General Topology and its Applications 1 (1970) 85–100. North-Holland Publishing Company

OPEN CONTINUOUS IMAGES OF CERTAIN KINDS OF *M*-SPACES AND COMPLETENESS OF MAPPINGS AND SPACES*

Howard H. WICKE
Sandia Laboratories, Albuquerque, New Mexico, USA

Received 16 July 1970

AMS Subj. Class.: Primary 54C10, 54E20, 54D99; Secondary 54E50

1. Introduction

This paper contributes to the problem of characterizing images of certain kinds of topological spaces under open continuous mappings or open continuous mappings which satisfy a completeness condition. The spaces under consideration belong to one of the following classes of *M*-spaces in the sense of Morita [11]: (1) Regular T_0 *M*-spaces. (2) T_2 paracompact *M*-spaces. (3) Regular T_0 complete *M*-spaces (cf. Definition 3.2). (4) Paracompact Čech complete spaces.[1] This article presents conditions characterizing those regular T_0-spaces which are images of spaces in the above classes under open mappings (in cases (3) and (4)) and open uniformly λ-complete mappings (in cases (1) and (2)) although only the proof that such mappings exist is given. The proof that the conditions are invariant under the appropriate mappings will be submitted elsewhere.

The regular T_0-spaces satisfying the conditions mentioned have a number of nice properties some of which follow directly from the main theorems proved here. These include invariance under certain open mappings (Theorems 4.3 and 4.4), invariance under perfect mappings (Theorem 4.7), and countable productivity. Characterizations of subspaces

* This work was supported by the United States Atomic Energy Commission.

[1] Such a space is an *M*-space since it is a paracompact *p*-space and any space of the latter kind is a paracompact *M*-space [12].

satisfying the same conditions will appear in another article. The conditions are hereditary with respect to closed subsets.

It may be instructive to compare the theorems proved here with certain other mapping theorems stemming from a common technique (cf. section 5) and with which they have strong analogies. In a first countable setting the following theorems have been proved previously:

Theorem 1.1 [7,14]. Every first countable T_0-space of infinite cardinality is an open continuous image of a metrizable space of the same weight.

Theorem 1.2 [17]. Every T_1-space having a base of countable order is an open continuous uniformly monotonically complete image of a metrizable space of the same weight.

Theorem 1.3 [15]. Every regular T_0-space having a monotonically complete base of countable order is an open continuous image of a complete metric space of the same weight.

In a non-first-countable setting analogues of Theorem 1.1 are:

Theorem 1.4 [16]. Every T_2-space of point-countable type [1] is an open continuous image of a T_2 paracompact p-space ($\equiv T_2$ paracompact M-space).

Theorem 1.5 [13]. Every regular T_0-space which is a q-space in the sense of Michael [10] is an open continuous image of an M-space.[2]

An analogue of Theorem 1.3 is

Theorem 1.6 [19]. A regular T_0-space which satisfies λ_b (cf. Definition 2.2) is an open continuous image of a paracompact Čech complete space of the same weight.

Theorem 4.1, part 1, is another analogue of 1.3 in which "complete metric space" is replaced by "complete M-space" while Theorem 4.2 provides two analogues of Theorem 1.2 in which "metric space" is re-

[2] Theorem 5.1 shows that the M-space may be taken to be of the same weight as the q-space.

placed respectively by "regular T_0 M-space" and "T_2 paracompact M-space".

The main theorems of the paper are stated in section 4. Theorems 4.3, 4.4 and 4.7 state (proofs will appear elsewhere) some fundamental invariance properties of the basic conditions defined in section 2. Theorems 4.5 and 4.6 follow from the Theorems 4.1, 4.3 and 4.2, 4.4, respectively, and give characterizations of open continuous images or open continuous uniformly λ-complete images of spaces belonging to the appropriate classes (1)–(4) above. The major technical part of the paper is devoted to the proofs of Theorems 4.1 and 4.2. These proofs are very closely related as may be seen from section 6. They are founded on the rather simple Lemma 5.3 which concerns a basic construction allowing realization of the open mappings in question as projections. Lemma 5.3 provides a common basis for Theorems 1.1–1.6 as well as Theorems 4.1 and 4.2. Individual differences in proofs arise in showing that the hypothesis of the lemma is satisfied and in demonstrating completeness of either spaces or mappings.

The completeness of spaces and mappings mentioned in the title refers to the completeness aspect of the definitions of λ_c, λ_b, and uniformly λ-complete mappings given below and to the involvement of completeness in the statements and proofs of the main theorems.

The techniques and concepts involved here are founded on base of countable order theory which Dr. J.M. Worrell Jr. and the author have elaborated in several publications [15, 17, 20, 22] and which they have long regarded as being fundamental for a theory (appropriate to the present stage of general topology) of non-first-countable structure [18, 23].

Notation, terminology and preliminaries. The notation and terminology used here conforms rather closely with that used by Kelley [9]. Certain conventions used below are noted here. The letter N denotes the set of positive integers and i, j, k and n are always used to denote members of N. The notation $(U_n)_{n \in N}$ denotes a sequence; i.e., a function U with domain N, and will be frequently abbreviated as (U_n). Occasionally a single letter such as α will also be used to denote a sequence. The term *decreasing sequence* refers to a sequence (U_n) such that $U_{n+1} \subset U_n$ for all $n \in N$. Quite often in the proofs if α is a sequence, $\cap\, \alpha_n$ is used to denote $\cap\, \{\alpha_n : n \in N\}$. A *quasi-perfect* [12] (*perfect*) mapping is a closed continuous mapping such that the inverse image of every point is countably compact (compact). The following three theorems are basic for the classes of M-spaces considered.

Theorem 1.7 [11]. A topological space is an M-space if and only if there exists a quasi-perfect mapping of it onto a metrizable space.

