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Invariance properties of a general bond-pricing equation

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Abstract

We perform the group classification of a bond-pricing partial differential equation of mathematical finance to discover the combinations of arbitrary parameters that allow the partial differential equation to admit a nontrivial symmetry Lie algebra. As a result of the group classification we propose "natural" values for the arbitrary parameters in the partial differential equation, some of which validate the choices of parameters in such classical models as that of Vasicek and Cox–Ingersoll–Ross. For each set of these natural parameter values we compute the admitted Lie point symmetries, identify the corresponding symmetry Lie algebra and solve the partial differential equation.

Keywords: Lie symmetry analysis; Group classification; Zero-coupon bond; Invariant solution; Interest rate model; Partial differential equation

1. Introduction

In recent years a number of studies have used Lie symmetries in finance to provide insight into the structure of associated partial differential equations. One of the earliest studies was done by Gazizov and Ibragimov [7], who dealt with the classical Black–Scholes–Merton equation. Since

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that time this approach has been applied to other partial differential equations from finance, for example Lo and Hui [15] and Carr et al. [4].

In this paper we present the *group classification* of the linear second-order parabolic partial differential equation

$$\frac{\partial u}{\partial t}+\frac{1}{2}\rho^2x^{2\gamma}\frac{\partial^2 u}{\partial x^2}+\left(\alpha+\beta x-\lambda\rho x^{\gamma}\right)\frac{\partial u}{\partial x}-xu=0, \tag{1.1}$$

where α, β, γ, ρ and λ are constants.[1] In financial mathematics Eq. (1.1), with the terminal condition

$$u(x,T)=1, \tag{1.2}$$

models the value, $u(x,t)$, of a zero-coupon bond with expiry at $t=T$ when the short-term interest rate (also called spot rate) is governed by a stochastic process of the form

$$dx=(\alpha+\beta x)\,dt+\rho x^{\gamma}\,dW(t), \tag{1.3}$$

where $W(t)$ is a standard Wiener process [13]. The parameter λ in (1.1) is called the *market price of risk*[2]; it is the extra increase in expected instantaneous rate of return on a bond per additional unit of risk. The stochastic differential equation (1.3) is a variant of a one-dimensional generalised square root process (GSR). Craddock and Platen [6] in 2004 applied symmetry group methods to finding fundamental solutions to a class of partial differential equations that had been derived from the GSR process. This process is associated with the minimum market model of Platen [19].

The study of (1.1) is important because many one-factor interest rate models that have been proposed in the literature can be nested in (1.3) with the precise forms of the arbitrary elements depending upon the particular model under consideration. For example the Vasicek [23] and Cox–Ingersoll–Ross (CIR) [5] models correspond to $\gamma=0$ and $\gamma=1/2$, respectively.

One would imagine that the values of the arbitrary parameters in (1.1) are determined through experimentation or by making some simplifying assumptions. Obviously we would like if possible to select values for these parameters that make the model tractable. From the point of view of symmetry methods (also called *group-theoretic modelling*) this means choosing parameter values that result in (1.1) possessing a maximal symmetry Lie algebra.

We know that Eq. (1.1), being a linear partial differential equation, admits $u\partial_u$ and $\phi\partial_u$, where $\phi(x,t)$ is any solution of (1.1). If, besides these two symmetries, (1.1) admits five additional symmetries, then it can be reduced to the standard heat equation [14], for which the solution is well known (see [21] and [12] for example).

Our goal in this paper is to determine systematically values of the arbitrary elements that make Eq. (1.1) interesting in the sense that the corresponding model equation admits a “large” symmetry Lie algebra and is consequently tractable. The problem of determining such elements is known as the *group classification problem* and the first systematic investigation of the problem was carried out by Sophus Lie [14] for general linear second-order partial differential equations with two independent variables.

[1] Typically α, β, γ and ρ are taken to be nonnegative, but in our study we remove this restriction and (as we explain later) use symmetry analysis to assign to these parameters (any) values that make the model tractable.

[2] In general $\lambda=\lambda(x,t)$, i.e., it depends upon both x and t.

The rest of this paper is organised as follows. In Section 2 we derive two equations, one for $\gamma \neq 1$ and the other for $\gamma = 1$, upon which the group classification of (1.1) depends. In Section 3 we present results of the group classification scheme and in Section 5 we conclude and make some observations.

2. Determination of classification equations of (1.1)

In this section we derive two equations, one for $\gamma \neq 1$ and the other for $\gamma = 1$, upon which the group classification of (1.1) depends. We use Lie's classical method (described, for example, in [3] and [18]) to determine the conditions that the constants α, β, γ, ρ and λ must fulfil for (1.1) to possess a "nontrivial" symmetry Lie algebra. We firstly introduce the idea of a symmetry of a differential equation.

Consider a general second-order partial differential equation,

$$\Delta(x, t, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0, \tag{2.1}$$

in one dependent variable u and two independent variables (x, t). Infinitesimal transformations,

$$\begin{aligned}
\tilde{t} &= f(t, x, u, \varepsilon) = t + \varepsilon\tau(t, x, u) + O(\varepsilon^2),\\
\tilde{x} &= g(t, x, u, \varepsilon) = x + \varepsilon\xi(t, x, u) + O(\varepsilon^2),\\
\tilde{u} &= h(t, x, u, \varepsilon) = u + \varepsilon\eta(t, x, u) + O(\varepsilon^2),
\end{aligned} \tag{2.2}$$

