Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 277 (2004) 298–317
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Constructible characters and canonical bases

Bernard Leclerc [a,*] and Hyohe Miyachi [b]

[a] *Laboratoire de Mathématiques Nicolas Oresme, Université de Caen, Campus II, Bld Maréchal Juin, BP 5186, 14032 Caen cedex, France*
[b] *Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan*

Received 17 April 2003

Communicated by Michel Broué

Abstract

We give closed formulas for all vectors of the canonical basis of a level 2 irreducible integrable representation of $U_v(\mathfrak{sl}_\infty)$. These formulas coincide at $v = 1$ with Lusztig's formulas for the constructible characters of the Iwahori–Hecke algebras of type B and D.

1. Introduction

In [17] we have obtained closed formulas for certain vectors of the canonical basis of the level 1 Fock space representation of $U_v(\widehat{\mathfrak{sl}}_n)$, inspired by similar formulas for decomposition numbers occurring in the modular representation theory of the groups $GL_m(\mathbb{F}_q)$. (See also [9] for closely related results obtained independently.)

In this paper we give closed formulas for all vectors of the canonical basis of an irreducible integrable representation of level 2 of $U_v(\mathfrak{sl}_\infty)$. These formulas are also inspired by some known results in the representation theory of finite Chevalley groups, namely by Lusztig's theory of families and constructible characters for Weyl groups of type B and D, and more generally for the corresponding Iwahori–Hecke algebras with unequal parameters [20,22].

Let $\Lambda = \Lambda_k + \Lambda_{k+r}$ $(k \in \mathbb{Z},\ r \in \mathbb{N})$ be a level 2 dominant integral weight of $U_v(\mathfrak{sl}_\infty)$ (here the Λ_i are the fundamental weights). Let $V(\Lambda)$ and $F(\Lambda)$ be respectively

* Corresponding author.
E-mail addresses: leclerc@math.unicaen.fr (B. Leclerc), miyachi@math.nagoya-u.ac.jp (H. Miyachi).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.02.023

the irreducible and the Fock space representation with highest weight Λ, and let $\Phi : V(\Lambda) \to F(\Lambda)$ be the natural embedding of $U_v(\mathfrak{sl}_\infty)$-modules (it is unique up to scalar multiplication). The Fock space $F(\Lambda)$ is endowed by construction with a standard basis $\mathcal{S}(\Lambda) = \{s_{(\lambda,\mu)}\}$ canonically labelled by all pairs (λ, μ) of partitions. Let $\mathcal{B}(\Lambda)$ denote the Kashiwara–Lusztig canonical basis of $V(\Lambda)$ [15,21].

On the other hand, let $H_m = H_m(q^r; q)$ be the Iwahori–Hecke algebra of type B_m over $\mathbb{C}(q^{1/2})$ with parameter q^r for the special generator T_0 and q for all other generators T_i $(1 \leqslant i \leqslant m-1)$. Let $\operatorname{Irr} H_m = \{\chi_{(\lambda,\mu)} \mid |\lambda| + |\mu| = m\}$ be the set of irreducible characters of H_m and $\operatorname{Con} H_m$ the set of constructible characters [20,22]. Then our main result states that the specialization at $v = 1$ of $\mathcal{B}(\Lambda)$ can be identified to $\bigcup_m \operatorname{Con} H_m$, in the following sense: if we write for a vector $b \in \mathcal{B}(\Lambda)$ of principal degree m,

$$\Phi(b) = \sum_{(\lambda,\mu)} \alpha^b_{(\lambda,\mu)}(v)\, s_{(\lambda,\mu)}, \tag{1}$$

then

$$\chi_b = \sum_{(\lambda,\mu)} \alpha^b_{(\lambda,\mu)}(1)\, \chi_{(\lambda,\mu)} \tag{2}$$

belongs to $\operatorname{Con} H_m$, and all constructible characters are obtained in this way. In particular, two irreducible characters $\chi_{(\lambda,\mu)}$ and $\chi_{(\lambda',\mu')}$ lie in the same family if and only if the corresponding vectors $s_{(\lambda,\mu)}$ and $s_{(\lambda',\mu')}$ have the same $U_v(\mathfrak{sl}_\infty)$-weight.

Similar results connect the constructible characters of type D_m and the canonical basis of the representation $V(2\Lambda_k)$.

The paper is organized as follows. In Section 2 we calculate explicitly the canonical basis of $V(\Lambda)$. In particular, we show that the number of nonzero coefficients $\alpha^b_{(\lambda,\mu)}(v)$ in the expansion of $b \in \mathcal{B}(\Lambda)$ is always a power of 2, and that all these coefficients are powers of v. It is remarkable that the combinatorics involved in these formulas is precisely the combinatorics of Lusztig's symbols, which were introduced to parametrize the irreducible unipotent characters of the Chevalley groups $G(\mathbb{F}_q)$ of classical type (see [20]). In Section 3 we briefly review Lusztig's work on constructible characters and families for Iwahori–Hecke algebras. In Section 4 we compare the canonical basis of $V(\Lambda)$ and the constructible characters of Iwahori–Hecke algebras of type B_m and D_m calculated by Lusztig [19,20,22], and we obtain our main result. Section 5 explains the relation between this result and a theorem of Gyoja [11] comparing constructible characters and decomposition matrices of Iwahori–Hecke algebras. Finally, Section 6 discusses a possible generalization of our results to Ariki–Koike algebras.

It is interesting to note that Brundan [8] has obtained similar formulas for the canonical basis of the level 0 module $\bigwedge^m \mathcal{V}^* \otimes \bigwedge^n \mathcal{V}$, where $\mathcal{V}$ denotes the vector representation of $\mathfrak{sl}_\infty$ and $\mathcal{V}^*$ the dual representation. His calculations were motivated by completely different problems in the representation theory of the Lie superalgebra $\mathfrak{gl}(m|n)$.

2. Canonical bases

2.1. Fix $n \geqslant 2$ and let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We consider the quantum enveloping algebra $U_v(\mathfrak{g})$ over $\mathbb{Q}(v)$ with Chevalley generators e_j, f_j, t_j $(1 \leqslant j \leqslant n)$ (see, for example, [12]).

The simple roots and the fundamental weights are denoted by α_k and Λ_k $(1 \leqslant k \leqslant n)$, respectively, and the fundamental representations of $U_v(\mathfrak{g})$ by $V(\Lambda_k)$. Let u_i $(1 \leqslant i \leqslant n+1)$ be the natural basis of the vector representation $V(\Lambda_1)$, that is,

$$e_j u_i = \delta_{i,j+1} u_{i-1}, \qquad f_j u_i = \delta_{ij} u_{i+1}, \qquad t_j u_i = v^{\delta_{ij} - \delta_{i,j+1}} u_i$$
$$(1 \leqslant i \leqslant n+1,\ 1 \leqslant j \leqslant n).$$

The fundamental representation $V(\Lambda_k)$ is obtained by v-deforming the kth exterior power of $V(\Lambda_1)$. It has a $\mathbb{C}(v)$-basis $\{u_\beta\}$ labelled by all sequences

$$\beta = (\beta_1, \ldots, \beta_k), \quad 1 \leqslant \beta_1 < \cdots < \beta_k \leqslant n+1.$$

For convenience, we sometimes identify such sequences with k element subsets of $\{1, \ldots, n+1\}$. Thus we may write non-ambiguously $j \in \beta$, $\beta \cup \{j\}$, and so on. The Chevalley generators act on $\{u_\beta\}$ by

$$e_j u_\beta = \begin{cases} 0, & \text{if } j+1 \notin \beta \text{ or } j \in \beta, \\ u_\gamma, & \text{otherwise, where } \gamma = \big(\beta \setminus \{j+1\}\big) \cup \{j\}; \end{cases} \tag{3}$$

$$f_j u_\beta = \begin{cases} 0, & \text{if } j+1 \in \beta \text{ or } j \notin \beta, \\ u_\gamma, & \text{otherwise, where } \gamma = \big(\beta \setminus \{j\}\big) \cup \{j+1\}; \end{cases} \tag{4}$$

$$t_j u_\beta = \begin{cases} v u_\beta, & \text{if } j \in \beta, \\ v^{-1} u_\beta, & \text{if } j+1 \in \beta, \\ u_\beta, & \text{otherwise.} \end{cases} \tag{5}$$

As is well known, since $V(\Lambda_k)$ is a minuscule representation $\{u_\beta\}$ is nothing but the canonical basis $\mathcal{B}(\Lambda_k)$ of $V(\Lambda_k)$. The highest weight vector is u_{β^k}, where $\beta^k := (1, 2, \ldots, k)$.