Theorem 1.8 [2]. A T_2-space is a paracompact p-space ($\equiv$ paracompact M-space) if and only if there exists a perfect mapping of it onto a metrizable space.

Theorem 1.9 [6]. A T_2-space is a paracompact Čech complete space if and only if there exists a perfect mapping of it onto a complete metric space.

Definition 1.1. Suppose X is a space and $A \subset X$. A collection $\mathfrak{N}$ of subsets of X is said to be a *network at* A if and only if $A \subset U$ and U open implies that $A \subset M \subset U$ for some $M \in \mathfrak{N}$. If each member of $\mathfrak{N}$ is open, $\mathfrak{N}$ is called a *base at* A. The set A is said to have *countable character* [1] if and only if there is a countable base at A. The set A is said to have *strongly countable character* if and only if there is a countable base $\{D_n : n \in N\}$ at A such that $\overline{D_{n+1}} \subset D_n$ for each n. If a collection $\mathfrak{N}$ is a network at every $\{x\}$ for all $x \in X$ then $\mathfrak{N}$ will be called a *network for* X [3].

2. The conditions $\lambda_c, \lambda_b, \beta_c, \beta_b$

The conditions fundamental for the paper are defined here. They all involve a sequence condition and can be considered as variations on a common theme. They have similarities with sequence conditions previously used in both first-countable cases, e.g., Moore spaces, complete Moore spaces; and in non-first-countable cases, e.g., Frolík's complete coverings [5], but as is pointed out in Remark 2.1 they can be considered as arising directly from certain kinds of bases of countable order.

Definition 2.1 [19,20]. A sequence $(\mathcal{C}_n)$ of collections of sets is said to be *monotonically contracting* if and only if, for each $n \in N$, if $x \in A \in \mathcal{C}_n$ there exists $B \in \mathcal{C}_{n+1}$ such that $x \in B \subset A$.

Definition 2.2. A topological space X is said to satisfy condition $\lambda_c(\lambda_b)$ if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ such that: (1) Each $\mathcal{G}_n$ is a collection of open subsets of X covering X, (2) If (G_n) is a decreasing sequence such that each $G_n \in \mathcal{G}_n$ and is nonempty, then $B = \cap \{ \overline{G_n} : n \in N \}$ is nonempty and countably compact (compact) and U open and $B \subset U$ implies that some $G_k \subset U$.

Definition 2.3. A topological space X is said to satisfy condition $\beta_c(\beta_b)$ if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ such that: (1) Each $\mathcal{G}_n$ is a collection of open subsets of X covering X. (2) If (G_n) is a decreasing sequence such that each $G_n \in \mathcal{G}_n$, then if $B = \cap \{ \overline{G_n} : n \in N \}$ is nonempty, it is countably compact (compact) and U open and $B \subset U$ implies that some $G_k \subset U$.

The concepts of λ-base and base closurewise of countable order were introduced and studied in [17,20]. Using methods employed in [15,17, 20], the following propositions may be proved.

Proposition 2.1. A T_2-space has a λ-base if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ such that: (1) Each $\mathcal{G}_n$ is a collection of open subsets of X covering X. (2) If (G_n) is a decreasing sequence such that each $G_n \in \mathcal{G}_n$ and is nonempty, then $B = \cap \{ \overline{G_n} : n \in N \}$ is nonempty and $B = \{ x \}$ for some $x \in X$; and U open and $x \in U$ implies that some $G_k \subset U$.

Proposition 2.2. A T_2-space X has a base closurewise of countable order if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ such that: (1) Each $\mathcal{G}_n$ is a collection of open subsets of X covering X. (2) If (G_n) is a decreasing sequence such that each $G_n \in \mathcal{G}_n$ and is nonempty, then if $B = \cap \{ \overline{G_n} : n \in N \}$ is nonempty, $B = \{ x \}$ for some $x \in X$ and U open and $x \in U$ implies that some $G_k \subset U$.

Remark 2.1. Thus it may be seen that conditions λ_c and λ_b are obtained from the condition of Proposition 2.1 by replacing the point x by a countably compact (respectively, compact) subset of X. A similar remark holds for conditions β_c and β_b and Proposition 2.2. The terminology used has its source in this. The letter λ is to suggest λ-base, and β is to suggest *base closurewise of countable order.* The letter c is to suggest *countably compact* and b is to suggest *compact* (≡ bicompact).

Remark 2.2. The conditions λ_b and β_b, in which compactness enters, have, in the context of complete regularity, extrinsic formulations involving Stone-Čech compactification. One of these extrinsic conditions has been studied in [19]. Another such condition giving rise to the class of μ-spaces has been reported on in [18]. For comparison, a definition is given here.

Definition 2.4. A T_1 topological space is called a *μ-space* if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ of collections of open subsets of ωX(the Wallman compactification of X [9]) covering X such that if (G_n) is a decreasing sequence such that, for each n, $G_n \in \mathcal{G}_n$ then $\cap \{G_n : n \in N\}$ is a subset of X if it contains a point of X.

In the completely regular case, where βX can be used, a space is a μ-space if and only if it satisfies condition β_b [21].

Remark 2.3. λ_b implies λ_c and β_b implies β_c.

Examples 2.1. Any Čech complete space satisfies λ_b.
2.2. Any Hausdorff paracompact M-space satisfies β_b.
2.3. Any complete M-space (cf. Definition 3.2) satisfies λ_c.
2.4. Any M-space satisfies β_c.

The following lemmas give equivalent forms of the conditions for the case of regular T_0-spaces. These forms will be used in the proofs of the main theorems. The lemmas have analogies with Theorem 1 of [15] where it is shown that a regular space has a monotonically complete base of countable order if and only if it satisfies a condition given by Aronszajn [4] which is a first countable analogue of the condition of Lemma 2.1.