depending on a continuous parameter, ε, are said to be a Lie point symmetry group of Eq. (2.1) if the equation has the same form in the new variables, $\tilde{t}$, $\tilde{x}$ and $\tilde{u}$, as in the original variables. The set of all such transformations forms a continuous group called the *Lie group*, $\mathcal{G}$, of (2.1). According to Lie's theory the construction of the symmetry group, $\mathcal{G}$, is equivalent to the determination of the associated operator,

$$G = \xi(t, x, u)\partial_x + \tau(t, x, u)\partial_t + \eta(t, x, u)\partial_u, \tag{2.3}$$

called in the literature the *generator* or *infinitesimal symmetry* (or simply *symmetry*) of the group $\mathcal{G}$. The requirement that (2.1) be invariant under the transformations (2.2) is determined by the *invariance condition*

$$G^{[2]}(\Delta)\big|_{\Delta=0} = 0, \tag{2.4}$$

where $G^{[2]}$ is the second-prolongation formula of G and is given by

$$G^{[2]} = G + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \eta^{tt} \frac{\partial}{\partial u_{tt}}$$

with $\eta^t, \eta^x, \eta^{xx}, \ldots$ given in the context of the transformations specified in [3]. The invariance condition and the fact that the derivatives of u are independent leads to a set of linear partial differential equations in τ, ξ and η called the *determining equations*, which are then solved to give the symmetries admitted by (2.1) [3,18]. Many computer algebra packages have been developed to find symmetries of differential equations [10].

2.1. Classification equation for $\gamma \neq 1$

Suppose that (1.1) admits a symmetry of the form (2.3). Application of the invariance condition, (2.4), to Eq. (1.1) yields the following determining equations[3]:

$$\tau_x = \tau_u = 0, \tag{2.5}$$

$$\tau_{uu} = 0, \tag{2.6}$$

$$\rho^2 x^{2\gamma} \tau_{xu} - 2\xi_u = 0, \tag{2.7}$$

$$\xi_{uu} - \left(\alpha + \beta x - \lambda\rho x^\gamma\right)\tau_{uu} = 0, \tag{2.8}$$

$$2\xi_{xu} - \eta_{uu} + xu\tau_{uu} - 2\left(\alpha + \beta x - \lambda\rho x^\gamma\right)\tau_{xu} = 0, \tag{2.9}$$

$$4\gamma\xi - 4\xi_{xx} + 2x\tau_t + 2x^2 u\tau_u + \rho^2 x^{1+2\gamma}\tau_{xx} + 2x\left(\alpha + \beta x - \lambda\rho x^\gamma\right)\tau_x = 0, \tag{2.10}$$

$$\begin{aligned} &-2\eta_t + 2u\xi + 2x\eta - 2xu\eta_u + 2xu\tau_t + 2x^2u^2\tau_u - \rho^2 x^{2\gamma}\eta_{xx} + \rho^2 x^{1+2\gamma} u\tau_{xx} \\ &\quad - 2\left(\alpha + \beta x - \lambda\rho x^\gamma\right)\eta_x + 2xu\left(\alpha + \beta x - \lambda\rho x^\gamma\right)\tau_x = 0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &-2x\xi_t - 2x^2 u\xi_u + 2\rho^2 x^{1+2\gamma}\eta_{xu} - \rho^2 x^{1+2\gamma}\xi_{xx} \\ &\quad + \left(2x\tau_t - 2x\xi_x + 2x^2u\tau_u + \rho^2 x^{1+2\gamma}\tau_{xx}\right)\left(\alpha + \beta x - \lambda\rho x^\gamma\right) \\ &\quad - 2\rho^2 x^{2+2\gamma} u\tau_{xu} + 2x\left(\alpha + \beta x - \lambda\rho x^\gamma\right)^2\tau_x + 2\left(\beta x - \gamma\lambda\rho x^\gamma\right)\xi = 0. \end{aligned} \tag{2.12}$$

From (2.5) we deduce that

$$\tau(x,t,u) = \tau(t), \tag{2.13}$$

whereby (2.7) is reduced to

$$\xi_u = 0 \tag{2.14}$$

and hence

$$\xi(x,t,u) = \xi(x,t). \tag{2.15}$$

If we now substitute (2.13) and (2.15) into (2.9), we find that

$$\eta_{uu} = 0 \tag{2.16}$$

and therefore

$$\eta(x,t,u) = C_1(x,t) + uC_2(x,t) \tag{2.17}$$

for some functions C_1 and C_2. If we substitute (2.13), (2.15) and (2.17) in (2.10) and solve the resulting partial differential equation for ξ, we find that

[3] We used program LIE [9] to generate the determining equations (2.5)–(2.12) and MATHEMATICA [24] to solve them.

$$\xi(x,t)=x^{\gamma}C_3(t)+\frac{x\dot{\tau}(t)}{2(1-\gamma)}, \quad \gamma\neq 1, \tag{2.18}$$

where C_3 is another function. If we now substitute (2.13), (2.15) and (2.17) into (2.11) and solve the resulting partial differential equation for C_2, we have

$$\begin{aligned}C_2(x,t)=\frac{1}{4\rho^2(\gamma-1)^2}\big[&2(\gamma-1)^2x^{-(1+\gamma)}\big(\gamma\rho^2x^{2\gamma}-2\alpha x-2\beta x^2\big)C_3(t)\\&-4(\gamma-1)x^{1-\gamma}\dot{C}_3(t)+2(\gamma-1)x^{1-2\gamma}\big(\alpha+\beta x-\lambda\rho x^{\gamma}\big)\dot{\tau}(t)\\&+x^{2(1-\gamma)}\ddot{\tau}(t)\big]+C_4(t),\end{aligned} \tag{2.19}$$