2.2. Let now $\Lambda = \Lambda_k + \Lambda_{k+r}$ $(1 \leqslant k \leqslant k+r \leqslant n)$ be a sum of two fundamental weights. Let $V(\Lambda)$ be the irreducible $U_v(\mathfrak{g})$-module with highest weight Λ and set $F(\Lambda) = V(\Lambda_{k+r}) \otimes V(\Lambda_k)$. We have a canonical embedding of $U_v(\mathfrak{g})$-modules $\Phi : V(\Lambda) \to F(\Lambda)$ defined by mapping the highest weight vector u_Λ of $V(\Lambda)$ to $u_{\beta^{k+r}} \otimes u_{\beta^k}$.

The basis $\mathcal{S}(\Lambda) = \{u_\beta \otimes u_\gamma \mid u_\beta \in \mathcal{B}(\Lambda_{k+r}),\ u_\gamma \in \mathcal{B}(\Lambda_k)\}$ of $F(\Lambda)$ will be called the standard basis. It is labelled by all symbols

$$S = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \beta_1, \ldots, \beta_{k+r} \\ \gamma_1, \ldots, \gamma_k \end{pmatrix} \tag{6}$$

with $1 \leqslant \beta_1 < \cdots < \beta_{k+r} \leqslant n+1$ and $1 \leqslant \gamma_1 < \cdots < \gamma_k \leqslant n+1$. We shall write for short $u_S = u_\beta \otimes u_\gamma$. The symbol attached to the highest weight vector is denoted by

$$S_0 = \begin{pmatrix} \beta^{k+r} \\ \beta^k \end{pmatrix} = \begin{pmatrix} 1, \ldots, k+r \\ 1, \ldots, k \end{pmatrix}. \tag{7}$$

The action of the Chevalley generators on this basis is obtained via the comultiplication of $U_v(\mathfrak{g})$, namely,

$$f_j(u_\beta \otimes u_\gamma) = u_\beta \otimes f_j u_\gamma + f_j u_\beta \otimes t_j u_\gamma, \tag{8}$$

$$e_j(u_\beta \otimes u_\gamma) = e_j u_\beta \otimes u_\gamma + t_j^{-1} u_\beta \otimes e_j u_\gamma, \tag{9}$$

$$t_j(u_\beta \otimes u_\gamma) = t_j u_\beta \otimes t_j u_\gamma. \tag{10}$$

2.3. Let A be the subring of $\mathbb{Q}(v)$ consisting of all rational functions regular at $v = 0$. Let L be the A-lattice of $F(\Lambda)$ spanned by $\mathcal{S}(\Lambda)$. Since crystal bases are compatible with tensor products [14], it is clear that $(\mathcal{S}(\Lambda), L)$ is a crystal basis of $F(\Lambda)$. Moreover, it is easy to see that the connected component of u_{S_0} in the crystal graph of $F(\Lambda)$ is the subgraph with vertices u_S where S is as in (6) with $\beta_i \leqslant \gamma_i$ for $1 \leqslant i \leqslant k$ [16]. Such symbols will be called standard. Clearly, this is the same as saying that the two rows of S form the two columns of a semistandard Young tableau.

2.4. Let $U_v^-(\mathfrak{g})$ be the subalgebra of $U_v(\mathfrak{g})$ generated by the f_i's. Let $x \mapsto \bar{x}$ denote the ring automorphism of $U_v^-(\mathfrak{g})$ defined by

$$\overline{f_i} = f_i, \quad \bar{v} = v^{-1}.$$

This induces a $\mathbb{C}$-linear map $u \mapsto \bar{u}$ on $V(\Lambda)$ given by

$$\overline{(x u_{S_0})} = \bar{x} u_{S_0} \quad \bigl(x \in U_v^-(\mathfrak{g})\bigr).$$

By 2.3, the canonical basis (or lower global basis) of $V(\Lambda)$ is parametrized by the set of standard symbols, and the element b_S attached to the symbol S is characterized by

$$\overline{b_S} = b_S \quad \text{and} \quad \Phi(b_S) \equiv u_S \mod vL. \tag{11}$$

2.5. Let $S = \binom{\beta}{\gamma}$ be a standard symbol. We define an injection $\psi : \gamma \to \beta$ such that $\psi(j) \leqslant j$ for all $j \in \gamma$. To do so it is enough to describe the subsets

$$\gamma^l = \bigl\{ j \in \gamma \bigm| \psi(j) = j - l \bigr\} \quad (0 \leqslant l \leqslant n).$$

We set $\gamma^0 = \gamma \cap \beta$ and for $l \geqslant 1$ we put

$$\gamma^l = \bigl\{ j \in \gamma - \bigl(\gamma^0 \cup \cdots \cup \gamma^{l-1}\bigr) \bigm| j - l \in \beta - \psi\bigl(\gamma^0 \cup \cdots \cup \gamma^{l-1}\bigr) \bigr\}.$$

Observe that the standardness of S implies that ψ is well-defined.

Example 1. Take

$$S = \begin{pmatrix} 1 & 3 & 5 & 8 & 9 \\ 3 & 6 & 7 & 10 & \end{pmatrix}.$$

Then

$$\gamma^0 = \{3\}, \qquad \gamma^1 = \{6, 10\}, \qquad \gamma^2 = \cdots = \gamma^5 = \emptyset, \qquad \gamma^6 = \{7\}.$$

Hence

$$\psi(3) = 3, \qquad \psi(6) = 5, \qquad \psi(7) = 1, \qquad \psi(10) = 9.$$

The pairs $(j, \psi(j))$ with $\psi(j) \neq j$ (that is, $j \notin \beta \cap \gamma$) will be called the pairs of S. Given a standard symbol S with p pairs, we denote by $\mathcal{C}(S)$ the set of all symbols obtained from S by permuting some pairs in S and reordering the rows. We consider S itself as an element of $\mathcal{C}(S)$, hence $\mathcal{C}(S)$ has cardinality 2^p. For $\Sigma \in \mathcal{C}(S)$ we denote by $n(\Sigma)$ the number of pairs permuted in S to obtain Σ.

Example 2. We continue Example 1. There are 3 pairs in S, namely $(6, 5)$, $(7, 1)$, $(10, 9)$. The $\Sigma \in \mathcal{C}(S)$ are given below, together with the corresponding $n(\Sigma)$:

Σ	$n(\Sigma)$	Σ	$n(\Sigma)$
$\begin{pmatrix} 1 & 3 & 5 & 8 & 9 \\ 3 & 6 & 7 & 10 & \end{pmatrix}$	0	$\begin{pmatrix} 1 & 3 & 6 & 8 & 9 \\ 3 & 5 & 7 & 10 & \end{pmatrix}$	1
$\begin{pmatrix} 3 & 5 & 7 & 8 & 9 \\ 1 & 3 & 6 & 10 & \end{pmatrix}$	1	$\begin{pmatrix} 1 & 3 & 5 & 8 & 10 \\ 3 & 5 & 6 & 9 & \end{pmatrix}$	1
$\begin{pmatrix} 3 & 6 & 7 & 8 & 9 \\ 1 & 3 & 5 & 10 & \end{pmatrix}$	2	$\begin{pmatrix} 1 & 3 & 6 & 8 & 10 \\ 3 & 5 & 7 & 9 & \end{pmatrix}$	2
$\begin{pmatrix} 3 & 5 & 7 & 8 & 10 \\ 1 & 3 & 6 & 9 & \end{pmatrix}$	2	$\begin{pmatrix} 3 & 6 & 7 & 8 & 10 \\ 1 & 3 & 5 & 9 & \end{pmatrix}$	3

We can now state the main result of this section.