Lemma 2.1. Suppose X is a regular T_0-space. Then in X the condition $\lambda_c(\lambda_b)$ is equivalent to: There exists a sequence $(\mathcal{G}_n)$ of collections of open sets covering X such that: (1) For each $n \in N$, if $x \in G \in \mathcal{G}_n$ there exists $G' \in \mathcal{G}_{n+1}$ such that $x \in G'$ and $\overline{G'} \subset G$. (2) If (G_n) is a sequence such that $\emptyset \neq \overline{G_{n+1}} \subset G_n \in \mathcal{G}_n$ for all $n \in N$, then $B = \cap \{G_n : n \in N\}$ is nonempty and countably compact (compact) and $\{G_n : n \in N\}$ is a base at B.[3]

Proof. Suppose X satisfies $\lambda_c(\lambda_b)$ and $(\mathcal{G}'_n)$ satisfies the corresponding condition of Definition 2.2. Then it may be seen (proof given in [20]) that there exists a sequence $(\mathcal{H}_n)$ of well-ordered collections covering /

[3] The condition of Lemma 2.1 involving compactness is identical with Condition $\mathcal{K}$ of [19].

X such that for each $n \in N$: (H1) $\mathcal{H}_n \subset \mathcal{G}'_n$. (H2) Each $H \in \mathcal{H}_n$ contains a point not in any predecessor of H in $\mathcal{H}_n$. (H3) If $n, k \in N$, $n < k$, and $x \in X$, then the first element of $\mathcal{H}_k$ that contains x is a subset of the first element of $\mathcal{H}_n$ doing so. Define $\mathcal{G}_1$ as $\mathcal{H}_1$. If $\mathcal{G}_1, \ldots, \mathcal{G}_n$ have been defined let $\mathcal{G}_{n+1}$ denote the collection of all sets of the form $U \cap G$ where $G \in \mathcal{H}_{n+1}$, U is open, U contains a point of G not in any predecessor of G in $\mathcal{H}_{n+1}$, and for some $G' \in \mathcal{G}_n$, $\overline{U} \subset G'$. Then $(\mathcal{G}_n)$ is a sequence of collections of open sets covering X satisfying (1) above. For $\mathcal{G}_1$ covers X. Suppose $\mathcal{G}_1, \ldots, \mathcal{G}_k$ cover X and satisfy (1) for $1 \leq n \leq k-1$. If $x \in G \in \mathcal{G}_k$ let H be the first element of $\mathcal{H}_{k+1}$ that contains x. Since X is regular, there exists an open U containing x such that $\overline{U} \subset G$. Therefore, $U \cap H \in \mathcal{G}_{k+1}$. Suppose (G_n) is a sequence such that $\emptyset \neq \overline{G_{n+1}} \subset G_n \in \mathcal{G}_n$ for all $n \in N$. Then for each n there exists a first $H_n \in \mathcal{H}_n$ that includes a term of (G_n). For each n there exists $j > n+1$ such that $G_j \subset H_n \cap H_{n+1}$. The set $G_j = U \cap H$ where $H \in \mathcal{H}_j$ and U contains $y \in H$ not in any predecessor of H. By (H3), H is a subset of the first element $H' \in \mathcal{H}_n$ that contains y. Therefore, H' does not precede H_n. Since $y \in H_n$, $H' = H_n$. Similarly H_{n+1} is the first element of $\mathcal{H}_{n+1}$ that contains y. Therefore, for each n, $H_{n+1} \subset H_n$. Since $H_n \in \mathcal{G}'_n$ it follows that $B' = \cap \{\overline{H_n} : n \in N\}$ is nonempty countably compact (compact) and any open $U \supset B'$ includes some H_n. By Lemma 5.1, $B = \cap \{G_n : n \in N\}$ is nonempty and countably compact (compact) and $\{G_n : n \in N\}$ is a base at B.

Suppose on the other hand the condition of the lemma is satisfied by a sequence $(\mathcal{G}'_n)$. It may be seen (proof given in [20]) that there exists a sequence $(\mathcal{H}_n)$ of well-ordered collections of open sets covering X such that for each $n \in N$ conditions (H1) and (H2) are satisfied as well as the following modification of (H3): If $n, k \in N$, $n < k$, and $x \in X$ then the closure of the first element of $\mathcal{H}_k$ that contains x is a subset of the first element of $\mathcal{H}_n$ doing so.

Let $\mathcal{H}'_1$ denote $\mathcal{H}_1$ and for $n > 1$ if $\mathcal{H}_{n-1}$ is defined let $\mathcal{H}'_n$ denote the collection of all sets $H \cap H'$ where $H \in \mathcal{H}_n$, $H' \in \mathcal{H}'_{n-1}$ and H' contains a point of H not in any predecessor of H in $\mathcal{H}_n$. For each n let $\mathcal{G}_n$ denote $\cup \{\mathcal{H}'_k : k \geq n\}$. Then $(\mathcal{G}_n)$ may be seen to be a monotonically contracting sequence of collections of open sets covering X. If (G_n) is a decreasing sequence such that each $G_n \in \mathcal{G}_n$ then, for each n, there exists $k_n \geq n$ such that $G_n \in \mathcal{H}'_{k_n}$. It follows that for each n there exists a first $H_n \in \mathcal{H}_n$ that includes a term of (G_n), and also, by an argument used in the first paragraph of this proof, that $\overline{H_{n+1}} \subset H_n$. Then $\cap \{H_n : n \in N\}$ is nonempty, countably compact (compact) and

$\{H_n : n \in N\}$ is a base at $\cap H_n$. Since for each n, there exists j such that $G_j \subset H_n$, it follows from Lemma 5.1 that $\cap \overline{G_n}$ is nonempty closed and countably compact (compact) and every open U which includes $\cap \overline{G_n}$ also includes some G_n.

