Distributed loop network with minimum transmission delay

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Abstract

Distributed loop networks are networks with at least one ring structure. They are widely used in the design of local area networks, multilibrary memory organizations, data alignments in parallel memory systems, and supercomputer architecture. In this paper, we give a systematic and unified method of solutions in the design and implementation of these networks. We show that doubly linked loop networks with transmission delay less than or equal to \((1 + \varepsilon)\sqrt{3N}\) can be constructed asymptotically for sufficiently large \(N\), the number of nodes in the network. This is close to the optimal value within a number which is small as compared to \(N\). We then give several infinite classes of values of \(N\) for which optimal doubly linked loop networks can be actually designed. The method is then generalized to obtain a new upper bound for possible transmission delays in multiply linked loop networks. Routing and rerouting algorithms are designed for the optimal loop networks.

1. Introduction

Advances in technology, especially the advent of VLSI circuit technology, have enabled us to construct very complex interconnection networks in recent years. These networks can be inter-PE communication networks which perform the necessary data routing and manipulation functions for data exchanges among the PEs in a number of array processor architectures. They can also be interprocessor-memory communication networks for multiprocessor systems, where all the processors share access to

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common sets of parallel memory modules. Various interconnection networks have been used in the design and implementation of local area networks, telecommunication networks and other distributed computer systems.

It is a common practice to increase the parallelism of operation of the high-speed memory in a high-performance computer system by combining several independent memory modules into a memory system to facilitate parallel block transfers. In this context, the network called memory circulator consists of a group of interconnected registers, one for each memory module, and control circuitry. Each register is connected to other registers and the pattern is cyclically symmetric. The pattern is completely determined once the connections are chosen. One of the most important issues is to choose the connections for a given number of registers such that the number of register-to-register transfers required for an arbitrary circulation is minimum. Here we assume that each register does not contribute much of the transmission delay during the transfer. For convenience, it is assumed that one of the connections is the one which connects each register to an adjacent register. If the \( N \) registers are labeled as \( 0, 1, 2, \ldots, N - 1 \), then register \( i \) is assumed to be connected to register \( i + 1 \mod N \). This Hamiltonian circuit \( 0 \to 1 \to 2 \to \cdots \to N - 1 \to 0 \) is called a ring (or loop) in the study of interconnection networks. An interconnection network is called a loop network if it has a ring, and is called a ring network if it is a ring.

In a number of array processors, for example the ILLIAC IV computer, the PE array can be operated as a circulator. When depicted as a ring of PEs, each PE of the ILLIAC IV network is connected to \( 2l \) other PEs. Moreover, each node \( i \) is connected to nodes \( i + 1 \) and \( i - 1 \mod N \), while if \( i \) is connected to \( i + s \), so is \( i - s \mod N \). The minimization problem is the same.

In the design of local area networks, loop topologies with unidirectional links are more frequently used than other topologies. They allow connections of high reliability and low transmission delay that can be made with optical fibers to reach the high speed required. The ring network has been one of the most popular network topologies used in the design and implementation of local area networks. This is due to its simplicity and expandability. The switching mechanism at each node can be implemented using standardized building block switching systems. However, the ring network is known to have a low degree of reliability and, hence, a low fault tolerance. In fact, the connectivity for a unidirectional ring network of \( N \) nodes is 1 since the breakdown of any node \( i \) would disable any path from node \( i - 1 \) to node \( i + 1 \mod N \). Moreover, the maximum distance between any two nodes (i.e., the diameter) is \( N - 1 \) since it would take \( N - 1 \) steps (ignoring the transmission delay at the nodes) to traverse from node \( i \) to node \( i - 1 \mod N \).