Theorem 3. *Let S be a standard symbol and let b_S be the element of the canonical basis of $V(\Lambda)$ such that $\Phi(b_S) \equiv u_S \bmod vL$. We have*

$$\Phi(b_S) = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_\Sigma.$$

Proof. Set $S = \binom{\beta}{\gamma}$. To simplify notation, we write b_S instead of $\Phi(b_S)$ throughout this proof. We proceed by induction on the principal degree of b_S, that is, on

$$d = \sum_i \beta_i + \sum_j \gamma_j - \binom{k+1}{2} - \binom{k+r+1}{2}.$$

Clearly, $d = 0$ if and only if

$$S = S_0 = \begin{pmatrix} \beta^{k+r} \\ \beta^k \end{pmatrix}$$

is the symbol attached to the highest weight vector of $F(\Lambda)$. In this case we have $C(S_0) = \{S_0\}$ and $b_{S_0} = u_{S_0}$, so the statement is true.

Otherwise, we can find $i \geqslant 2$ in S such that $\{i, i-1\} \cap \beta = \{i\}$ or $\{i, i-1\} \cap \gamma = \{i\}$. Let j be the smallest of these i's.

(a) Suppose that $j \in \beta \cap \gamma$ and $j-1 \notin S$. We denote by S' the symbol obtained by changing in S the two occurrences of j into $j-1$. Clearly S' is again standard, and the pairs of S' are the same as those of S. By induction we may assume that

$$b_{S'} = \sum_{\Sigma' \in \mathcal{C}(S')} v^{n(\Sigma')} u_{\Sigma'}. \tag{12}$$

The element $j-1$ occurs in both rows of each $\Sigma' \in \mathcal{C}(S')$. By (8), it follows that

$$f_{j-1}^{(2)} u_{\Sigma'} = u_{\Sigma},$$

where Σ is obtained from Σ' by changing the two occurrences of $j-1$ into j. Therefore,

$$f_{j-1}^{(2)} b_{S'} = \sum_{\Sigma' \in \mathcal{C}(S')} v^{n(\Sigma')} u_{\Sigma} = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_{\Sigma}.$$

In particular, $f_{j-1}^{(2)} b_{S'} \equiv u_S \bmod vL$. Since $\overline{f_{j-1}^{(2)}} = f_{j-1}^{(2)}$, it follows from (11) that $f_{j-1}^{(2)} b_{S'} = b_S$, and the result is proved in this case.

(b) Suppose that $j \in \beta \cap \gamma$ and that $j-1$ occurs in one of the two rows of S (it cannot occur in both rows by definition of j). We denote by S' the symbol obtained by changing j into $j-1$ in the other row. S' is again standard. By induction we may assume that (12) holds. For $\Sigma' \in \mathcal{C}(S')$ we have $f_{j-1} u_{\Sigma'} = u_{\Sigma}$ where Σ is obtained by changing $j-1$ into j in the row of Σ' that does not contain j. Indeed, if this row is the bottom row by (8) no power of v occurs in $f_{j-1} u_{\Sigma}$, and if this is the top row then the bottom row has both j and $j-1$, so the contribution of t_{j-1} applied to this row is $v^{1-1} = 1$. As in (a), it follows that $f_{j-1} b_{S'} = b_S$. On the other hand, we also have

$$f_{j-1} b_{S'} = \sum_{\Sigma' \in \mathcal{C}(S')} v^{n(\Sigma')} u_{\Sigma}.$$

Let us now compare the pairs of S' and S. In S' we have $1, \ldots, j-1$ in both rows, and j must be in the top row. Then, either j does not belong to a pair of S' or it belongs to a pair $(j+k, j)$. The pairs of S are the same as those of S' in the first case, and in the second case they are the same except $(j+k, j)$ which becomes $(j+k, j-1)$. Therefore, in the first case we obviously have

$$\sum_{\Sigma' \in \mathcal{C}(S')} v^{n(\Sigma')} u_{\Sigma} = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_{\Sigma}.$$

This also holds in the second case since $j-1$ is changed into j in Σ' in the row which does not contain j, that is, in the row containing $j+k$. Hence we always have $j-1$ and $j+k$ lying in different rows in Σ.

(c) Suppose that there is a single occurrence of j in S, and let S' be the symbol obtained by changing this j into $j-1$. Again, S' is standard, and by induction we may assume that (12) holds.

(c1) If $j-1$ occurs on both rows of S', then it also occurs in both rows of all $\Sigma' \in \mathcal{C}(S')$, and no j occurs in any Σ'. Hence by (8),

$$f_{j-1} u_{\Sigma'} = u_{\Sigma_1} + v\, u_{\Sigma_2},$$

where Σ_1 (respectively Σ_2) is obtained from Σ' by changing $j-1$ into j in the bottom (respectively top) row. In particular, when $\Sigma' = S'$, $\Sigma_1 = S$ because Σ_2 is not standard. Hence we have again $f_{j-1} b_{S'} \equiv u_S \bmod vL$ and therefore $f_{j-1} b_{S'} = b_S$. On the other hand, the pairs of S are all the pairs of S' plus the new pair $(j, j-1)$, thus we also have

$$b_S = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_\Sigma.$$

(c2) If $j-1$ occurs in only one row of S', then all $\Sigma' \in \mathcal{C}(S')$ contain also one $j-1$ and no j. It follows that $f_{j-1} u_{\Sigma'} = u_{\Sigma'}$ where Σ' is obtained by changing this $j-1$ into j. Hence, $f_{j-1} b_{S'} = b_S$. Finally, $j-1$ belongs necessarily to the top row of S' (otherwise, by the definition of j and the standardness of S' we should have $1, 2, \ldots, j-1$ in both rows of S' and we would be in case (c1)). If $j-1$ does not belong to a pair of S', then the pairs of S are exactly the same as those of S' and

$$\sum_{\Sigma' \in \mathcal{C}(S')} v^{n(\Sigma')} u_\Sigma = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_\Sigma.$$

If S' has a pair $(j+k, j-1)$, this pair becomes $(j+k, j)$ in S and all other pairs are preserved, so the same formula still holds. □

The proof of Theorem 3 also shows the following:

Proposition 4. *Let Λ be a sum of two fundamental weights. Then each element b of the canonical basis $\mathcal{B}(\Lambda)$ is of the form*

$$b = f_{i_1}^{(r_1)} \cdots f_{i_s}^{(r_s)} u_\Lambda$$

for some $i_j \in \{1, \ldots, n\}$ and $r_j \in \{1, 2\}$.

Note that this proposition can easily be proved directly by using the fact that all i-strings of the crystal graph of $V(\Lambda)$ have length $\leqslant 2$ (see, for example, [8, 3.19]).

2.6. There is an alternative way to index the standard basis $\mathcal{S}(\Lambda)$, namely by pairs of Young diagrams (or partitions) with shifted content. Let $\mathrm{Sy}(n,k,r)$ denote the set of all symbols S as in Eq. (6). To $S \in \mathrm{Sy}(n,k,r)$ we attach the pair of partitions (λ,μ) (written in weakly increasing order) defined by

$$\lambda_i = \beta_i - i, \quad \mu_j = \gamma_j - j \quad (1 \leqslant i \leqslant k+r,\ 1 \leqslant j \leqslant k). \tag{13}$$

This establishes a one-to-one correspondence between $\mathrm{Sy}(n,k,r)$ and the set $\mathrm{Pa}(n,k,r)$ of pairs $(\lambda,\mu) = ((\lambda_1,\ldots,\lambda_{k+r}),(\mu_1,\ldots,\mu_k))$ such that

$$0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_{k+r} \leqslant n+1-k-r, \qquad 0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_k \leqslant n+1-k. \tag{14}$$

An element of $\mathrm{Pa}(n,k,r)$ is conveniently represented by the pair of Young diagrams corresponding to λ and μ in which the cell of λ with coordinates (i,j) is filled with $i-j+k+r$ and the cell of μ with coordinates (i,j) is filled with $i-j+k$. Thus the symbol

$$S = \begin{pmatrix} 1 & 2 & 3 & 6 & 8 \\ 2 & 3 & 5 & & \end{pmatrix} \in \mathrm{Sy}(7,3,2)$$

corresponds to

$$(\lambda,\mu) = \left(\begin{array}{|c|c|c} \hline 4 & 5 & \multicolumn{1}{|c}{} \\ \hline 5 & 6 & \multicolumn{1}{|c|}{7} \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|} \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline \end{array} \right) \in \mathrm{Pa}(7,3,2).$$

Condition (14) is equivalent to the fact that all cells of λ and μ contain integers between 1 and n.