Lemma 2.2. Suppose X is a regular T_0-space. Then in X the condition $\beta_c(\beta_b)$ is equivalent to: There exists a sequence $(\mathcal{G}_n)$ of collections of open sets covering X such that: (1) For each $n \in N$, if $x \in G \in \mathcal{G}_n$ there exists $G' \in \mathcal{G}_{n+1}$ such that $x \in G'$ and $\overline{G'} \subset G$. (2) If (G_n) is a sequence such that $\overline{G_{n+1}} \subset G_n \in \mathcal{G}_n$ for all $n \in N$, then $B = \cap\{G_n : n \in N\}$ is countably compact (compact) and if B is nonempty, $\{G_n : n \in N\}$ is a base at B.

Proof. The constructions given in the proof of Lemma 2.1 will provide a proof for this lemma when obvious modifications are made concerning the nonemptiness of the sets B and $\cap \overline{G}_n$.

3. Uniformly λ-complete mappings and complete M-spaces

The following definition presents an analogue of the concept of uniformly monotonically complete mapping introduced in [17]. The uniformly λ-complete mappings have properties in a non-first-countable context analogous to those possessed by uniformly monotonically complete mappings in a first countable one. An extrinsic formulation appropriate to the case of μ-spaces has also been given.

Definition 3.1. A mapping f of a space X into a space Y is said to be *uniformly λ-complete* if and only if there exists a monotonically contracting sequence $(\mathcal{G}_n)$ in X of collections of open sets such that for every $y \in Y$ if (G_n) and (W_n) are decreasing sequences of open sets such that for each n, there exists j such that $W_j \subset G_n \in \mathcal{G}_n$ and $W_n \cap f^{-1}(y) \neq \emptyset$, then $\cap \{\overline{W_n} : n \in N\} \cap \overline{f^{-1}(y)} \neq \emptyset$.

Examples 3.1. Any continuous mapping $f: X \to Y$ such that $f^{-1}(y)$ is countably compact is uniformly λ-complete.

3.2. Uniformly monotonically complete mappings of essentially T_1-spaces are uniformly λ-complete.

3.3. Any mapping on a T_1-space satisfying λ_c is uniformly λ-complete. (This motivates the λ-terminology in Definition 3.1.)

The following concept involves a simple modification of Morita's characterization of M-spaces (Theorem 1.7).

Definition 3.2. A space will be called a *complete M-space* if and only if there exists a quasi-perfect mapping of it onto a complete metric space.

Proposition 3.1. A space is a complete M-space if and only if there exists a normal sequence $(\mathcal{U}_n)$ for the space such that if (K_n) is a decreasing sequence of nonempty closed sets for which there exists a sequence (U_n) such that for each $n \in N$, $U_n \in \mathcal{U}_n$ and there exists j such that $K_j \subset U_n$, then $\cap \{K_j : j \in N\} \neq \emptyset$.

4. The main theorems

Theorem 4.1. Suppose X is a regular T_0-space satisfying one of the conditions λ_c or λ_b. Then X is an open continuous image of a space Y having the same weight as X which is a subspace of a product space $\Gamma \times X$ where Γ is a zero-dimensional complete metric space (Γ is a closed subspace of a Baire space). If X satisfies λ_c then Y is a regular T_0 complete M-space. If X satisfies λ_b then Y is a paracompact Čech complete space.[4]

Theorem 4.2. Suppose X is a regular T_0-space satisfying one of the conditions β_c or β_b. Then X is a uniformly λ-complete open continuous image of a space Y having the same weight as X which is a subspace of a product space $\Gamma \times X$ where Γ is a zero-dimensional metric space (Γ is a subspace of a Baire space). If X satisfies β_c then Y is a regular T_0 M-space. If X satisfies β_b then Y is a T_2 paracompact M-space.

The following two theorems are stated but not proved here. The proofs will be submitted elsewhere.

Theorem 4.3. Suppose X is a regular T_0-space satisfying either λ_c or λ_b. Then any regular T_0 open continuous image of X satisfies the same condition.

Theorem 4.4. Suppose X is a regular T_0-space satisfying either β_c or

[4] The part of Theorem 4.1 relating to λ_b has been proven in [19].

β_b. Then any regular T_0 uniformly λ-complete open continuous image of X satisfies the same condition.

Combining Theorem 4.1 with 4.3 and Examples 2.1 and 2.3 and Theorem 4.2 with 4.4 and Examples 2.2 and 2.4 the following characterizations are obtained.

Theorem 4.5. A regular T_0-space is an open continuous image of a paracompact Čech complete space[5] (regular T_0 complete M-space) if and only if it satisfies $\lambda_b(\lambda_c)$.

Theorem 4.6. A regular T_0-space is an open continuous uniformly λ-complete image of a T_2 paracompact p-space (regular T_0 M-space) if and only if it satisfies $\beta_b(\beta_c)$.

A proof of the following theorem is contained in a joint work of the author and J.M. Worrell Jr. [21].

Theorem 4.7. If a regular T_0-space X satisfies one of the conditions λ_c, λ_b, β_b, or β_c then any perfect image of X satisfies the same condition.

5. The mapping lemma

Lemmas 5.1 and 5.2 are simple statements useful in the sequel. Lemma 5.3 is the main lemma. Theorem 5.1 is given as an illustration of the general application of 5.3.

Lemma 5.1. Suppose X is a T_1-space and (B_n) is a decreasing sequence of subsets of X such that $B = \cap \{\overline{B_n} : n \in N\}$ is nonempty and countably compact and if U is open and $U \supset B$ then some $B_k \subset U$.

Suppose (A_n) is a decreasing sequence of nonempty sets such that for each n there exists j such that $A_j \subset B_n$.

Then $A = \cap \{\overline{A_n} : n \in N\}$ is a nonempty, closed, and countably compact subset of B and if U is open and $U \supset A$ then some $A_k \subset U$.

