} L_2$ (the *projection of* L_2 *onto* L_1). In this way, two elements a, b of the generalized n-gon $\mathscr{P}$ which are not at distance n define two unique other elements $\mathrm{proj}_a b$ and $\mathrm{proj}_b a$. But we can go on and consider $\mathrm{proj}_{\mathrm{proj}_a b} b$, etc. This suggests the following definition (calling two elements at distance n *opposite*).

Definition 6.1. Let $\mathscr{P}=(P,B,I)$ be a (thick) generalized n-gon. A *fibered generalized n-gon* $\mathscr{FP}$ consists of a set $\mathscr{FP}$ of f-points and a set $\mathscr{FB}$ of f-lines such that every point and every line of $\mathscr{P}$ is the base point and base line of some f-point and f-line, respectively, and such that $(\mathscr{FP}, \mathscr{FB})$ satisfies the following axiom:

(FGP) Let $(a,\alpha), (b,\beta)$ be any pair of f-elements of $\mathscr{FP}$ with a and b not opposite. Then $(\mathrm{proj}_a b, \alpha \wedge \beta)$ belongs to $\mathscr{FP} \cup \mathscr{FB}$.

It is now clear why we consider generalized polygons in this paper: because a generalized polygon provides a maximum number of pairs (a,b) of elements for which $\mathrm{proj}_a b$ is well-defined.

Let us remark that two points (or two lines) of a point-line geometry (and thus of any generalized n-gon) are always at even distance, and that a point and a line are always at odd distance.

Let $\mathscr{FP}$ be a fibered generalized n-gon with base generalized n-gon $\mathscr{P}$. Let (a,α) and (b,β) be two f-elements of $\mathscr{FP}$, with a not opposite b in $\mathscr{P}$. Then there is a unique path $a=a_0,a_1,a_2,\ldots,a_i=b$, with $a_{j-1}Ia_j$, $j=1,2,\ldots,i$, and with i equal to the distance from a to b. From axiom (FGP) we readily deduce that $(a_j,\alpha\wedge\beta)$ is an element of $\mathscr{FP}$ for all j, $0<j<i$.

6.2. Construction

A universal construction for fibered generalized n-gons is now completely similar to the case $n=3$ treated before. One considers a subset P' of the set of points and a subset B' of the set of lines such that the only closed configuration containing $P'\cup B'$ is $\mathscr{P}$ itself (here a *closed configuration* is a set S of points and lines closed under the projection, i.e., if $a,b\in S$ and a is not opposite b, then $\mathrm{proj}_a b\in S$). Then we consider for each element a of $P'\cup B'$ a nonempty set of f-elements (f-points or f-lines, respectively) with base element a. The set of all such f-elements is denoted by X_1. This is step 1 of the construction. For $i>1$, we may describe step i as follows. For any pair (a,b) of nonopposite base elements of f-elements $(a,\alpha),(b,\beta)$ of X_{i-1}, we add $(c,\alpha\wedge\beta)$ to the set X_i' whenever c lies on the unique shortest path connecting a and b (but we require $a\neq c\neq b$). Then we put $X_i=X_i'\cup X_{i-1}$. The union of X_i, i ranging over the set of natural numbers, forms a fibered generalized n-gon. As for $n=3$, this construction gives us all possible examples.

For any subset X of $P\cup B$, we denote by $\langle X\rangle$ the *closure of* X, i.e. the intersection of all closed configurations containing X. We also say that X *spans* $\langle X\rangle$.

6.3. Contagious values

When dealing with contagious values, we will again assume that the fibered generalized n-gon we consider is mono-point-generated, i.e., in step 1 of the construction above, we put X_1 equal to P, and we initially assign exactly one degree of membership to every point. These values are called the *initial values* of the corresponding points. A value x is *contagious* if every point p with initial value $x_p>x$ catches the value x in the course of the construction (hence if (p,x) belongs to $\mathscr{FP}$).

An *apartment* in $\mathscr{P}$ is an ordinary n-gon in $\mathscr{P}$ (in the incidence graph of $\mathscr{P}$, this is a cycle of length $2n$). An f-apartment $\mathscr{F}\Sigma$ in $\mathscr{FP}$ consists of n f-points and n f-lines whose base elements form an apartment Σ in $\mathscr{P}$, and such that axiom (FGP) holds in $\mathscr{F}\Sigma$. As for the case $n=3$, it is again easy to see that all elements of $\mathscr{F}\Sigma$ have a constant degree of membership. This degree is called the *degree of membership of* $\mathscr{F}\Sigma$.

Theorem 6.1. *Suppose we have a fibered generalized n-gon $\mathscr{FP}$ with base generalized n-gon $\mathscr{P}$. If there exists an f-apartment $\mathscr{F}\Sigma$ with degree of membership x, then the value x is contagious.*

Proof. If there exists no value $z > x$, then x is trivially contagious. So suppose there exists an f-point (p, z), with p not a point of Σ, and with $z > x$.

Let a be an element of Σ opposite p (this exists, see e.g. [8], Lemma 1.5.9). Let b, c be the two neighbors of a in Σ. Then $(\text{proj}_p b, x)$ and $(\text{proj}_p c, x)$ belong to $\mathscr{FP}$, they have different base elements, and hence (p, x) also belongs to $\mathscr{FP}$ since p is the projection of $\text{proj}_p b$ onto $\text{proj}_p c$. □

This theorem is the generalized n-gon version of Lemma 5.2 in Section 5. Like in the case of fibered projective planes, the condition in the previous theorem is not necessary. It is sufficient for x to be contagious that there exist some elements that 'generate' an f-apartment. First we state the following lemma, the proof of which is left as an easy exercise.