Example 5. Take $n=2$, $k=r=1$. The correspondence is given in the table below, where we have denoted by $\emptyset_i$ the empty partition regarded as the highest weight vector in $V(\Lambda_i)$:

S	(λ,μ)	S	(λ,μ)	S	(λ,μ)
$\begin{pmatrix}1&2\\1&\end{pmatrix}$	$(\emptyset_2,\emptyset_1)$	$\begin{pmatrix}1&3\\1&\end{pmatrix}$	$\left(\boxed{2},\emptyset_1\right)$	$\begin{pmatrix}1&2\\2&\end{pmatrix}$	$\left(\emptyset_2,\boxed{1}\right)$
$\begin{pmatrix}2&3\\1&\end{pmatrix}$	$\left(\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array},\emptyset_1\right)$	$\begin{pmatrix}1&3\\2&\end{pmatrix}$	$\left(\boxed{2},\boxed{1}\right)$	$\begin{pmatrix}1&2\\3&\end{pmatrix}$	$\left(\emptyset_2,\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}\right)$
$\begin{pmatrix}2&3\\2&\end{pmatrix}$	$\left(\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array},\boxed{1}\right)$	$\begin{pmatrix}1&3\\3&\end{pmatrix}$	$\left(\boxed{2},\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}\right)$	$\begin{pmatrix}2&3\\3&\end{pmatrix}$	$\left(\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array},\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}\right)$

2.7. Theorem 3, which is valid for any rank n, extends readily to the quantum algebra $U_v(\mathfrak{sl}_\infty)$ associated to the doubly infinite Dynkin diagram of type A_∞, as we shall now explain.

In this case the simple roots α_i and the fundamental weights Λ_i are indexed by $i \in \mathbb{Z}$. Let $\Lambda = \Lambda_{k+r} + \Lambda_k$ where $k \in \mathbb{Z}$ and $r \in \mathbb{N}$. The standard basis of $F(\Lambda) = V(\Lambda_{k+r}) \otimes V(\Lambda_k)$ is labelled by the set $\mathrm{Sy}(k,r)$ of all symbols $S = \binom{\beta}{\gamma}$, where

$$\beta = (\beta_i;\ i \in \mathbb{Z},\ i \leqslant k+r), \qquad \gamma = (\gamma_i;\ i \in \mathbb{Z},\ i \leqslant k)$$

are now semi-infinite increasing sequences satisfying $\beta_i = \gamma_i = i$ for $i \ll 0$. Such sequences can be regarded as the semi-infinite wedges arising in the classical construction of the fundamental representations of $\mathfrak{sl}_\infty$ by Kac and Peterson (see [13, §14.9]).

We have a trivial bijection $T : \mathrm{Sy}(k,r) \to \mathrm{Sy}(k+1,r)$ given by

$$T(\beta)_i = \beta_{i-1} + 1, \quad T(\gamma)_j = \gamma_{j-1} + 1 \quad (i \leqslant k+1+r,\ j \leqslant k+1).$$

This comes from the diagram automorphism $k \to k+1$ of A_∞, which implies that the representations $F(\Lambda_{k+r} + \Lambda_k)$ and $F(\Lambda_{k+r+1} + \Lambda_{k+1})$ of $U_v(\mathfrak{sl}_\infty)$ are essentially the same.

We also have a notion of standard symbol $S \in \mathrm{Sy}(k,r)$, namely when $\beta_i \leqslant \gamma_i$ $(i \leqslant k)$. As in the finite rank case, standard symbols label the crystal basis of the irreducible module $L(\Lambda)$. Like in 2.5, we can define the pairs of a standard symbol, and since β_i and γ_i coincide for i small enough, there is a finite number of them, say p. This yields the subset $\mathcal{C}(S) \subset \mathrm{Sy}(k,r)$ of cardinality 2^p obtained by permuting these pairs in S in all possible ways, and for $\Sigma \in \mathcal{C}(S)$ the integer $n(\Sigma)$ of pairs in which Σ differs from S.

Having adapted in this way the notation, Theorem 3 holds without modification for the algebra $U_v(\mathfrak{sl}_\infty)$.

Alternatively, following 2.6, we can replace $\mathrm{Sy}(k,r)$ by the set $\mathrm{Pa}(k,r)$ of all pairs (λ, μ), where

$$(\lambda_i;\ i \in \mathbb{Z},\ i \leqslant k+r), \qquad (\mu_i;\ i \in \mathbb{Z},\ i \leqslant k),$$

are weakly increasing sequences of nonnegative integers with finitely many nonzero elements. Equivalently, an element of $\mathrm{Pa}(k,r)$ can be regarded as a pair of Young diagrams with contents shifted by k and $k+r$, but now without the restriction (14) on the sizes of λ and μ. The correspondence $\mathrm{Sy}(k,r) \to \mathrm{Pa}(k,r)$ is again given by

$$\lambda_i = \beta_i - i, \quad \mu_j = \gamma_j - j \quad (i \leqslant k+r,\ j \leqslant k). \tag{15}$$

When indexed by $\mathrm{Pa}(k,r)$, the vectors of the standard basis $\mathcal{S}(\Lambda)$ will be denoted by $s_{(\lambda,\mu)}$. The elements $(\lambda,\mu) \in \mathrm{Pa}(k,r)$ corresponding to the standard symbols are characterized by

$$\lambda_i \leqslant \mu_i \quad (i \leqslant k). \tag{16}$$

Example 6. Take $r = 1$ and $k \in \mathbb{Z}$. The vectors of the standard basis $\mathcal{S}(\Lambda)$ of weight

$$\nu = \Lambda_k + \Lambda_{k+1} - \alpha_{k-1} - 2\alpha_k - 2\alpha_{k+1} - \alpha_{k+2}$$

are labelled by pairs $(\lambda, \mu) \in \mathrm{Pa}(k, 1)$ with $\lambda_i = \mu_i = 0$ for $i \leqslant k - 2$. Thus, ignoring the infinite common initial string of zeros, we can write for short $(\lambda, \mu) = ((\lambda_{k-1}, \lambda_k, \lambda_{k+1}), (\mu_{k-1}, \mu_k))$. The vectors of the canonical basis $\mathcal{B}(\Lambda)$ of weight ν are labelled by the following pairs (λ, μ):

$$\big((0,0,0),(3,3)\big),\ \big((0,0,1),(2,3)\big),\ \big((0,0,2),(2,2)\big),\ \big((0,1,1),(1,3)\big),\ \big((0,1,2),(1,2)\big).$$

By Theorem 3, the expansion of these vectors on the standard basis is given by the columns of the matrix:

$$\begin{array}{r|ccccc}
((0,0,0),(3,3)) & 1 & & & & \\
((0,0,1),(2,3)) & v & 1 & & & \\
((0,0,2),(2,2)) & & v & 1 & & \\
((0,1,1),(1,3)) & & v & & 1 & \\
((0,1,2),(1,2)) & v & v^2 & v & v & 1 \\
((1,1,1),(0,3)) & & & & v & \\
((0,2,2),(1,1)) & v^2 & & & & v \\
((1,1,2),(0,2)) & & & & v^2 & v \\
((1,2,2),(0,1)) & & & v & & v^2 \\
((2,2,2),(0,0)) & & & v^2 & &
\end{array}$$

3. Constructible characters, left cell representations, and families

In this section we review following [20,22] the definition of constructible characters and families for Iwahori–Hecke algebras.

3.1. Let (W, S) be a finite Coxeter group, and let $H(W)$ be the corresponding Iwahori–Hecke algebra over $\mathbb{C}(q^{1/2})$ with parameters $q^{k(s)}$. Here $k(s) \in \mathbb{N}$ and $k(s) = k(s')$ when s and s' are conjugate in W. The standard basis of $H(W)$ is denoted by $\{T_w \mid w \in W\}$.