Proof. For any k, $\overline{A_n} \cap B \neq \emptyset$. For if $B \subset X \setminus \overline{A_k}$ then for some n,

[5] This part of the theorem was obtained in the completely regular case in [19].

$B_n \subset X \setminus A_k$. There exists $j > \max(n, k)$ such that $A_j \subset B_n$. Since $A_j \subset A_k$ this involves a contradiction. Therefore $(\overline{A_k} \cap B)_{k \in N}$ is a decreasing sequence of nonempty closed subsets of B so that $A = \cap \{\overline{A_k} \cap B: k \in N\} = \cap \{\overline{A_k}: k \in N\}$ is nonempty, closed, and countably compact. Suppose U is open and $A \subset U$. If no $A_k \subset U$ there exists a sequence (x_n) of distinct points of X such that for each n $x_n \in (A_n \setminus U) \cap B_n$. For there exists $j \geq 1$ and $x_1 \in X$ such that $A_j \subset B_1$ and $x_1 \in A_j \setminus U$. Suppose $x_1, ..., x_n$ have been defined. There exists $j \geq n+1$ such that $A_j \subset B_{n+1}$. If $A_j \setminus U \subset \{x_1, ..., x_n\}$ then for all $k > j$, $A_k \setminus U \subset \{x_1, ..., x_n\}$. Therefore some $x_i \in A_k$ for all $k \in N$ and thus $x_i \in A \setminus U$ which is impossible. Therefore there exists $x_{n+1} \in (A_j \setminus U) \setminus \{x_1, ..., x_n\}$. The set $C = \{x_i : i \in N\}$ has a limit point in B. If $C \cap B$ is infinite this is clear. Let C_k denote $\{x_i : i \geq k\}$. Suppose there exists k such that $B \subset X \setminus C_k$. Any limit point of C is also a limit point of C_k since X is T_1. If B contains no limit point of C then $B \subset X \setminus C_k$. Hence some $B_n \subset X \setminus C_k$ which involves a contradiction. But if x is a limit point of C, $x \in \overline{A_n} \setminus U$ for all n and thus $x \in A \setminus U$ which is impossible. Therefore some $A_k \subset U$.

Remark 5.1. Suppose X is T_1, B is closed and countably compact, and (B_n) is a decreasing sequence such that U open and $U \supset B$ implies that some $B_k \subset U$. If $x_n \in B_n$ for all n, then if $\{x_n : n \in N\}$ is infinite it has a limit point in B.

Proof. This follows directly from the proof of Lemma 5.1.

Lemma 5.2. Suppose X is a regular T_0-space covered by a family of countably compact closed (compact) sets of strongly countable character. If $\mathcal{W}$ is a base for X, then X has a network $\mathfrak{N}$ of countably compact closed (compact) sets such that every $A \in \mathfrak{N}$ has a base $\{D_n : n \in N\}$ with $\overline{D_{n+1}} \subset D_n$ and $D_n \in \mathcal{W}$ for all n.

Proof. Suppose U is open in X, $x \in U$, and $\mathcal{W}$ is a base for X. There exists a countably compact closed (compact) set B of strongly countable character which contains x. Let $\{B_n : n \in N\}$ denote a base at B such that for each n, $\overline{B_{n+1}} \subset B_n$. By induction and the axiom of choice it may be seen that there exists a sequence (A_n) of members of $\mathcal{W}$ containing x such that $\overline{A_{n+1}} \subset B_{n+1} \cap A_n \cap U$ for each n. Thus (A_n) and (B_n) satisfy the hypothesis of Lemma 5.1. Therefore $A = \cap A_n$ is a countably compact closed (compact) set of strongly countably character containing x, and $\{A_n : n \in N\}$ is the desired base at A, and $A \subset U$. Using these considerations, the collection $\mathfrak{N}$ may be defined in an obvious way.

Lemma 5.3. Suppose X is a topological space and $\mathcal{W}$ is a collection of open subsets of X.

Suppose there exists a collection Γ of sequences $\alpha: N \to \mathcal{W}$ satisfying (Γ1) $\cap \{\alpha_n : n \in N\}$ is nonempty and has $\{\alpha_n : n \in N\}$ as a base. (Γ2) $\mathfrak{N} = \{B \subset X$: There exists $\alpha \in \Gamma$ and $B = \cap \{\alpha_n : n \in N\}\}$ is a network for X. (Γ3) If $B \in \mathfrak{N}$ and $\beta \in \Gamma$ and some $\beta_n \supset B$ then there exists $\alpha \in \Gamma$ such that $B = \cap \{\alpha_n : n \in N\}$ and $\alpha_i = \beta_i$ for $1 \leqslant i \leqslant n$.

Let Γ have the Baire space topology [8] and $\Gamma \times X$ the product topology. Let $Y = \{(\alpha, x) \in \Gamma \times X : x \in \cap \{\alpha_n : n \in N\}\}$. Let π_1 and π_2 denote the projection mappings of $\Gamma \times X$ onto Γ and X respectively. Then

(a) $\pi_2 | Y$ is an open continuous mapping of Y onto X.

(b) The weight of $Y \leqslant |\mathcal{W}| \cdot (\text{weight of } X) \cdot \aleph_0$.

(c) $\pi_1 | Y$ maps Y onto Γ and for each $\alpha \in \Gamma$, $(\pi_1 | Y)^{-1}(\alpha)$ is homeomorphic to an element of $\mathfrak{N}$.

If, in addition, each element of $\mathfrak{N}$ is countably compact, X is T_1, and each element of Γ is decreasing, then

(d) $\pi_1 | Y$ is a closed continuous mapping of Y onto Γ.

(e) Y is an M-space.