One way to increase the connectivity and decrease the diameter is to add links to nodes of the ring network. It is practical to add only as few links as possible since more links at each node would be costly and complex. For example, the crossbar switch used in the multiprocessor architecture possesses complete connectivity with respect to memory modules and the PEs. It has the potential for the highest bandwidth and system efficiency. However, it is not cost-effective for a large multiprocessor
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The idea is to design regular loop networks of low degree and small maximum-message path lengths (or small graph diameter). The first loop network with indegree = outdegree = 2 was proposed by Wolf and Liu [24] and called a **distributed double-loop computer network** (DDLCN). A DDLCN is a network of \( N \) nodes where node \( i \) is adjacent to nodes \( i+1 \) and \( i-1 \mod N \). Clearly, the diameter of such a network is \( \lceil N/2 \rceil \), where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). Similarly, \( \lceil x \rceil \) is defined to be the least integer greater than or equal to \( x \). Later, Grnarov et al. [9] proposed a more reliable network called a **daisy chain network** where node \( i \) is adjacent to nodes \( i+1 \) and \( i-2 \mod N \). This loop network has diameter \( \lceil N/3 \rceil + 1 \). Imase and Itoh [15] gave an algorithm to design networks with minimum diameter which do not have the ring structure. The networks proposed by Pradham and Reddy [19] and Pradham [18] are variations of loop networks, and the degree at each node is not the same. These networks have good performance and reliability as compared to the ring networks. Another variation is the **chordal ring network** proposed by Arden and Lee [1], where the network is an undirected degree-3 graph formed by adding chords to a single cycle, or an undirected ring. The diameter is shown to be of \( O(N^{1/2}) \).

In this paper, we investigate loop networks where each node has the same number \( l \) of in-links and out-links. We also take into consideration the regularity and symmetry of the network. The topology is completely determined once the \( l \) connections are chosen. Wong and Coppersmith [25] formulated the problem and established lower bounds for the diameters and upper bounds when \( N = u^l \) for some integer \( u \). For such networks of \( N \) nodes, the lower bound is \((l!N)^{1/l} - \frac{1}{2}(l+1)\) and the upper bound is \( lN^{1/l} - l \) when \( N = u^l \) for some \( u \). Note that when \( l = 1 \), these two bounds coincide and the number is the diameter of the ring network of \( N \) nodes. When \( l = 2 \), they improved the lower bound to \( \lceil (3N)^{1/2} \rceil - 2 \). Raghavendra et al. [21] proposed a doubly linked (i.e., \( l = 2 \)) loop network architecture called **forward loop backward hop** (FLBH) topology. It has a ring in the forward direction connecting all the neighboring nodes and a backward hop connecting nodes that are separated by a distance \( s = \lceil N^{1/2} \rceil \). The diameter of this network is shown to be \((N/\lceil s+1 \rceil) + (s-1)\), an improvement over the DDLCN and the daisy chain networks. However, these loop topologies achieve the lower bound \( \lceil (3N)^{1/2} \rceil - 2 \) only for small values of \( N \). Fiol et al. [8] gave an exhaustive search for the optimal values of the diameters \( k \) for a given \( N \) and a fixed-step connection other than the ring. The precise bounds are \( \lceil (3N)^{1/2} \rceil - 2 \leq d(N) \leq \lceil (3N)^{1/2} \rceil - 1 \) for \( N \leq 256 \), where \( d(N) \) is the optimal diameter. By using a geometrical approach and tessellation on the plane, they obtain some infinite families of optimal loop networks. Hwang and Xu [14] gave a heuristic method which finds a topology with diameter roughly \((3N)^{1/2} + 2(3N)^{1/4} + q - 1\) for large \( N \), where \( q = \lceil (N-1)^{1/2}/q^* \rceil - 3q^* \) and \( q^* = \lfloor (N/3)^{1/2} \rfloor \). They also give some infinite classes of \( N \) for which optimal topologies are found.

What other networks can achieve the optimal diameter in the case \( l = 2 \)? Furthermore, what networks are optimal or nearly optimal loop networks when \( l \geq 3 \)?
questions remain largely unexplored. In this paper, we take the first step to investigate
good loop topologies for the cases $l \geq 3$. One of the main results is a number-theoretic
method employed to give a unified and systematic approach for the study of optimal
diameters in the design of regular fixed-step loop networks. For the doubly linked (i.e.,
$l = 2$) networks, we give an approximation result that for every $\varepsilon > 0$ and $N > N_0(\varepsilon)$,
there exists a number $s = (1 + o(1))\sqrt{3N}$ so that the diameter $d(N; 1, s) < (1 + \varepsilon)\sqrt{3N}$.
This technique is then applied to give many infinite classes of $N$ for which optimal and
nearly optimal topologies are found. The technique is also generalized to higher-
dimension cases to give new upper bounds for the diameters of loop networks with
$l \geq 3$. The result is recursive in nature. It is shown that if good (or optimal) topologies
exist for the $l$-linked loop networks, then fairly good $(l+1)$-linked loop topologies can
be constructed for sufficiently large $N$. Moreover, the diameter can be calculated
explicitly once the diameter of the $l$-linked loop topology is given.