The algebra $H(W)$ is semisimple, and its irreducible characters are in natural bijection with those of $\mathbb{C}W$ via the specialization map $\chi \in \mathrm{Irr}\, H(W) \mapsto \psi \in \mathrm{Irr}\, \mathbb{C}W$ given by

$$\psi(w) = \chi(T_w)|_{q=1}. \tag{17}$$

Let $\tau : H(W) \to \mathbb{C}(q^{1/2})$ be the symmetrizing trace defined by $\tau(T_w) = \delta_{w,1}$. Write

$$\tau = \sum_{\chi \in \mathrm{Irr}\, H(W)} c_\chi^{-1} \chi. \tag{18}$$

The Schur elements c_χ are known to be Laurent polynomials in $q^{1/2}$. Moreover, the lowest exponent of q in c_χ is of the form $-a_\chi$ for some nonnegative integer a_χ. Hence, using the bijection (17), we attach to each $\psi \in \mathrm{Irr}\, W$ an integer $a_\psi := a_\chi$ called the a-invariant of ψ.

3.2. Let $I \subset S$ and let (W_I, I) be the corresponding parabolic subgroup. Let $H(W_I)$ be its Iwahori–Hecke algebra with parameters $q^{k(s)}$ $(s \in I)$. By the above construction we can assign to $\xi \in \operatorname{Irr} W_I$ an a-invariant a_ξ. One can prove that if $\psi \in \operatorname{Irr} W$ occurs as an irreducible constituent of $\operatorname{Ind}_{W_I}^W \xi$, then $a_\psi \geqslant a_\xi$. This suggests the definition of the truncated induction, which is the linear map given for $\xi \in \operatorname{Irr} W_I$ by

$$\mathbf{j}_{W_I}^W(\xi) = \sum_{\psi \in \operatorname{Irr} W,\, a_\psi = a_\xi} \langle \chi, \operatorname{Ind}_{W_I}^W \xi \rangle \psi,$$

where $\langle \psi, \operatorname{Ind}_{W_I}^W \xi \rangle$ is the multiplicity of ψ in the induced character $\operatorname{Ind}_{W_I}^W \xi$. Note that the a-invariants and the map $\mathbf{j}_{W_I}^W$ depend on the choice of parameters $q^{k(s)}$.

3.3. Using the truncated induction, the constructible characters of W (or of $H(W)$) are defined inductively in the following way:

(1) If $W = \{1\}$, only the trivial character is constructible.
(2) If $W \neq \{1\}$, the set of constructible characters of W consists of all characters of the form

$$\mathbf{j}_{W_I}^W(\varphi) \quad \text{or} \quad \operatorname{sgn} \otimes \mathbf{j}_{W_I}^W(\varphi),$$

where sgn is the sign character of W, and φ is a constructible character of W_I for some proper subset I of S.

3.4. Using the Kazhdan–Lusztig basis of $H(W)$, Lusztig has defined a partition of W into subsets called left cells, and has associated to each left cell a representation of $H(W)$. In the equal parameter case, that is, when all $k(s)$ are equal, there is an identification theorem between constructible representations and left cell modules as follows.

Theorem 7 (Lusztig). *Assume that W is a finite Weyl group and that all $k(s)$ are equal. Then, the constructible characters coincide with the characters of the left cell representations of $H(W)$.*

Lusztig conjectures a similar result in the unequal parameter case.

3.5. In his study of irreducible characters of Chevalley groups $G(\mathbb{F}_q)$, Lusztig has obtained a division of the irreducible unipotent characters of $G(\mathbb{F}_q)$ into families. This induces for the corresponding finite Weyl group W a partition of $\operatorname{Irr} W$ into certain subsets also called families (see [20]).

Later, Lusztig has introduced the constructible characters of W in order to obtain a more direct way of describing the families of $\operatorname{Irr} W$. This goes as follows. Consider the graph $\mathcal{G}_W$ with set of vertices $\operatorname{Irr} W$ in which two irreducible characters χ and χ' are joined if and only if there exists a constructible character ψ such that χ and χ' both appear in the decomposition of ψ. Then the families of $\operatorname{Irr} W$ are the connected components of $\mathcal{G}_W$.

4. Constructible characters of type *B* and *D*

4.1. We denote by W_m the Weyl group of type B_m. The Coxeter generators $s_0, s_1, \ldots, s_{m-1}$ fall into two conjugacy classes $\{s_0\}$ and $\{s_1, \ldots, s_{m-1}\}$.

Let $r \in \mathbb{N}$. We are going to describe, following Lusztig [19,20,22], the constructible characters of W_m for the choice of parameters $k(s_0) = r$, $k(s_1) = \cdots = k(s_{m-1}) = 1$.

4.1.1. First, recall that the irreducible characters of W_m are labelled by the bipartitions (λ, μ) of m. Let $k \geqslant m$ and $n \geqslant m - 1 + k + r$. Then, by 2.6, $\mathrm{Pa}(n, k, r)$ contains all bipartitions of m. Hence the irreducible characters may be indexed by symbols in $\mathrm{Sy}(n, k, r)$. More precisely, they are parametrized by all symbols $S = \binom{\beta}{\gamma}$ such that

$$\sum_i \beta_i + \sum_j \gamma_j - \binom{k+1}{2} - \binom{k+r+1}{2} = m.$$

Let $\mathrm{Sy}(n, k, r, m)$ denote the subset of these symbols. For $S \in \mathrm{Sy}(n, k, r, m)$ we denote by χ_S the corresponding element of $\mathrm{Irr}\, W_m$.

4.1.2. For $S = \binom{\beta}{\gamma} \in \mathrm{Sy}(n, k, r, m)$ we define

$$Z = \{z_1 < z_2 < \cdots < z_M\} := (\beta \cup \gamma) - (\beta \cap \gamma), \qquad \widetilde{Z} := (Z; \beta \cap \gamma),$$

and we denote by π the map $S \mapsto \widetilde{Z}$. Note that $M + 2|\beta \cap \gamma| = 2k + r$, hence $M - r$ is even.

4.1.3. An involution ι of Z is called r-admissible if

(1) ι has r fixed points;
(2) if $M = r$ there is no further condition; otherwise one requires that there exist two consecutive elements $z, z' \in Z$ such that $\iota(z) = z'$ and that the restriction of ι to $Z - \{z, z'\}$ is an r-admissible involution of $Z - \{z, z'\}$.

4.1.4. We now fix $\widetilde{Z}$ and we consider the set $\pi^{-1}(\widetilde{Z})$ of all symbols $S \in \mathrm{Sy}(n, k, r, m)$ such that $\pi(S) = \widetilde{Z}$. Let ι be an r-admissible involution of Z. Define $\mathcal{C}(\widetilde{Z}, \iota)$ to be the set of all symbols $S = \binom{\beta}{\gamma} \in \pi^{-1}(\widetilde{Z})$ such that for each nontrivial orbit O of ι, β contains one element of O and γ the other. Clearly, we have that $\mathcal{C}(\widetilde{Z}, \iota)$ has cardinality $2^{(M-r)/2}$. The following is [22, 22.24].

Theorem 8 (Lusztig). *For every r-admissible involution ι of Z, the character*

$$\psi_\iota = \sum_{S \in \mathcal{C}(\widetilde{Z}, \iota)} \chi_S$$

is constructible. Moreover, all constructible characters of W_m arise in this way.

4.2. We now relate constructible characters to the canonical bases of Section 2. Recall that we have defined in 2.5 the set of pairs of a standard symbol S.

Lemma 9.