Proof. If $\alpha \in \Gamma$, let $S(\alpha | n) = \{\alpha' \in \Gamma : \alpha'_j = \alpha_j, j = 1, ..., n\}$. Then $\{S(\alpha | n) : n \in N \text{ and } \alpha \in \Gamma\}$ is a base for the topology of Γ. Let $\mathcal{V}$ denote a base for the topology of X of minimum cardinality. For $\alpha \in \Gamma$ and $V \in \mathcal{V}$ such that $V \subset \alpha_n$ let $D(\alpha | n; V)$ denote $(S(\alpha | n) \times V) \cap Y$. Then $\mathcal{B} = \{D(\alpha | n; V) : n \in N, \alpha \in \Gamma, V \in \mathcal{V}, \text{ and } V \subset \alpha_n\}$ is a base for Y. Clearly $|\mathcal{B}| \leqslant |\mathcal{W}| \cdot (\text{weight of } X) \cdot \aleph_0$. Let θ denote $\pi_1 | Y$ and φ denote $\pi_2 | Y$.

Suppose $D(\alpha | n; V) \in \mathcal{B}$. If $x \in V$ there exists $B \in \mathfrak{N}$ such that $x \in B \subset V \subset \alpha_n$. By ($\Gamma$3) there exists $\alpha' \in \Gamma$ such that $B = \cap \alpha'_n$ and $\alpha'_i = \alpha_i$, $i = 1, ..., n$. Thus $(\alpha', x) \in D(\alpha | n; V)$. It follows that $\varphi[D(\alpha | n; V)] = V$ and thus φ is open and onto.

Since $\cap \alpha_n$ is nonempty for all $\alpha \in \Gamma$ and since $\theta^{-1}(\alpha) = \{\alpha\} \times \cap \alpha_n$, (c) follows. Under the additional assumptions of the lemma suppose that $\alpha \in \Gamma$ and $\theta^{-1}(\alpha) \subset U$ is open. If $D(\alpha | n; \alpha_n) \not\subset U$ for any n there exists a sequence $((\alpha^n, x_n))_{n \in N}$ such that, for each n, $(\alpha^n, x_n) \in D(\alpha | n; \alpha_n) \backslash U$ and the values of (x_n) are distinct. This may be seen using induction and the fact that α is a decreasing sequence. Let $E = \{x_n : n \in N\}$ and $B = \cap \alpha_n$. By Remark 5.1, B contains a limit point y of E. It may easily be seen that (α, y) is a limit point of $\{(\alpha^n, x_n) : n \in N\}$. Thus $(\alpha, y) \in \theta^{-1}(\alpha) \backslash U$ which involves a contradiction. Thus some $D(\alpha | n; \alpha_n) \subset U$. Since $D(\alpha | n; \alpha_n)$ includes every set of the form $\theta^{-1}(\alpha')$ that it intersects it follows that θ is closed. By (c) and the countable compactness of the elements of $\mathfrak{R}$, $\theta^{-1}(\alpha)$ is countably compact. Morita's Theorem 1.7 implies that Y is an M-space.

By appropriate choices of the collection Γ the Theorems 1.1–1.6 of section 1 can be proved using Lemma 5.3. The following theorem adds two details to Theorem 1.5 of Nagata. E. Michael also observed (oral communication) that the method of [16] can be used to obtain Theorem 1.5 and pointed out that this gives regularity of the M-space.

Theorem 5.1. A regular T_0 q-space is an open continuous image of a regular T_0 M-space of the same weight.

Proof. As noted in [13], a regular T_0-space X is a q-space if and only if each point has a sequence (U_n) of open neighborhoods such that, for each n, $\overline{U_{n+1}} \subset U_n$ and if $x_n \in U_n$ for each n, then if $\{x_n : n \in N\}$ is infinite, it has a limit point. It is easy to see that $\cap \{U_n : n \in N\}$ is closed and countably compact and has $\{U_n : n \in N\}$ as a base. Suppose $\mathcal{W}$ is a base for X of minimum cardinality and Γ is the collection of sequences $\alpha : N \to \mathcal{W}$ such that: (1) For each n, $\overline{\alpha_{n+1}} \subset \alpha_n$. (2) $\cap\, \alpha_n$ is countably compact, nonempty, and has $\{\alpha_n : n \in N\}$ as a base. Then, with the use of Lemma 5.2, it follows that Γ satisfies (Γ1)–(Γ3). Thus Lemma 5.3 implies that X is an open continuous image of an M-space Y of the same weight as X. Since Y is a subset of the regular T_0-space $\Gamma \times X$ the conclusion follows.

6. Proofs of Theorems 4.1 and 4.2

The various cases treated will be designated by the symbols λ_c, λ_b, β_c, or β_b. Since X is assumed to be regular and T_0, the forms of the basic conditions given in Lemmas 2.1 and 2.2 can and will be used. Lemma 5.3 will also be used and the first part of the proofs will be devoted to defining an appropriate collection Γ. In each of the four cases there exists a monotonically contracting sequence $(\mathcal{G}_n)$ having the appropriate properties of the corresponding definition. It may be seen (proof given in [20]) that there exists a sequence $(\mathcal{H}_n)$ of well-ordered collections covering X such that for each $n \in N$: (P1) $\mathcal{H}_n \subset \mathcal{G}_n$. (P2) Each $H \in \mathcal{H}_n$ contains a point not in any predecessor of H in $\mathcal{H}_n$. (P3) If $n, k \in N$, $n < k$, and $x \in X$, the closure of the first element of $\mathcal{H}_k$ that contains x is a subset of the first element of $\mathcal{H}_n$ that contains x. Let $\mathcal{W}$ denote a base for X of minimum cardinality.