where C_4 is yet another arbitrary function of integration. If we now suppose that C_1 is a solution of (1.1), then substitution of (2.13), (2.17) (with C_2 given by (2.19)) and (2.18) into (2.11) gives what we call the *classification equation* of (1.1) for $\gamma\neq 1$:

$$\begin{aligned}&h_0(t)+x^{-2+3\gamma}h_1(t)+x^{-\gamma}h_2(t)+x^{-1+2\gamma}h_3(t)+x^{\gamma}h_4(t)+x^2h_5(t)\\&\quad+x^{2(1-\gamma)}h_6(t)+x^{2-\gamma}h_7(t)+x^{1+\gamma}h_8(t)+xh_9(t)+x^{1-2\gamma}h_{10}(t)\\&\quad+x^{1-\gamma}h_{11}(t)+x^{3-2\gamma}h_{12}(t)+x^{-1+\gamma}h_{13}(t)=0,\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}h_0(t)&=-8\alpha(\gamma-1)^2\gamma\lambda\rho C_3(t)+4\alpha(\gamma-1)\gamma(2\gamma-1)\rho^2\dot{\tau}(t),\\h_1(t)&=2(\gamma-2)(\gamma-1)^3\gamma\rho^4C_3(t),\\h_2(t)&=8\alpha^2(\gamma-1)^2\gamma C_3(t),\\h_3(t)&=-4(\gamma-1)^3\gamma\lambda\rho^3C_3(t),\\h_4(t)&=-2(\gamma-1)^2\gamma\lambda\rho^3\dot{\tau}(t),\\h_5(t)&=-4(\gamma-1)(2\gamma-3)\rho^2\dot{\tau}(t),\\h_6(t)&=-4\alpha\beta(\gamma-1)(4\gamma-3)\dot{\tau}(t),\\h_7(t)&=8\beta^2(\gamma-1)^3C_3(t)+12\beta(\gamma-1)^2\lambda\rho\dot{\tau}(t)-8(\gamma-1)\ddot{C}_3(t),\\h_8(t)&=-8(\gamma-1)^2\rho^2C_3(t),\\h_9(t)&=-8\beta(\gamma-1)^3\lambda\rho C_3(t)-8(\gamma-1)^2\lambda\rho\dot{C}_3(t)+8(\gamma-1)^2\rho^2\dot{C}_4(t)\\&\quad+(\gamma-1)^2\big(\beta(8\gamma-4)-4\lambda^2\big)\rho^2\dot{\tau}(t)+2(\gamma-1)(2\gamma-1)\rho^2\ddot{\tau}(t),\\h_{10}(t)&=-4\alpha^2(\gamma-1)(2\gamma-1)\dot{\tau}(t),\\h_{11}(t)&=8\alpha\beta(\gamma-1)^2(2\gamma-1)C_3(t)+4\alpha(\gamma-1)(3\gamma-2)\lambda\rho\dot{\tau}(t),\\h_{12}(t)&=-8\beta^2(\gamma-1)^2\dot{\tau}(t)+2\dddot{\tau}(t),\\h_{13}(t)&=-8\alpha\gamma(\gamma-1)^2\rho^2C_3(t).\end{aligned}$$

2.2. Classification equation for $\gamma = 1$

In the case of $\gamma = 1$ we proceed as in Section 2.1 and deduce that the coefficient functions, ξ, τ and η, of the symmetry (2.3) are given by

$$\xi(x,t,u) = xC_3(t) + \frac{1}{2}\dot{\tau}(t)x\ln x, \tag{2.21}$$

$$\tau(x,t,u) = \tau(t) \tag{2.22}$$

and

$$\begin{aligned}\eta(x,t,u) = \phi(x,t) + u\bigg[&C_4(t) - \frac{\alpha C_3(t)}{\rho^2 x} + \frac{\dot{C}_3(t)\ln x}{\rho^2} \\ &+ \frac{\ln x[\ddot{\tau}(t)x\ln x - (2\alpha + (2\beta + 2\lambda\rho - \rho^2)x)]\dot{\tau}(t)}{4\rho^2 x}\bigg],\end{aligned} \tag{2.23}$$

where $\phi(x,t)$ is any solution of (1.1) and the functions C_3, C_4 and τ satisfy the classification equation

$$\begin{aligned}&h_1(t) + h_0(t)\ln x + h_8(t)(\ln x)^2 + \frac{h_2(t) + h_3(t)\ln x}{x} \\ &\quad + \frac{h_4(t) + h_5(t)\ln x}{x^2} + x\big(h_6(t) + h_7(t)\ln x\big) = 0\end{aligned} \tag{2.24}$$

with

$$\begin{aligned}
h_0(t) &= \frac{8}{\rho^2}\ddot{C}_3(t), \\
h_1(t) &= 8\dot{C}_4(t) + \frac{4(2\beta - 2\lambda\rho - \rho^2)}{\rho^2}\dot{C}_3(t) \\
&\quad - \frac{(\rho(2\lambda + \rho) - 2\beta)^2}{\rho^2}\dot{\tau}(t) + 2\ddot{\tau}(t), \\
h_2(t) &= \frac{8\alpha(\beta - \rho(\lambda + \rho))}{\rho^2}\big(C_3(t) - \dot{\tau}(t)\big), \\
h_3(t) &= \frac{4\alpha(\beta - \rho(\lambda + \rho))}{\rho^2}\dot{\tau}(t), \\
h_4(t) &= \frac{4\alpha^2}{\rho^2}\big(2C_3(t) - \dot{\tau}(t)\big), \\
h_5(t) &= \frac{4\alpha^2}{\rho^2}\dot{\tau}(t), \\
h_6(t) &= -8\big(C_3(t) + \dot{\tau}(t)\big), \\
h_7(t) &= -4\dot{\tau}(t), \\
h_8(t) &= \frac{2}{\rho^2}\dddot{\tau}(t).
\end{aligned}$$