Lemma 6.2. *Suppose we have a fibered generalized n-gon $\mathscr{FP}$ with base generalized n-gon $\mathscr{P}$. If there exist three f-elements (a_1, x), (a_2, y) and (a_3, z), such that $y, z \geqslant x$ and a_1, a_2 and a_3 span an apartment Σ of $\mathscr{P}$, then every element of Σ will catch the value x.*

We can now characterize contagious values, giving at the same time a characterization of apartments in the fibered context.

Theorem 6.3. *Suppose we have a fibered generalized n-gon $\mathscr{FP}$ with base generalized n-gon $\mathscr{P}$. A value x is contagious if and only if one of the following holds.*

(1) *The value x is the highest initial value.*

(2) *There exists an element a of $\mathscr{P}$ such that all elements of P_x are at distance at most $n/2$ from a, and such that $P_{>x}$ is contained in the "tree" consisting of all shortest paths from a to the elements of P_x.*

(3) *For every point p with initial value x, there exist elements (a, y) and (b, z) of $\mathscr{FP}$ with $y, z \geqslant x$ and such that $\langle\{p, a\}\rangle \cup \langle\{a, b\}\rangle \cup \langle\{p, b\}\rangle$ is an apartment Σ of $\mathscr{P}$. Hence for every such point p, there exists an f-apartment containing (p, x).*

Proof. ⇐ If x is the highest occurring value, then x is trivially contagious. If (2) holds, then clearly x is contagious. If (3) is satisfied, then Lemma 6.2 and Theorem 6.1 show that x is contagious.

⇒ If x is trivially contagious then x is the highest initial value, so (1) is satisfied. Suppose now that x is (not trivially) contagious so that there exists at least one f-point (q, z) with a degree of membership $z > x$. Suppose further that (2) is not satisfied.

Suppose first that $\langle P_{>x}\rangle$ contains an apartment Σ. Then we can take for a and b the two neighbors of an element of Σ opposite to p.

Now suppose that $\langle P_{>x}\rangle$ does not contain an apartment. We claim that there exist two elements q_1, q_2 of $P_{>x}$ at distance $r \geqslant n-3$ from each other.

Therefore, let m be the diameter of $P_{>x}$ and suppose in order to find a contradiction that $m \leqslant n-4$. Let p_1, p_2 be two points of $P_{>x}$ at distance m. Let c be the unique element of $\mathscr{P}$ at distance $m/2$ from both p_1 and p_2. If an element p' of $P_{>x}$ were at distance $m' > m/2$ from c, then it would be at distance $m' + m/2$ from p_1 or p_2 (see e.g. Lemma 1.5.6 of [8]), contradicting the fact that m is the diameter of $P_{>x}$. Hence all elements of $P_{>x}$ are at distance $\leqslant (n-4)/2$ from c, a contradiction. Our claim is proved.

Since x is contagious, q_1 must catch the value x. Hence two distinct neighbors of q_1 must also have degree of membership x. Similarly for q_2. But this means that we have found a path of length $r+4 \geqslant n+1$ of which all elements have some degree of membership $\geqslant x$. It is now easily seen that we can find an apartment Σ having $n+1$ elements in common with this path. All elements of Σ catch the value x, and hence we can again take for a and b the two neighbors of an element of Σ opposite p. □

By the appearance of (2) in the previous theorem, we are not anymore able to show that, if a value x is nontrivially contagious, then every value $y < x$ is contagious. There are indeed counterexamples. For example, consider a generalized n-gon with $n > 8$. Let p_1, p_2, p_3, p_4 be four consecutive collinear points, with p_1 not collinear with p_3, and with p_2 not collinear with p_4. Suppose all points of the generalized n-gon are given the same initial value x_0, except for p_1 and p_4 (that we both give the value x_1), p_2 (which we give the initial value x_2) and p_3 (that we give x_3). Suppose that $x_0 < x_2 < x_1 < x_3$. Then x_1 is contagious, because p_3 catches x_1. But x_2 is not contagious, because p_1 can never catch the value x_2.

If we would call a value x *globally contagious* if (3) of the previous theorem is satisfied, then one can show that there are values which are globally contagious (by considering any f-apartment), and that every value strictly smaller than a globally contagious one is also globally contagious. We leave the (easy) proof to the reader. Also, it is clear that the fibers of the globally contagious values are generalized sub-n-gons contained in one another. This puts a restriction on the number of (essentially distinct) globally contagious values, especially in the finite case.

7. Some remarks

We have not covered every possible aspect of fibered geometries. Instead, we have tried to give the flavor of what can be done, and which kind of questions arise when dealing with these things. We get different problems when changing the initial conditions, e.g., when considering different possibilities of step 1 of the construction of fibered generalized polygons.

For more properties, examples and applications of fibered geometries, and notably of fibered generalized n-gons, we refer to the Ph.D. Thesis of the first author. It turns out that fibered buildings are a main step towards a general definition of fuzzy buildings, which, on their turn, give rise to interesting geometric and combinatorial problems.

Also, the definition of a fibered generalized polygon is ready-made for generalization to other geometries, also of higher rank, notably building geometries. This is interesting, because our main motivation for introducing fibered geometries is to find new applications of finite geometry. Via the incidence or collinearity graph, we obtain fibered or fuzzy graphs, and these have a wide range of practical applications. This will be treated elsewhere.

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