(a) *Let $S \in \pi^{-1}(\widetilde{Z})$ be a standard symbol. The involution ι of Z whose nontrivial orbits are the pairs of S is r-admissible.*
(b) *Conversely, let ι be an r-admissible involution of Z. Let $S = \binom{\beta}{\gamma} \in \pi^{-1}(\widetilde{Z})$ be the symbol such that, if $\iota(z) = z' > z$, then $z \in \beta$ and $z' \in \gamma$. Then S is standard and the pairs of S are the nontrivial orbits of ι.*

Proof. We shall use repeatedly the following easy remark:

(R) if $S = \binom{\beta}{\gamma} \in \pi^{-1}(\widetilde{Z})$ is a symbol and if $z \in \beta$ and $z' \in \gamma$ are equal, or are consecutive in Z with $z < z'$, then S is standard if and only if the symbol $S' = \binom{\beta'}{\gamma'}$ with $\beta' = \beta - \{z\}$ and $\gamma' = \gamma - \{z'\}$ is standard.

(a) We argue by induction on the number $(M - r)/2$ of pairs of S. If $M = r$ the claim is trivial, so assume that $M > r$. Recall the notation of 2.5. Let $l \geqslant 1$ be minimal such that $\gamma^l \neq \emptyset$. By the minimality of l, for all $0 < j < l$ and all $x \in \gamma - \beta$ we have $x - j \notin \beta - \gamma$. Let $z' \in \gamma^l$ and set $z = \psi(z') = z' - l$. Then z and z' are consecutive in Z. Indeed, otherwise there would exist $z'' \in Z$ with $z < z'' < z'$. But this would contradict the minimality of l, since if $z'' \in \beta$ we would have $z' - j \in \beta - \gamma$ for $j = z' - z'' < l$, and if $z'' \in \gamma$ we would have $z'' - j \in \beta - \gamma$ for $j = z'' - z < l$.

Let $S' = \binom{\beta'}{\gamma'}$ with $\beta' = \beta - \{z\}$ and $\gamma' = \gamma - \{z'\}$. By (R), S' is again standard, so by induction we know that the involution ι' of $Z - \{z, z'\}$ whose non trivial orbits are the pairs of S' is r-admissible. Thus, using the definition of r-admissibility we see that ι is r-admissible.

(b) If $M = r$, it follows immediately from (R) that S is standard. Moreover, S has no pair and the claim is proved. If $M > r$, by definition of r-admissibility, there exists a pair z, z' of consecutive elements of Z such that $\iota(z) = z'$, $z < z'$ and the restriction ι' of ι to $Z - \{z, z'\}$ is r-admissible. By induction the symbol $S' = \binom{\beta'}{\gamma'}$ built from ι' as in the statement of the lemma is standard and its pairs are the nontrivial orbits of ι'. Hence, by (R) the symbol $S = \binom{\beta}{\gamma}$ with $\beta = \beta' \cup \{z\}$ and $\gamma = \gamma' \cup \{z'\}$ is standard. Moreover, since z and z' are consecutive in Z, they form a pair in S and the other pairs are the pairs of S'. Thus, the pairs of S are the nontrivial orbits of ι. □

It follows from Theorem 8 and Lemma 9 that the constructible characters of W_m can be parametrized by the set $\mathrm{SSy}(n, k, r, m)$ of standard symbols of $\mathrm{Sy}(n, k, r, m)$. Moreover, if $S \in \pi^{-1}(\widetilde{Z})$ is standard and if ι is the r-admissible involution corresponding to S as in Lemma 9, we have

$$\mathcal{C}(S) = \mathcal{C}(\widetilde{Z}, \iota). \tag{19}$$

Therefore, comparing Theorems 8 and 3, we obtain immediately the main result of this section:

Theorem 10. *Let $k \geqslant m$ and $n \geqslant m-1+k+r$. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and consider the canonical basis*

$$\mathcal{B}_m = \{ b_S \mid S \in \mathrm{SSy}(n,k,r,m) \}$$

of the principal degree m component of the irreducible $U_v(\mathfrak{g})$-module $V(\Lambda_{k+r} + \Lambda_k)$. For $b_S \in \mathcal{B}_m$ write as in Theorem 3

$$\Phi(b_S) = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_\Sigma .$$

Then the set of constructible characters of W_m for the parameters $q^r, q, \ldots, q$ is obtained "by specializing $v \mapsto 1$ in $\Phi(\mathcal{B}_m)$," namely it consists of the

$$\psi_S = \sum_{\Sigma \in \mathcal{C}(S)} \chi_\Sigma \quad \big(S \in \mathrm{SSy}(n,k,r,m) \big) .$$

Taking into account the remarks of 2.7, one can reformulate Theorem 10 as a statement about $U_v(\mathfrak{sl}_\infty)$, as we did in the introduction. It is in fact more natural, since in this way one gets rid of k and n which are irrelevant as long as they are large enough. This amounts to replace symbols by certain inductive limits of them, as in [22, 22.7].

4.3. Recall from 3.5 the partition of $\mathrm{Irr}\, W_m$ given by the connected components of the graph $\mathcal{G}_{W_m}$. Lusztig has shown [22, 22.2, 23.1] that χ_S and $\chi_{S'}$ belong to the same class if and only if the symbols $S, S' \in \mathrm{Sy}(n,k,r,m)$ have the same content, that is, the same elements with the same multiplicities. Using Eq. (5), it is easy to deduce the following alternative description.

Corollary 11. *χ_S and $\chi_{S'}$ belong to the same component of $\mathcal{G}_{W_m}$ if and only if u_S and $u_{S'}$ belong to the same weight space of the $U_v(\mathfrak{sl}_\infty)$-module $F(\Lambda)$.*

4.4. Let W'_m denote the Weyl group of type D_m. The irreducible characters of W'_m are indexed by unordered bipartitions $\{\lambda, \mu\}$ of m, with the convention that pairs of the form $\{\lambda, \lambda\}$ label two irreducible characters. Taking k and n large enough, we can equivalently index the elements of $\mathrm{Irr}\, W'_m$ by the orbits on the set of symbols $\mathrm{Sy}(n,k,0,m)$ of the involution $\sharp$ exchanging the two rows of a symbol, except that one point orbits label in fact two characters. We denote the elements of $\mathrm{Irr}\, W'_m$ by $\chi_S = \chi_{S^\sharp}$ in the first case, and by χ_S^I, χ_S^{II} in the second case.

Note that all generators s of W'_m are conjugate, hence all parameters $k(s)$ of the Iwahori–Hecke algebra $H(W'_m)$ must be equal.

4.4.1. Retain the notation of 4.1.2–4.1.4, with $r = 0$.

Theorem 12 (Lusztig). *Suppose that $Z \neq \emptyset$ and let ι be a 0-admissible involution of Z. Then the character*

$$\psi_\iota = \frac{1}{2} \sum_{S \in \mathcal{C}(\iota)} \chi_S$$

is constructible. On the other hand, if $Z = \emptyset$, the characters χ_S^I and χ_S^{II} are both constructible. Moreover, all constructible characters of W'_m arise in this way.

4.4.2. It follows from Theorems 12, 3, and Lemma 9 that:

Theorem 13. *Let $k \geqslant m$ and $n \geqslant m - 1 + k$. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and consider the canonical basis*

$$\mathcal{B}_m = \{ b_S \mid S \in \mathrm{SSy}(n, k, 0, m) \}$$

of the principal degree m component of the irreducible $U_v(\mathfrak{g})$-module $V(2\Lambda_k)$. For $b_S \in \mathcal{B}_m$ write as in Theorem 3

$$\Phi(b_S) = \sum_{\Sigma \in \mathcal{C}(S)} v^{n(\Sigma)} u_\Sigma.$$

(*Note that if $S^\sharp = S$, then $\Phi(b_S) = u_S$.*) *Then the set of constructible characters of W'_m is obtained by "specializing $v \mapsto 1$ in $\Phi(\mathcal{B}_m)$," namely it consists of the characters*

$$\psi_S = \frac{1}{2} \sum_{\Sigma \in \mathcal{C}(S)} \chi_\Sigma,$$

for $S^\sharp \neq S$, and of the characters χ_S^I and χ_S^{II} for $S^\sharp = S$.

5. Relation with results of Gyoja

5.1. Let R be an integral domain and assume that q and Q are invertible elements of R. We shall denote by $H_m(Q; q)_R$ the Iwahori–Hecke algebra of type B_m over R with parameters Q for the special generator T_0 and q for the remaining generators T_i $(1 \leqslant i \leqslant m - 1)$.