In Section 2 of this paper, we give some definitions and develop some preliminary
results. Section 3 deals mainly with loop networks for $l = 2$. The cases for $l \geq 3$ are
included in Section 4. In Section 5, we design the routing algorithms for the construc-
ted optimal loop networks. We also study the transmission delay when a node
becomes faulty in the loop network. For other notations and terminologies on parallel
and distributed processing not defined in this paper, the reader is referred to the book
by Hwang and Briggs [13].

2. Definitions and preliminary results

Let $N$ be the number of nodes in the loop network. Since the network has a
ring, we denote it as $G(N; 1, s_2, s_3, \ldots, s_l)$, where each node $i$ is adjacent to
$i+1, i+s_2, \ldots, i+s_l \mod N$, respectively. Let $d(N; 1, s_2, s_3, \ldots, s_l)$ be the diameter
of the network $G(N; 1, s_2, s_3, \ldots, s_l)$. The main problem is to find loop topologies
$G(N; 1, s_2, s_3, \ldots, s_l)$ such that the diameter would be minimized. Let $d(N) = \min\{d(N; 1, s_2, s_3, \ldots, s_l) : 2 \leq s_i \leq N - 1, i = 2, 3, \ldots, l\}$. When $l = 2$, these loop networks
are called doubly linked loop networks. Hence, the words “triply” and “multiply” would
imply the cases $l = 3$ and $l \geq 3$, respectively.

By considering the first quadrant in the $l$-dimensional Euclidean space, Wong and
Coppersmith [25] showed the following lemma.

Lemma 2.1. In the loop network $G(N; 1, s_2, s_3, \ldots, s_l)$ we have $d(N) \geq (l! N)^{1/l} - \frac{1}{2}(l+1)$. When $N = u^l$ for some integer $u$, $d(N; 1, u, u^2, \ldots, u^{l-1}) = lN^{1/l} - l$. Moreover, when $l = 2$, the lower bound can be tightened to $lb(N) = \lceil (3N)^{1/2} \rceil - 2$.

A loop network $G(N; 1, s_2, s_3, \ldots, s_l)$ is said to be optimal if it has the minimum
diameter among all values of $s_2, s_3, \ldots, s_l$. We note that a loop network which achieves
the diameter of the lower bound in Lemma 2.1 has to be optimal. However, the
converse is not true. The lower bound stated in Lemma 2.1 fails to be tight for some $N$. This will become clear in the later sections.

Let us distribute the $N$ nodes of the loop network evenly on a circle. They are labeled as $0, 1, 2, \ldots, N - 1$. When $l = 2$, let $s_2 = s$. In this case, take an integer $s$, $1 < s < N$. Connect node $i$ to nodes $i + 1$ and $i + s \mod N$. An example is $G(N; 1, 3)$ with $N = 9$ and $d(N; 1, 3) = 4$. Note that this loop topology is optimal since $\lceil (3N)^{1/2} \rceil - 2 = 4$ for $N = 9$. Here the fixed-step connection $s = 3$ is taken as the square root of $N = 9$. The general fixed-step connection $s = \lfloor N^{1/2} \rfloor$ was studied by Raghavendra and Gerla [20] and Raghavendra et al. [22]. However, Lemma 2.1 shows that the resulting diameter $d(N; 1, N^{1/2})$ is not optimal in general. In fact, the number is far away from the lower bound $\lceil (3N)^{1/2} \rceil - 2$. In this section, we introduce an infinite class of $N$ for which loop topologies can be constructed to realize the diameter lower bound stated in Lemma 2.1. Let $N = 3t^2 + 3t, t \geq 1$, a positive integer. Let $s = 3t + 2$. We will show that $d(N; 1, 3t + 2) = 3t$. Therefore, the loop network $G(N; 1, 3t + 2)$ is optimal since $\lceil (3N)^{1/2} \rceil - 2 = 3t$.