3. Results of the group classification of Eq. (1.1)

3.1. The Lie point symmetries

It is clear that

$$\tau(t) = \epsilon_1, \quad C_3(t) = 0 \quad \text{and} \quad C_4(t) = \epsilon_2 \tag{3.1}$$

for constants ϵ_1 and ϵ_2 solve both classification equations (2.20) and (2.24), i.e.

$$\xi = 0, \quad \tau = \epsilon_1 \quad \text{and} \quad \eta = u\epsilon_2 + \phi(x,t), \tag{3.2}$$

where $\phi(x,t)$ is any solution of (1.1), solves the determining equations. Thus we have:

Case 0. α, β, γ, ρ, λ arbitrary.

$$G_1 = \partial_t, \qquad G_2 = u\partial_u, \qquad G_\phi = \phi(x,t)\partial_u, \tag{3.3}$$

where $\phi(x,t)$ is any solution of (1.1). The symmetries in (3.3) generate the *principal Lie algebra*, $L_{\mathcal{P}}$, of (1.1).

We deduce from the form of Eq. (2.20) that extensions of the principal Lie algebra are only possible for those values of γ for which the coefficients of the functions, $h_i(t)$, in (2.20) are not distinct. Therefore for possible extension of the principal Lie algebra we investigate among these values of γ:

$$0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1, \frac{4}{3}, \frac{3}{2}, 2, 3.$$

In fact we see that the principal Lie algebra is extended for only $\gamma \in \{0, \frac{1}{2}, \frac{3}{2}, 2\}$ as follows:

Case 1. $\gamma = 2$, $\alpha = 0$, $\beta \neq 0$, $\lambda = -\frac{1}{\rho}$.

$$\begin{aligned}
G_3 &= \mathrm{e}^{\beta t}\left[x^2\partial_x - \left(\frac{2}{\rho^2} + \frac{2\beta}{\rho^2 x} - x\right)u\partial_u\right],\\
G_4 &= \mathrm{e}^{-\beta t}\left[x^2\partial_x + xu\partial_u\right],\\
G_5 &= \mathrm{e}^{2\beta t}\left[\frac{1}{2}\left(x + \frac{x^2}{\beta}\right)\partial_x - \frac{1}{2\beta}\partial_t + \left(\frac{3}{2} - \frac{1}{\beta\rho^2} - \frac{\beta}{\rho^2 x^2} - \frac{2}{\rho^2 x} + \frac{x}{2\beta}\right)u\partial_u\right],\\
G_6 &= \mathrm{e}^{-2\beta t}\left[\frac{1}{2}\left(x + \frac{x^2}{\beta}\right)\partial_x + \frac{1}{2\beta}\partial_t + \frac{xu}{2\beta}\partial_u\right].
\end{aligned}$$

Case 2. $\gamma = \frac{3}{2}$, $\alpha = 0$, $\beta \neq 0$, $\lambda = 0$.

$$G_3 = e^{\beta t}\left[x\partial_x - \frac{1}{\beta}\partial_t + 2\left(1 - \frac{\beta}{\rho^2 x}\right)u\partial_u\right],$$
$$G_4 = e^{-\beta t}\left[x\partial_x + \frac{1}{\beta}\partial_t\right].$$

Case 3a. $\gamma = 1/2$, $\lambda = 0$ ($\mu = \sqrt{\beta^2 + 2\rho^2}$).

$$G_3 = e^{\mu t}\left[x\partial_x + \frac{1}{\mu}\partial_t + \frac{\beta - \mu}{\rho^2}\left(\frac{\alpha}{\mu} - x\right)u\partial_u\right],$$
$$G_4 = e^{-\mu t}\left[x\partial_x - \frac{1}{\mu}\partial_t - \frac{\beta + \mu}{\rho^2}\left(\frac{\alpha}{\mu} + x\right)u\partial_u\right].$$

Case 3b. $\gamma = 1/2$, $\alpha = \rho^2/4$ ($\mu = \sqrt{\beta^2 + 2\rho^2}$).

$$G_3 = e^{\mu t/2}\left[\sqrt{x}\partial_x - \frac{\beta - \mu}{\rho^2}\left(\frac{\lambda\rho}{\mu} + \sqrt{x}\right)u\partial_u\right],$$
$$G_4 = e^{-\mu t/2}\left[\sqrt{x}\partial_x + \frac{\beta + \mu}{\rho^2}\left(\frac{\lambda\rho}{\mu} - \sqrt{x}\right)u\partial_u\right],$$
$$G_5 = e^{\mu t}\left[\left(x - \frac{\beta\lambda\rho}{\mu^2}\sqrt{x}\right)\partial_x + \frac{1}{\mu}\partial_t + \left\{a_0(\mu) + \sqrt{x}a_1(\mu) + xa_2(\mu)\right\}u\partial_u\right],$$
$$G_6 = e^{\mu t}\left[\left(x - \frac{\beta\lambda\rho}{\mu^2}\sqrt{x}\right)\partial_x - \frac{1}{\mu}\partial_t + \left\{a_0(-\mu) + \sqrt{x}a_1(-\mu) + xa_2(-\mu)\right\}u\partial_u\right],$$

where

$$a_0(\zeta) = \frac{\beta - \zeta}{4\zeta} - \frac{\lambda^2[\rho^2 + (\beta - \zeta)\zeta]}{\zeta^3},$$
$$a_1(\zeta) = \frac{\lambda(\beta - \zeta)^2}{\rho\zeta^2},$$
$$a_2(\zeta) = \frac{\zeta - \beta}{\rho^2}.$$