Let Γ_c denote the collection of all sequences $\alpha: N \to \mathcal{W}$ such that for each n: (A1) $\overline{\alpha_{n+1}} \subset \alpha_n$. (A2) α_n contains a point y such that the first element of $\mathcal{H}_n$ that contains y includes α_n. (A3) $\cap\, \alpha_n$ is nonempty and $\{\alpha_n : n \in N\}$ is a base at $\cap\, \alpha_n$. (A4) $\cap\, \alpha_n$ is countably compact. Let Γ_b denote the collection of all sequences $\alpha: N \to \mathcal{W}$ satisfying (A1)–(A4) and additionally: (A5) $\cap\, \alpha_n$ is compact. In the cases λ_c, β_c the set $\mathfrak{N}_c = \{B$: For some $\alpha \in \Gamma_c$, $B = \cap\, \alpha_n\}$ is a network for X. For, by Lemma 5.2, X has a network of countably compact closed sets of strongly countably character. If $x \in U$ where U is open let C be a member of such a network which contains x such that $C \subset U$. Let (D_n) denote a decreasing sequence such that $\{D_n : n \in N\}$ is a base at C and $\overline{D_{n+1}} \subset D_n$ for all n. By induction a sequence $\alpha: N \to \mathcal{W}$ may be defined such that for each n: (A1) and (A2) are satisfied and $\alpha_n \subset D_n$. By Lemma 5.1, $\cap\, \alpha_n$ is countably compact and has $\{\alpha_n : n \in N\}$ as a base. Thus Γ_c satisfies (Γ1) and (Γ2). Suppose that $B \in \mathfrak{N}_c$ and $\beta \in \Gamma_c$ and some $\beta_n \supset B$. Suppose $B = \cap\, \alpha'_k$ for $\alpha' \in \Gamma_c$. Then for some $j > n$, $\alpha_j \subset \beta_n$. Define $\alpha: N \to \mathcal{W}$ by $\alpha_i = \beta_i$ for $1 \leqslant i \leqslant r$ and $\alpha_{n+1} = \alpha'_{j+i-1}$ for $i \in N$. Then (A1) is satisfied by α and $\cap\, \alpha_k = \cap\, \alpha'_k = B$ so (A3) and (A4) are satisfied. If $1 \leqslant j \leqslant n$, α_j satisfies (A2). If $j = n + i$ then $\alpha_j = \alpha'_{j+i-1}$ and there exists $y \in \alpha_j$ such that the first element H of $\mathcal{H}_{j+i-1}$ containing y includes α_j. But the first element of $\mathcal{H}_j$ that contains y includes H so that (A2) is satisfied for all n. Therefore $\alpha \in \Gamma_c$ so that (Γ3) is satisfied. In cases λ_b and β_b the above argument applies directly and in addition one has that all the sets $\cap\, \alpha_n$, where $\alpha \in \Gamma_b$, are compact. Thus the set $\mathfrak{N}_b = \{B$: For some $\alpha \in \Gamma_b$, $B = \cap\, \alpha_n\}$ is a network and Γ_b satisfies (Γ1)–(Γ3). Therefore conclusions (a)–(e) of Lemma 5.3 hold in all cases. Let $\theta = \pi_1 | Y$ and $\varphi = \pi_2 | Y$.

Cases λ_c *and* λ_b. Here it must be shown that the spaces Γ_b and Γ_c are complete with respect to some metric. Let ρ denote the Baire metric [8] in Γ_c. It may be shown as in [15] that if (α^n) is a Cauchy sequence in Γ_c with respect to ρ there exists $\alpha: N \to \mathcal{W}$ and an increasing sequence $(n_j)_{j \in N}$ of positive integers such that, for all $k \in N$, the first k values of α_{n_k} are $\alpha_1, ..., \alpha_k$. Since each $\alpha^n \in \Gamma_c$ it follows that α satisfies (A1) and (A2). By (A2) each $\mathcal{H}_n$ contains a first H_n that includes a term of α. For each n there exists $j > n + 1$ such that $\alpha_j \subset H_n \cap H_{n+1}$. There exists $y \in \alpha_j$ such that the first element H' of $\mathcal{H}_j$ that contains y includes α_j. If H is the first element of $\mathcal{H}_n$ containing y then $H \supset H' \supset \alpha_j$. Thus H does not precede H_n. Since H_n contains y it follows that $H = H_n$. Similarly, H_{n+1} is the first element of $\mathcal{H}_{n+1}$ that contains y. Thus for each n, $\overline{H_{n+1}} \subset H_n$ by (P3). Since each $H_n \in \mathcal{G}_n$ and λ_c is satisfied it follows that $\cap H_n$ is nonempty, closed, countably compact, and $\{H_n : n \in N\}$ is a base at $\cap H_n$. Since for each n there exists j such that $\alpha_j \subset H_n$, Lemma 5.1 implies that $\cap \alpha_n$ is nonempty, closed, countably compact, and has $\{\alpha_n : n \in N\}$ as a base. Thus (A3) and (A4) are satisfied so that $\alpha \in \Gamma_c$. Clearly $\alpha = \lim_{n\to\infty} \alpha_n$ so that Γ_c is complete. Replacing "λ_c", "Γ_c", and "countably compact" in the above proof by "λ_b", "Γ_b", and "compact", respectively, shows that Γ_b is complete. Thus in case λ_c, Y is a quasi-perfect preimage of a complete metric space and in case λ_b, Y is a perfect preimage of a complete metric space. In the latter case, Frolík's Theorem 1.9 shows that Y is paracompact and Čech complete.