Since the network is node-symmetric, we only have to calculate the maximum number of steps for the node $0$ to reach any other node in the network. We divide the circle counterclockwise into the following segments:

$$0, \ldots, 2t + 2, \ldots, s, \ldots, x, \ldots, (i + 1)s, \ldots, (t - 1)s, \ldots, 3t^2 + t, \ldots, ts, \ldots, N,$$

where $ts = t(3t + 2) = 3t^2 + 2t$ and $(t - 1)s = 3t^2 - t$. Moreover, $2t + 1 = (t + 1)s = (t + 1)(3t + 2) \mod N$ and $3t^2 + t = (2t)s = 2t(3t + 2) \mod N$. Also $x = hs \mod N$, $t + 1 < h < 3t$ and $i < t - 1$. We underline a node if the value of that node is calculated modulo $N$. The idea is to use the jump sizes $1$ and $s$ efficiently such that any node on the circle can be reached in steps as few as possible. Hence, we have:

(a) Any node between $0$ and $2t + 2$ can be reached from $0$ in less than $2t + 2$ steps by using the jump size $1$ on the ring.

(b) Nodes between $2t + 2$ and $s = 3t + 2$ can be reached in at most $(t + 1) + (t - 1) = 2t$ steps since the node $2t + 2$ can be reached from $0$ using jump size $s$ in $(t + 1)$ steps and the other nodes can be reached from the node $2t + 2$ in less than $t - 1$ steps on the ring.

(c) Nodes between $ts = 3t^2 + 2t$ and $N = 3t^2 + 3t$ can be reached in less than $t + (t - 1) = 2t - 1$ steps by using $t$ of the $s$ jumps and at most $t - 1$ steps on the ring.

In general, we have to consider only the nodes between $(t - 1)s = 3t^2 - t - 2$ and $ts = 3t^2 + 2t$. This is because any node between $is$ and $(i + 1)s$, where $i < t - 1$, can be reached in less number of steps. Now if a node is between $(t - 1)s$ and $3t^2 + t$, it can be reached by the jump size $s$ in $t - 1$ steps plus $2t + 1$ steps on the ring. Hence, the maximal number of steps is $(t - 1) + (2t + 1) = 3t$. On the other hand, any node between $3t^2 + t = (2t)s \mod N$ and $ts$ can be reached in at most $2t + (t - 1) = 3t - 1$ steps by using the jump size $s$ in $2t$ steps. Summarizing, we have the following theorem.

**Theorem 2.2.** If $N = 3t^2 + 3t, t \geq 1$, then $d(N; 1, 3t + 2) = 3t$. This loop topology $G(N; 1, 3t + 2)$ is optimal, that is, $d(G(3t^2 + 3t; 1, 3t + 2)) = d(3t^2 + 3t)$. 
To illustrate our design with an example, consider \( t = 3 \) and \( N = 36 \). The partition of the circle would be as follows:

\[ 0, \ldots, 8, \ldots, s, \ldots, t, \ldots, x, \ldots, (i + 1)s, \ldots, 2s, \ldots, 30, \ldots, 3s, \ldots, N, \]

where \( s = 3t + 2 = 11, 8 = (t + 1)s = 44 \pmod{36} \) and \( 30 = (2t)s = 6 \cdot 11 = 66 \pmod{36} \). It is readily verified that \( d(36; 1, 11) = 9 \). Note that \( d(36; 1, (36)\frac{1}{2}) = 10 \).

Theorem 2.2 suggests that taking \( s = 3t + (k - 1), k \geq 1 \) as jump size when \( N \) is of the form \( 3t^2 + kt \) might lead to the construction of optimal loop topologies. This turns out to be not true in general as will be seen in Section 3. However, it suggests a close relation between the jump size \( s \) and the number \( N \) of nodes. In fact, we will show that by suitably choosing the jump size \( s \), loop topologies with fairly good diameters can be obtained.