Case 3c. $\gamma = 1/2$, $\alpha = 3\rho^2/4$, $\lambda = 0$ ($\mu = \sqrt{\beta^2 + 2\rho^2}$).

$$G_3 = e^{\mu t/2}\left[\sqrt{x}\partial_x - \left(\frac{1}{2\sqrt{x}} + \frac{\beta - \mu}{\rho^2}\sqrt{x}\right)u\partial_u\right],$$
$$G_4 = e^{-\mu t/2}\left[\sqrt{x}\partial_x - \left(\frac{1}{2\sqrt{x}} + \frac{\beta + \mu}{\rho^2}\sqrt{x}\right)u\partial_u\right],$$
$$G_5 = e^{\mu t}\left[x\partial_x + \frac{1}{\mu}\partial_t + \frac{\beta - \mu}{\rho^2}\left(\frac{3\rho^2}{4\mu} - x\right)u\partial_u\right],$$
$$G_6 = e^{-\mu t}\left[x\partial_x - \frac{1}{\mu}\partial_t - \frac{\beta + \mu}{\rho^2}\left(\frac{3\rho^2}{4\mu} + x\right)u\partial_u\right].$$

Case 4a. $\gamma = 0$, $\beta \neq 0$ ($\kappa = \alpha - \lambda\rho$, $\theta = \frac{\kappa}{\beta} + \frac{\rho^2}{2\beta^2}$).

$$
\begin{aligned}
G_3 &= \mathrm{e}^{-\beta t}\left[\frac{\rho^2}{2\beta}\partial_x - (x + l_1)u\partial_u\right],\\
G_4 &= \mathrm{e}^{\beta t}\left[\partial_x + \frac{1}{\beta}u\partial_u\right],\\
G_5 &= \mathrm{e}^{-2\beta t}\left[\left(l_2 + \frac{\rho^2}{2\beta}x\right)\partial_x - \frac{\rho^2}{2\beta^2}\partial_t + \left(l_3 - l_4 x - x^2\right)u\partial_u\right],\\
G_6 &= \mathrm{e}^{2\beta t}\left[(l_5 + \beta x)\partial_x + \partial_t + \left(\frac{\rho^2}{2\beta^2} + x\right)u\partial_u\right],
\end{aligned}
$$

where

$$
l_1 = \theta, \qquad l_2 = \frac{\beta\kappa\rho^2 + \rho^4}{2\beta^3}, \qquad l_3 = -\left(\frac{\rho^2}{2\beta} + \theta^2\right),
$$
$$
l_4 = 2\theta + \frac{\rho^2}{2\beta^2}, \qquad l_5 = \kappa + \frac{\rho^2}{\beta}.
$$

Case 4b. $\gamma = 0$, $\beta = 0$ ($\kappa = \alpha - \lambda\rho$).

$$
\begin{aligned}
\Sigma_1 &= \partial_t,\\
\Sigma_2 &= u\partial_u,\\
\Sigma_3 &= \partial_x + ut\partial_u,\\
\Sigma_4 &= \rho^2 t\partial_x - \left(\kappa t - \rho^2 t^2/2 - x\right)u\partial_u,\\
\Sigma_5 &= \left(x + 3\rho^2 t^2/2 + \kappa t\right)\partial_x + 2t\partial_t + \left(3tx + \rho^2 t^3/2 - \kappa t^2\right)u\partial_u,\\
\Sigma_6 &= \left[\rho^4 t^4/4 - \kappa\rho^2 t^3 + \kappa^2 t^2 - \rho^2 t + x\left(3\rho^2 t^2 - 2\kappa t\right) + x^2\right]u\partial_u\\
&\quad + \left(\rho^4 t^3 + 2\rho^2 t x\right)\partial_x + 2\rho^2 t^2\partial_t.
\end{aligned}
$$

4. Solution of (1.1) and (1.2) in Cases 1–4 of Section 3

From Lie's result of group classification of linear second-order partial differential equations with two independent variables [14] it follows that, in the cases in which (1.1) admits the maximal seven Lie point symmetry algebra, (1.1) can be mapped into the standard heat equation by an invertible transformation of the form

$$
z = f(x,t), \qquad \tau = g(t), \qquad w = h(x,t)u \tag{4.1}
$$

for some functions f, g and h, with $f_x \neq 0$ and $g_t \neq 0$. The transformation (4.1) may be obtained with the help of two quadratures as was done in Gazizov and Ibragimov [7]. Another (and perhaps simpler) method is to exploit the algorithm of Bluman and Kumei (see Chapter 6 of [3]) whereby symmetries admitted by a variable coefficient partial differential equation are used to transform the equation into one with constant coefficients. The application of this method in each of the

Cases 1, 3b, 3c, 4a and 4b maps (1.1) into an equation of the form $w_\tau + \delta w_{zz} = 0$ for some constant δ, which can easily be scaled away. Further one can choose (some of) the free parameters in (4.1) suitably so that (1.1) and (1.2) is mapped by (4.1) into the standard heat equation Cauchy problem,

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2}, \tag{4.2}$$

$$w(z,0) = \eta(z), \tag{4.3}$$

for some function η. Provided η is "well-behaved" the solution to (4.2) and (4.3) is well known [12,21]; it is

$$w(z,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \eta(\zeta) \exp\left\{-\frac{(z-\zeta)^2}{4\tau}\right\} \mathrm{d}\zeta. \tag{4.4}$$

The solution obtained from (4.4) is (of course) transformable back into a solution to (1.1) and (1.2) by the inverse of (4.1).