Cases β_c and β_b. In these cases it must be shown that the mapping φ is uniformly λ-complete. To this end let $\mathcal{G}_n$ denote $\{D(\alpha|n; \alpha_n) : \alpha \in \Gamma\}$ where Γ denotes either Γ_c or Γ_b depending on the case. Clearly each $\mathcal{G}_n$ covers Y. Suppose $(\beta, x) \in D(\alpha|n; \alpha_n)$. Then $(\beta, x) \in D(\beta|n+1; \beta_{n+1}) \subset D(\alpha|n; \alpha_n)$. Thus $(\mathcal{G}_n)$ is a monotonically contracting sequence of collections of open sets covering Y. Suppose $x \in X$ and (W_n) and (G_n) are such that for each n: (1) W_n is open. (2) $W_{n+1} \subset W_n$. (3) There exists j such that $W_j \subset G_n$. (4) $W_n \cap \varphi^{-1}(x) \neq \emptyset$. (5) $G_{n+1} \subset G_n \in \mathcal{G}_n$. Each $G_n = D(\alpha^n | n; \alpha^n_n)$ for some α^n. Let α_n denote α^n_n. For each n, the first n terms of α^n are $\alpha_1, ..., \alpha_n$. For $n = 1$, $\alpha^1_1 = \alpha_1$. Assume the statement for n. Since $G_{n+1} \subset G_n$ it follows that $\alpha^{n+1}_i = \alpha^n_i = \alpha_i$ for $1 \leqslant i \leqslant n$. Since $\alpha^{n+1}_{n+1} = \alpha_{n+1}$ the conclusion follows. Also (4) implies that $G_n \cap \varphi^{-1}(x) \neq \emptyset$ so that $x \in \cap \alpha_n$. By (A2) each $\mathcal{H}_n$ contains a first H_n that includes a term of α. Clearly $x \in \cap H_n$. With the use of condition β_c or β_b and an argument used in the third paragraph of this proof it follows that $\cap H_n$ is closed and countably compact (or compact, depending on case β_c or β_b) and has $\{H_n : n \in N\}$ as a base. Lemma 5.1 implies that $\cap \alpha_n$ is nonempty, closed, countably compact (respectively, compact), and has $\{\alpha_n : n \in N\}$ as a base. Thus $\alpha \in \Gamma$ and $(\alpha, x) \in \varphi^{-1}(x)$. For each j there exists $\beta^j \in \Gamma$ such that $(\beta^j, x) \in W_j \cap \varphi^{-1}(x)$. Suppose $(\alpha, x) \in D(\alpha|n; V)$ where $V \subset \alpha_n$. There exists $j \geqslant n$ such that $W_k \subset G_n$ for all $k \geqslant j$. Hence $\beta^k_i = \alpha_i$ for $1 \leqslant i \leqslant n$. Thus $(\beta^k, x) \in D(\alpha|n; V)$ for all $k \geqslant j$. It follows that $(\alpha, x) \in \cap \overline{W_n} \cap \varphi^{-1}(x)$, so that φ is uniformly λ-complete.

Acknowledgement

I wish to thank J.M. Worrell Jr., for his valuable contributions to this article.

References

[1] Arhangel'skiĭ, A.V., Bicompact sets and the topology of spaces, Tr. Mosk. Mat. Obšč. 13 (1965) 3-55; Transl. Moscow Math. Soc. 13 (1965) 1-62.
[2] Arhangel'skiĭ, A.V., On a class of spaces containing all metric and all locally bicompact spaces, Mat. Sb. 67 (109) (1965) 55-88.
[3] Arhangel'skiĭ, A.V., Mappings and spaces, Usp. Mat. Nauk 21 (1966) 133-184; Transl. Russ. Math. Surv. 21 (1966) 115-162.
[4] Aronszajn, N., Über die Bogenverknüpfung in topologischen Räumen, Fund. Math. 15 (1930) 228-241.
[5] Frolík, Z., Generalizations of the G_δ property of complete metric spaces, Czech. Mat. J. 10 (85) (1960) 359-378.
[6] Frolík, Z., On the topological product of paracompact spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 8 (1960) 747-750.
[7] Hanai, S., On open mappings II, Proc. Japan Acad. 37 (1961) 233-238.
[8] Hausdorff, F., Mengenlehre (3rd. ed., Gruyter, Berlin-Leipzig, 1935).
[9] Kelley, J.L., General topology (Van Nostrand, Princeton, N.J., 1955).
[10] Michael, E., A note on closed maps and compact sets, Israel J. Math. 2 (1964) 1 3-176.
[11] Morita, K., Products of normal spaces with metric spaces, Math. Ann. 154 (1964) 365-382.
[12] Morita, K., Some properties of M-spaces, Proc. Japan Acad. 43 (1967) 869-872.
[13] Nagata, J., Mappings and M-spaces, Proc. Japan Acad. 45 (1969) 140-144.
[14] Ponomarev, V.I., Axioms of countability and continuous mappings, Bull. Polon. Acad. Sci., Sér. Sci. Math. Astronom. Phys. 8 (1960) 127-133 (Russian).
[15] Wicke, H.H., The regular open continuous images of complete metric spaces, Pac. J. Math. 23 (1967) 621-625.
[16] Wicke, H.H., On the Hausdorff open continuous images of Hausdorff paracompact p-spaces, Proc. Amer. Math. Soc. 22 (1969) 136-140.
[17] Wicke, H.H. and J.M. Worrell Jr., Open continuous mappings of spaces having bases of countable order, Duke Math. J. 34 (1967) 255-272.
[18] Wicke, H.H. and J.M. Worrell Jr., On a class of spaces containing Arhangel'skiĭ's p-spaces, Notices Amer. Math. Soc. 14 (1967) 687.
[19] Wicke, H.H. and J.M. Worrell Jr., On the open continuous images of paracompact Čech complete spaces, Pac. J. Math., to appear.
[20] Wicke, H.H. and J.M. Worrell Jr., On topological completeness of first-countable Hausdorff spaces, submitted for publication.
[21] Wicke, H.H. and J.M. Worrell Jr., Perfect mappings and certain interior images of M-spaces, to be submitted.
[22] Worrell Jr., J.M. and H.H. Wicke, Characterizations of developable topological spaces, Can. J. Math. 17 (1965) 820-830.
[23] Worrell Jr., J.M. and H.H. Wicke, Non-first-countable topological structure, Notices Amer. Math. Soc. 14 (1967) 935.