We denote by $S_R^{(\lambda,\mu)}$ the Specht $H_m(Q; q)_R$-module corresponding to a bipartition (λ, μ) of m [10]. Let $\{D_R^{(\lambda,\mu)}\}$ be a complete set of non-isomorphic simple $H_m(Q; q)_R$-modules parametrized as in [2] by the Kleshchev bipartitions (λ, μ) of m. Let $P_R^{(\lambda,\mu)}$ be the projective indecomposable $H_m(Q; q)_R$-module corresponding to a Kleshchev bipartition (λ, μ), that is, the projective cover of $D_R^{(\lambda,\mu)}$.

Note that when $H_m(Q;q)_R$ is semisimple, every bipartition is a Kleshchev bipartition. On the other hand, in the case where $Q = -q^r$ and q has infinite multiplicative order, the Kleshchev bipartitions are precisely the elements of $\mathrm{Pa}(k,r)$ corresponding to standard symbols described in Section 2, Eq. (16).

5.2. Let $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$, where $q^{1/2}$ is an indeterminate. We consider the following modular system $(K, \mathcal{O}, \Bbbk)$. Let $\mathcal{A}_2$ be the localization of $\mathcal{A}$ at the prime ideal $2\mathcal{A}$, let $\mathcal{O}$ be its completion, K the fractional field of $\mathcal{O}$, and $\Bbbk = \mathbb{F}_2(q^{1/2})$ the residue field of $\mathcal{O}$. Then, K contains $\mathbb{Z}[q^{1/2}]$ and q has clearly infinite order in $\mathcal{O}$ and $\Bbbk$.

For a finite-dimensional associative algebra $\mathfrak{A}$ let $R_0(\mathfrak{A})$ be the Grothendieck group of the category mod-$\mathfrak{A}$. We denote by $[M]$ the class of M in $R_0(\mathfrak{A})$.

Since $-q^r = q^r$ in $\Bbbk$, the natural map $\mathcal{A} \to \Bbbk$ gives rise to two decomposition maps

$$\mathbf{d}^2_{r,+} : R_0\big(H_m\big(q^r;q\big)_K\big) \to R_0\big(H_m\big(q^r;q\big)_{\Bbbk}\big),$$
$$\mathbf{d}^2_{r,-} : R_0\big(H_m\big(-q^r;q\big)_K\big) \to R_0\big(H_m\big(q^r;q\big)_{\Bbbk}\big).$$

Note that $H_m(q^r;q)_K$ is semisimple but $H_m(-q^r;q)_K$ is not semisimple in general.

Let $\mathbb{F}$ be a field containing $\mathbb{Z}[q^{1/2}, Q^{1/2}]$ where Q is an indeterminate. Then $H_m(Q;q)_{\mathbb{F}}$ is semisimple. In particular, we know that $R_0(H_m(Q;q)_{\mathbb{F}})$ is isomorphic to $R_0(H_m(q^r;q)_K)$. We denote this isomorphism by $\mathbf{i}$. Also, we denote by $\mathbf{d}$ the decomposition map

$$\mathbf{d} : R_0\big(H_m(Q;q)_{\mathbb{F}}\big) \to R_0\big(H_m\big(-q^r;q\big)_K\big).$$

So, we have the following commutative diagram:

$$\begin{array}{ccc} R_0\big(H_m(Q;q)_{\mathbb{F}}\big) & \xrightarrow{\ \mathbf{i}\ } & R_0\big(H_m\big(q^r;q\big)_K\big) \\ \Big\downarrow{\scriptstyle \mathbf{d}} & & \Big\downarrow{\scriptstyle \mathbf{d}^2_{r,+}} \\ R_0\big(H_m\big(-q^r;q\big)_K\big) & \xrightarrow{\ \mathbf{d}^2_{r,-}\ } & R_0\big(H_m\big(q^r;q\big)_{\Bbbk}\big). \end{array}$$

Combining Proposition 4 and [4, Corollary 3.7], we deduce immediately that

Proposition 14. *For every Kleshchev bipartition* (λ, μ),

$$\mathbf{d}^2_{r,-}\big(\big[P_K^{(\lambda,\mu)}\big]\big) = \big[P_{\Bbbk}^{(\lambda,\mu)}\big].$$

Hence, the decomposition matrices of $\mathbf{d}^2_{r,+}$ *and* $\mathbf{d}$ *are the same and preserve their canonical indices.*

5.3. By Ariki's theorem [1], the decomposition matrix of $\mathbf{d}$ is given by the canonical basis of the $U_v(\mathfrak{sl}_\infty)$-module $V(\Lambda_{r+k} + \Lambda_k)$ calculated in Section 2.

More precisely, write again $\Lambda = \Lambda_{r+k} + \Lambda_k$. Let $\{s_{(\lambda,\mu)}\}$ be the standard basis of the degree m component of $F(\Lambda)$, and for a Kleshchev bipartition (λ, μ) let $b_{(\lambda,\mu)}$ be the element of the canonical basis of $V(\Lambda) \subset F(\Lambda)$ such that $b_{(\lambda,\mu)} \equiv s_{(\lambda,\mu)} \bmod vL$. Let $\overline{F(\Lambda)}$ and $\overline{V(\Lambda)}$ denote the $\mathfrak{sl}_\infty$-modules obtained by specializing v to 1 in $F(\Lambda)$ and $V(\Lambda)$ using the $\mathbb{Z}[v, v^{-1}]$-lattices spanned by $\{s_{(\lambda,\mu)}\}$ and $\{b_{(\lambda,\mu)}\}$, respectively. By abuse of notation we continue to write $\{s_{(\lambda,\mu)}\}$ and $\{b_{(\lambda,\mu)}\}$ for the images of these bases in $\overline{F(\Lambda)}$ and $\overline{V(\Lambda)}$.

Then the complexified Grothendieck group $R_{\mathbb{C}}(H_m(Q; q)_{\mathbb{F}})$ is isomorphic to the degree m component of $\overline{F(\Lambda)}$, the basis $\{[S_{\mathbb{F}}^{(\lambda,\mu)}]\}$ being mapped to $\{s_{(\lambda,\mu)}\}$. Similarly, $R_{\mathbb{C}}(H_m(-q^r; q)_K)$ is isomorphic to the degree m component of $\overline{V(\Lambda)}$, the basis $\{[P_K^{(\lambda,\mu)}]\}$ being mapped to $\{b_{(\lambda,\mu)}\}$. Finally, the map $\mathbf{d}$ corresponds to the natural homomorphism of $\mathfrak{sl}_\infty$-modules from $\overline{F(\Lambda)}$ to $\overline{V(\Lambda)}$.

5.4. In the equal parameter case, the decomposition map $\mathbf{d}^2_{1,+}$ was first investigated by Gyoja [11] in terms of Kazhdan–Lusztig cell representations. Taking into account Theorem 7, he proved the following theorem.

Theorem 15 (Gyoja). *The decomposition matrix of $\mathbf{d}^2_{1,+}$ is equal to the matrix whose columns give the expansion of the constructible characters of $H_m(q; q)$ in terms of the irreducible ones.*

Thus we see that using Gyoja's theorem, Ariki's theorem, and Proposition 14, we obtain a more conceptual proof of Theorem 10 in the equal parameter case, which does not use explicit combinatorial calculations.

On the other hand, using our approach and Ariki's theorem, we obtain that Theorem 15 also holds in the unequal parameter case:

Theorem 16. *The decomposition matrix of $\mathbf{d}^2_{r,+}$ is equal to the matrix whose columns give the expansion of the constructible characters of $H_m(q^r; q)$ in terms of the irreducible ones.*

Note that recently, Bonnafé and Iancu [5] have determined the left cells of $H_m(q^r; q)$ in the asymptotic case $r \gg m$. They proved that in this case the characters supported by the left cells are irreducible and coincide with the constructible characters.

6. Cyclotomic algebras

The results of Gyoja [11] together with some recent work of Rouquier [23] provide a way of generalizing the definition of families of characters of a Weyl group to complex reflection groups and their cyclotomic Hecke algebras. For the groups $W = G(d, 1, m) = \mathbb{Z}_d \wr \mathfrak{S}_m$, these generalized families have been explicitly described by Broué and Kim [6].