To construct (4.1) we used the symmetries G_4 and G_6 in Cases 1 and 4a, Σ_4 and Σ_6 in Case 4b, and G_3 and G_5 in Cases 3b and 3c. We now present explicitly in each of these cases the function η in (4.3), the functions f, g and h in (4.1), and the solution to (1.1) and (1.2). In the expressions that follow K_1 and K_2 are arbitrary constants.

Case 1.

$$f(x,t) = K_1 - \mathrm{e}^{\beta t}\left(\frac{1}{\beta} + \frac{1}{x}\right), \qquad g(t) = \frac{\rho^2(\mathrm{e}^{2\beta T} - \mathrm{e}^{2\beta t})}{4\beta}, \qquad h(x,t) = \frac{K_2 \mathrm{e}^{\beta t}}{x},$$

$$\eta(\zeta) = \frac{K_2(\beta K_1 - \mathrm{e}^{\beta T} - \beta\zeta)}{\beta},$$

$$u(x,t) = 1 + \frac{x(1-\mathrm{e}^{\beta(T-t)})}{\beta}.$$

Case 3b. ($\mu = \sqrt{\beta^2 + 2\rho^2}$, $\Omega = \frac{\lambda^2(\beta^2+\rho^2)}{\mu^2} + \frac{\mu-\beta}{4} - \lambda^2$).

$$f(x,t) = K_1 + \mathrm{e}^{-\mu t/2}\left(2\sqrt{x} - \frac{2\beta\lambda\rho}{\mu^2}\right),$$

$$g(t) = \frac{\rho^2(\mathrm{e}^{-\mu t} - \mathrm{e}^{-\mu T})}{2\mu},$$

$$h(x,t) = K_2 \exp\left[\Omega t + \frac{(\beta-\mu)x}{\rho^2} + \frac{2\lambda(\beta-\mu)\sqrt{x}}{\rho\mu}\right],$$

$$\eta(\zeta) = \exp\left\{\Omega T + (\mu-\beta)\left[\frac{\mathrm{e}^{\mu T/2}\lambda(\mu+\beta)(K_1-\zeta)}{\rho\mu^2} - \frac{\mathrm{e}^{\mu T}(K_1-\zeta)^2}{4\rho^2} - \frac{\beta\lambda^2(\beta+2\mu)}{\mu^4}\right]\right\},$$

$$u(x,t) = \frac{\mathrm{e}^{\varphi(x,t)}}{\sqrt{1 + (\frac{\beta-\mu}{2})\psi_2(T-t)}},$$

where

$$\varphi(x,t) = \frac{1}{1+(\frac{\beta-\mu}{2})\psi_2(T-t)}\Big\{2\lambda\rho\psi_1(T-t)^2\sqrt{x} + x\psi_2(T-t) + \frac{\Omega}{2}\big[\beta\psi_2(T-t)+\mu\psi_3(T-t)\big](T-t) - \frac{\lambda^2\rho^2}{\mu^2}\big[2\beta\psi_1(T-t)^2+\psi_2(T-t)\big]\Big\}$$

and

$$\psi_1(y) = \frac{1-e^{\mu y/2}}{\mu}, \qquad \psi_2(y) = \frac{1-e^{\mu y}}{\mu}, \qquad \psi_3(y) = \frac{1+e^{\mu y}}{\mu}.$$

Case 3c. ($\mu = \sqrt{\beta^2+2\rho^2}$).

$$f(x,t) = K_1 + 2e^{-\mu t/2}\sqrt{x}, \qquad g(t) = \frac{\rho^2(e^{-\mu t}-e^{-\mu T})}{2\mu},$$

$$h(x,t) = K_2\sqrt{x}\exp\left[\frac{(\beta-\mu)x}{\rho^2} + \frac{(\mu-3\beta)t}{4}\right],$$

$$\eta(\zeta) = \frac{K_2(K_1-\zeta)}{2}\exp\left[\frac{(3\rho^2T-{K_1}^2e^{\mu T})(\mu-\beta)}{4\rho^2} + \frac{e^{T\mu}(\mu-\beta)(2K_1-\zeta)\zeta}{4\rho^2}\right],$$

$$u(x,t) = \exp\left\{\frac{3(\mu-\beta)}{4}\big[\mu\psi_2(T-t)-\beta\psi_1(T-t)\big](T-t) - 2x\psi_1(T-t)\right\} \times\left(\frac{2\mu}{2\beta+(\mu-\beta)\psi_2(T-t)}\right)^{3/2},$$

where

$$\psi_1(y) = e^{\mu y}-1, \qquad \psi_2(y) = e^{\mu y}+1.$$

Case 4a. ($\kappa = \alpha-\lambda\rho$, $\theta = \frac{\kappa}{\beta}+\frac{\rho^2}{2\beta^2}$).

$$f(x,t) = K_1 + \frac{e^{-\beta t}(\beta\kappa+\rho^2+\beta^2 x)}{\beta^2}, \qquad g(t) = \frac{\rho^2(e^{-2\beta t}-e^{-2\beta T})}{4\beta},$$

$$h(x,t) = K_2\exp\left\{\frac{(2\kappa\beta+\rho^2)t}{2\beta^2} - \frac{x}{\beta}\right\},$$

$$\eta(\zeta) = K_2\exp\left\{\frac{\kappa\beta+\rho^2}{\beta^3} + \theta T + \frac{e^{\beta T}(K_1-\zeta)}{\beta}\right\},$$

$$u(x,t) = \exp\left\{\theta(T-t) + \frac{(\theta+x)\psi(T-t)}{\beta} + \frac{\rho^2\psi(T-t)^2}{4\beta^3}\right\},$$

where

$$\psi(y) = 1-e^{\beta y}.$$

Case 4b. ($\kappa = \alpha - \lambda\rho$).

$$f(x,t) = K_1 - \frac{t}{2} + \frac{x}{\rho^2 t}, \qquad g(t) = \frac{T-t}{2\rho^2 tT},$$

$$h(x,t) = K_2\sqrt{t}\exp\left\{\frac{\rho^2 t^3}{24} - \frac{\kappa^2 t}{2\rho^2} + \left(\frac{\kappa}{\rho^2} - \frac{t}{2}\right)x - \frac{x^2}{2\rho^2 t}\right\},$$

$$\eta(\zeta) = K_2\sqrt{T}\exp\left\{\frac{T^2[\kappa + 2\rho^2(K_1-\zeta)]}{2} - \frac{\rho^2 T^3}{3} - \frac{T[\kappa + \rho^2(K_1-\zeta)]^2}{2\rho^2}\right\},$$

$$u(x,t) = \exp\left\{\frac{(T-t)[\rho^2(T-t)^2 - 3\kappa(T-t) - 6x]}{6}\right\}.$$

In Cases 2 and 3a (1.1) and (1.2) does not lend itself to solution via the heat equation. We proceed to find the solution in each of these cases as an invariant solution [7,8,17,20,22]. One starts by finding a symmetry admitted by both (1.1) and (1.2) and then uses this symmetry to construct a general invariant solution of (1.1). The free parameters in the general invariant solution are then chosen suitably so that the solution satisfies the auxiliary condition (1.2).

Case 2. Using the routine outlined in [17,22] it turns out that in this case

$$\Gamma = \mathrm{e}^{-\beta(t-T)}\left[x\partial_x + \frac{1-\mathrm{e}^{\beta(t-T)}}{\beta}\partial_t\right] \tag{4.5}$$

is admitted by both (1.1) and (1.2). Therefore the general invariant solution of (1.1) arising from (4.5) can be written in the form

$$u(x,t) = y(\zeta), \quad \zeta = x\left(1 - \mathrm{e}^{\beta(T-t)}\right), \tag{4.6}$$

where $y(\zeta)$ satisfies the ordinary differential equation

$$\rho^2\zeta^2 y'' + 2\beta y' - 2y = 0. \tag{4.7}$$

The solution to (4.7) in terms of a *Whittaker function* (as the equation does not seem to admit a solution in ordinary functions) is obtained as follows: We make the change of variable

$$z = \frac{2\beta}{\rho^2\zeta}, \qquad w(z) = y(\zeta) \tag{4.8}$$

followed by

$$w(z) = \frac{v(z)\exp\{z/2\}}{z}, \tag{4.9}$$

after which we obtain the *Liouville standard form* (or *normal form*),

$$v'' + \left(\frac{1/4 - m^2}{z^2} + \frac{1}{z} - \frac{1}{4}\right)v = 0, \tag{4.10}$$

where

$$m = \frac{\sqrt{\rho^2 + 8}}{2\rho}.$$

The general solution of (4.10) is [2] (see also [1])

$$v = C_1 M_{1,m}(z) + C_2 M_{1,-m}(z), \quad 2m \neq 0, \pm 1, \pm 2, \ldots, \tag{4.11}$$

where C_1 and C_2 are arbitrary constants, and $M_{k,m}(z)$ is a *Whittaker function of the first kind* defined by

$$M_{k,m}(z) = \mathrm{e}^{-z/2} z^{m+1/2} M\left(\frac{1}{2} + m - k, 2m+1, z\right), \quad 2m \neq 0, -1, -2, -3, \ldots. \tag{4.12}$$

In view of (4.8), (4.9) and (4.11) we have that

$$y(\zeta) = \zeta \exp\left\{\frac{\beta}{\rho^2 \zeta}\right\}\left[D_1 M_{1,m}\left(\frac{2\beta}{\rho^2 \zeta}\right) + D_2 M_{1,-m}\left(\frac{2\beta}{\rho^2 \zeta}\right)\right],$$
$$2m \neq 0, \pm 1, \pm 2, \ldots, \tag{4.13}$$

where $D_i = \frac{\rho^2}{2\beta} C_i$, $i = 1, 2$, is the solution to (4.7). In conclusion we state simply that the invariant solution of (1.1) compatible with the terminal condition, (1.2), is obtained from (4.6) and (4.13) after a suitable choice of the free parameters D_1 and D_2. This may be done for specific (numerical) values of the parameters.

Case 3a. In Case 3a we note that an obvious renaming of the parameters in (1.1) converts the problem (1.1) and (1.2) into the CIR problem [5], the solution of which was constructed as an invariant solution in [22]. So, after adjusting the parameters in the CIR problem appropriately, we have the following solution to (1.1) and (1.2):

$$u(x,t) = \exp\left\{\frac{2x\psi(T-t)}{2\mu + (\mu - \beta)\psi(T-t)}\right\}\left\{\frac{2\mu \exp[(\mu - \beta)(T-t)/2]}{2\mu + (\mu - \beta)\psi(T-t)}\right\}^{2\alpha/\rho^2}, \tag{4.14}$$

where

$$\psi(y) = \mathrm{e}^{\mu y} - 1$$

and

$$\mu = \sqrt{\beta^2 + 2\rho^2}.$$

Table 1
Scaling factors for "standardising" the commutation relations of the symmetries in Section 3.1, $\omega_1 = \frac{2\beta\kappa+\rho^2-\beta^3}{2\beta^3}$, $\omega_2 = 4\lambda^2(\frac{\rho^2}{\mu^2} - 1) + \frac{4\beta^2\lambda^2}{\mu^2} - \beta$