6.1. Let $d, m \in \mathbb{N}^*$ and $\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{Z}^d$. Let $\mathbb{F}$ be a field containing $\mathcal{A}$ and a primitive dth root of unity ζ. We denote by $H_{(d,m,\mathbf{r})}$ the unital associative algebra over $\mathbb{F}$ generated by $T_0, \dots, T_{m-1}$ subject to the relations

$$(T_i + 1)(T_i - q) = 0 \quad \text{for } 1 \leqslant i \leqslant m-1,$$
$$\prod_{j=0}^{d-1} \bigl(T_0 - \zeta^j q^{r_j}\bigr) = 0, \qquad T_0T_1T_0T_1 = T_1T_0T_1T_0,$$
$$T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i \quad \text{for } 1 \leqslant i \leqslant m-2,$$
$$T_iT_j = T_jT_i \quad \text{for } 0 \leqslant i < j-1 \leqslant m-2.$$

Note that the Hecke algebra $H_m(q^r; q)_{\mathbb{F}}$ of type B_m is isomorphic to $H_{(2,m,\mathbf{r})}$ with $\mathbf{r} = (r+k, k)$ for any $k \in \mathbb{Z}$. The algebras $H_{(d,m,\mathbf{r})}$ have been introduced independently by Ariki and Koike [3] and Broué and Malle [7].

The algebra $H_{(d,m,\mathbf{r})}$ is semisimple and its irreducible modules are naturally labelled by d-tuples of partitions $\underline{\lambda} = (\lambda^0, \dots, \lambda^{(d-1)})$ with $\sum_j |\lambda^{(j)}| = m$ [3]. Denote the irreducible character of $H_{(d,m,\mathbf{r})}$ indexed by $\underline{\lambda}$ by $\chi_{\underline{\lambda}}$. To a d-partition $\underline{\lambda}$, one can associate a d-tuple of Young diagrams whose cells are filled with their content shifted by the parameters r_j. More precisely, the cell with row number i and column number j belonging to the sth Young diagram is filled with the integer $i - j + r_s$. Reading all the cells of $\underline{\lambda}$, one obtains a multiset of integers called the content of $\underline{\lambda}$, that we shall denote by $c(\underline{\lambda})$.

Theorem 17 (Broué–Kim). *The characters $\chi_{\underline{\lambda}}$ and $\chi_{\underline{\mu}}$ belong to the same Rouquier family of* $\operatorname{Irr} H_{(d,m,\mathbf{r})}$ *if and only if $c(\underline{\lambda}) = c(\underline{\mu})$.*

6.2. To the same data we associate the $\mathfrak{sl}_\infty$-weight $\Lambda = \sum_{j=0}^{d-1} \Lambda_{r_j}$ and the $U_v(\mathfrak{sl}_\infty)$-modules $F(\Lambda)$ and $V(\Lambda)$. Here, $V(\Lambda)$ is the irreducible integrable module with highest weight Λ and $F(\Lambda) = \bigotimes_{j=0}^{d-1} V(\Lambda_{r_j})$. Moreover, as before for $d = 2$, the standard basis of $F(\Lambda)$ is indexed in a natural way by all d-tuples of Young diagrams with contents shifted by $r_0, \dots, r_{d-1}$. We shall denote this basis by $\{s_{\underline{\lambda}}\}$. The $U_v(\mathfrak{sl}_\infty)$ weight of $s_{\underline{\lambda}}$ is equal to

$$\operatorname{wt} s_{\underline{\lambda}} = \Lambda - \sum_{j \in \mathbb{Z}} c_j \alpha_j,$$

where c_j denotes the number of elements equal to j in $c(\underline{\lambda})$ and the α_j are the simple roots of $\mathfrak{sl}_\infty$. From this and Theorem 17 it is easily deduced that

Proposition 18. *The characters $\chi_{\underline{\lambda}}$ and $\chi_{\underline{\mu}}$ belong to the same Rouquier family of* $\operatorname{Irr} H_{(d,m,\mathbf{r})}$ *if and only if $s_{\underline{\lambda}}$ and $s_{\underline{\mu}}$ belong to the same weight space of $F(\Lambda)$.*

6.3. By analogy with the case $d = 2$, one can then consider the canonical basis of $V(\Lambda)$ and its expansion on the standard basis of $F(\Lambda)$, which can be calculated via a simple algorithm [18]. Attached to each element

$$b = \sum_{\underline{\lambda}} \alpha^b_{\underline{\lambda}}(v) u_{\underline{\lambda}}$$

of this basis, we have a certain character of $H_{(d,m,\mathbf{r})}$:

$$\psi = \sum_{\underline{\lambda}} \alpha^b_{\underline{\lambda}}(1) \chi_{\underline{\lambda}}.$$

It would be interesting to understand whether these characters are good analogues for $G(d, 1, m)$ of the constructible or left cell characters of the Weyl groups of type B_m.

Acknowledgments

We are grateful to M. Broué, M. Geck, and R. Rouquier for several stimulating discussions.

This work was done while the second author was visiting IHES (France) from April 2002 to March 2003. He is very grateful to IHES and its staff for their kind hospitality and support.

References

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(n, 1, m)$, J. Math. Kyoto Univ. 36 (1996) 789–808.
[2] S. Ariki, On the classification of simple modules for cyclotomic Hecke algebras of type $G(m, 1, n)$ and Kleshchev multipartitions, Osaka J. Math. 38 (2001) 827–837.
[3] S. Ariki, K. Koike, A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations, Adv. Math. 106 (1994) 216–243.
[4] S. Ariki, A. Mathas, The representation type of Hecke algebras of type B, Adv. Math. 181 (2004) 134–159.
[5] C. Bonnafé, L. Iancu, Left cells in type B_n with unequal parameters, Represent. Theory 7 (2003) 587–609.
[6] M. Broué, S. Kim, Familles de caractères des algèbres de Hecke cyclotomiques, Adv. Math. 172 (2002) 53–136.
[7] M. Broué, G. Malle, Zyklotomische Heckealgebren, Astérisque 212 (1993) 119–189.
[8] J. Brundan, Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$, J. Amer. Math. Soc. 16 (2002) 185–231.
[9] J. Chuang, K.M. Tan, Some canonical basis vectors in the basic $U_q(\widehat{\mathfrak{sl}}_n)$-module, J. Algebra 248 (2002) 765–779.
[10] R. Dipper, G.D. James, Representations of Hecke algebras of type B_n, J. Algebra 146 (1992) 454–481.
[11] A. Gyoja, Cells and modular representations of Hecke algebras, Osaka J. Math. 33 (1996) 307–341.
[12] J.C. Jantzen, Lectures on Quantum Groups, in: Grad. Stud. Math., vol. 6, Amer. Math. Soc., Providence, RI, 1996.
[13] V.G. Kac, Infinite Dimensional Lie Algebras, third ed., Cambridge Univ. Press, Cambridge, UK, 1990.
[14] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, Comm. Math. Phys. 133 (1990) 249–260.

[15] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991) 465–516.
[16] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, J. Algebra 165 (1994) 295–345.
[17] B. Leclerc, H. Miyachi, Some closed formulas for canonical bases of Fock spaces, Represent. Theory 6 (2002) 290–312.
[18] B. Leclerc, P. Toffin, A simple algorithm for computing the global crystal basis of an irreducible $U_q(\mathfrak{sl}_n)$-module, Internat. J. Algebra Comput. 10 (2000) 191–208.
[19] G. Lusztig, A class of irreducible representations of a Weyl group II, Indag. Math. 44 (1982) 219–226.
[20] G. Lusztig, Characters of Reductive Groups over a Finite Field, in: Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1984.
[21] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990) 447–498.
[22] G. Lusztig, Hecke Algebras with Unequal Parameters, in: CRM Monogr. Ser., vol. 18, Amer. Math. Soc., Providence, RI, 2003.
[23] R. Rouquier, Familles et blocs d'algèbres de Hecke, C. R. Acad. Sci. 329 (1999) 1037–1042.