Case	Scaling factors					
	G_1	G_2	G_3	G_4	G_5	G_6
1	$\frac{1}{\beta}$	$\frac{3}{2}$	1	$\frac{3\rho^2}{4\beta}$	$-\frac{4\beta}{3\rho^2}$	$-\frac{3\rho^2}{4\beta}$
2	$\frac{1}{\beta}$	1	1	$\frac{1}{2}$		
3a	$\frac{1}{\mu}$	$-\frac{\alpha\beta}{\mu\rho^2}$	1	$\frac{1}{2}$		
3b	$\frac{2}{\mu}$	$\frac{\omega_2}{2\mu}$	$\frac{\rho}{\mu\sqrt{2}}$	$\frac{\omega_2\rho}{\mu\sqrt{2}}$	$\frac{1}{\omega_2}$	ω_2
3c	$\frac{2}{\mu}$	$-\frac{3\beta}{2\mu}$	$-\frac{3\beta\rho^2}{2\mu}$	$\frac{1}{\mu}$	$-\frac{3\beta\rho^2}{2}$	$-\frac{2}{3\beta\rho^2}$
4a	$-\frac{1}{\beta}$	$-\omega_1$	ω_1	1	$\frac{\omega_1}{2}$	$\frac{1}{\omega_1\rho^2}$
4b	1	1	1	1	1	1

4.1. *Identification of the associated Lie symmetry algebras*

In this section we identify the finite-dimensional symmetry Lie algebras of Section 3.1 obtained by excluding the solution symmetries. Firstly, for the purpose of identification, we change the set of basis operators of the Lie algebra generated by the symmetries in Case 4b to

$$G_1 = \Sigma_5, \qquad G_2 = \frac{1}{2}\Sigma_2, \qquad G_3 = \frac{1}{2}\Sigma_4, \qquad G_4 = \Sigma_3,$$
$$G_5 = \frac{1}{4}\Sigma_6, \qquad G_6 = -\frac{2}{\rho^2}(\Sigma_1 + \kappa\,\Sigma_3 + \Sigma_4).$$

Secondly, we standardise the commutation relations in the various cases by multiplying the symmetries with suitable scaling factors given in Table 1. Then, denoting the Heisenberg–Weyl algebra with

$$[\Gamma_1, \Gamma_2] = 0, \qquad [\Gamma_1, \Gamma_3] = 0, \qquad [\Gamma_2, \Gamma_3] = \Gamma_1$$

by W, we identify the following two types of symmetry Lie algebra:

Cases 1, 3b, 3c, 4a, 4b. $sl(2,\mathbb{R}) \oplus_s W$.

$$\begin{array}{lll} [G_1, G_3] = G_3, & [G_1, G_4] = -G_4, & [G_1, G_5] = 2G_5, \\ [G_1, G_6] = -2G_6, & [G_3, G_4] = -G_2, & [G_3, G_6] = G_4, \\ [G_4, G_5] = G_3, & [G_5, G_6] = G_1 - G_2. & \end{array}$$

Cases 2, 3a. $sl(2,\mathbb{R}) \oplus_s A_1$.

$$[G_1, G_3] = G_3, \qquad [G_1, G_4] = -G_4, \qquad [G_3, G_4] = G_1 - G_2.$$

5. Summary and discussion

We have performed the group classification of the partial differential equation (1.1) and shown that the Lie algebra of this equation essentially depends upon the parameters α, β, γ, ρ and λ. Every member of the family (1.1) admits the symmetries $G_1 = \partial_t$, $G_2 = u\partial_u$ and the solution symmetries. However, admission by (1.1) of additional symmetries is only possible for $\gamma \in \{0, 1/2, 3/2, 2\}$ as is presented in Cases 1 to 4 in Section 3. For each of these cases we have computed the Lie point symmetries, calculated their commutation relations and identified the corresponding Lie algebras. Further we have used the admitted symmetries to solve (1.1) and (1.2) completely in each of Cases 1 to 4.

It is noteworthy that through the method of group classification we have rediscovered (or validated parameter choices in) some of the interesting models that have been proposed in the literature:

- Case 4a with $\gamma = 0$ and $\beta \neq 0$ is the Vasicek model [23],
- Case 3a with $\gamma = 1/2$ and $\lambda = 0$ is the Cox–Ingersoll–Ross model [5],
- Case 3b with $\gamma = 1/2$ and $\alpha = \rho^2/4$ is the Longstaff model [16].

The other models that have arisen from the group classification procedure such as Case 3c have probably not been considered before. We defer to future research investigation of the possible usefulness of these models in the wonderful world of financial mathematics.

In the light of the theory on relating different differential equations [3] we observe an interesting relationship between the algebras $sl(2,\mathbb{R}) \oplus_s A_1$ (admitted in Cases 2, 3a) and the algebra $sl(2,\mathbb{R}) \oplus_s W$ (admitted in Cases 1, 3b, 3c, 4a, 4b); the former is a subalgebra of the latter, with the following correspondence establishing an isomorphism between the corresponding (scaled) symmetries:

$$2G_1 \longleftrightarrow G_1, \quad 2G_2 \longleftrightarrow G_2, \quad \nu G_3 \longleftrightarrow G_5, \quad \frac{2}{\nu} G_4 \longleftrightarrow G_6, \quad \nu \neq 0.$$

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