Substituting \( t \) by \( t + 1 \) in the formula \( 3t^2 + 3t \), we have \( 3(t + 1)^2 + 3(t + 1) = 3t^2 + 9t + 6 \). Hence, any positive integer \( N \) can be represented as \( N = 3t^2 + kt + h \), where \( 0 \leq h < t \) and \( 3 \leq k < 9 \). We now state the following lemma which concludes this section.

**Lemma 2.3.** Any positive integer \( N \) can be represented as \( N = 3t^2 + kt + h \), where \( 0 \leq h < t \) and \( 3 \leq k < 9 \).

### 3. Doubly linked loop networks

In this section, we continue to study doubly linked loop networks. That is, the fixed-step \( I \)-linked topologies with \( I = 2 \). First, we give a general asymptotic result. Then we use the developed method to obtain optimal loop networks.

Since every positive integer lies between \( 3t^2 + 3t \) and \( 3t^2 + 9t + 6 \) for some \( t \geq 0 \), by Lemma 2.3, let \( N = 3t^2 + kt + h \), where \( 0 \leq h < t \) and \( 3 \leq k < 9 \). The expression of \( N \) as \( 3t^2 + kt + h \) enables us to properly choose the jump size \( s \) so that after \( t \) jumps of size \( s \) per jump, \( ts \) is approximately two-thirds of \( s \) from 0 mod \( N \). First we state and prove the following general result.

**Theorem 3.1.** Let \( N \) be the number of nodes in the doubly linked loop network \( G(N; 1, s) \). Let \( b = \lfloor \frac{N}{s} \rfloor \) and \( m = N - bs \). Then we have \( d(N; 1, s) \leq \max \{ 2b + m - 1, b + s - m - 2 \} \).

**Proof.** We only have to show that any node in the network \( G(N; 1, s) \) can be reached from the node 0 in less than or equal to \( \max \{ 2b + m - 1, b + s - m - 2 \} \) steps using jump sizes 1 and \( s \).

For convenience, we place the nodes of the network as follows:

\[ 0, \ldots, (b + 1)s, \ldots, s, \ldots, (b - 1)s, \ldots, 2bs, \ldots, bs, \ldots, N, \]
where the sequence of jumps for the size $s$ is

$$0, s, 2s, \ldots, bs, (b + 1)s, (b + 2)s, \ldots, 2bs,$$

and the node $x$ is the node with label $x \mod N$. Let $[i, j]$ denote the set of integers from $i$ to $j$ inclusively. Since $bs$ is the last node before we reach the node $N = 0$, we can partition $[(i - 1)s, is]$, where $i \leq b$, into the following two segments: $A = [(i - 1)s, (b + i)s]$ and $B = [(b + i)s, is]$. The length of $A$ is equal to $(b + i)s - (i - 1)s = N + s - m = s - m \mod N$ and the length of $B$ is equal to $is - (b + i)s = -bs \equiv m \mod N$. Therefore, all the nodes in $A$ except $(b + 1)s$ can be reached from node 0 in less than or equal to $(i - 1) + (s - m - 1)$ steps and those in $B$ except $is$ can be reached in less than or equal to $(b + i) + m - 1$ steps. Since $i \leq b$, we have $d(N; 1, s) \leq \max \{2b + m - 1, b + s - m - 2\}$. □

Theorem 3.1 gives a general guideline as to how large the diameter of the network $G(N; 1, s)$ can be. The problem of optimization can be reduced to suitably selecting the number $s$ such that $2b + m$ and $b + s - m$ would be as small as possible.

We observe that if we go around the circle using $s = (1 + O(1))\sqrt{3N}$ and when we pass $N$ for the first time we are at about $2s/3$, then $b = \lfloor N/s \rfloor = (1 + O(1))/3)s$ and $s - m = 2s/3$. Therefore, $m = s - (s - m) = s/3$. Hence, we have

$$2b + m - 1 = 2\left(\frac{1 + O(1)}{3}\right)s + \frac{s}{3} - 1$$

$$= (1 + O(1))\sqrt{3N} - 1$$

and

$$b + s - m - 2 = \left(\frac{1 + O(1)}{3}\right)s + \frac{2s}{3} - 2$$

$$= (1 + O(1))\sqrt{3N} - 2.$$ 

This means that if we use 1 and $s$ to jump around the circle at most twice, the diameter of the network $G(N; 1, s)$ is $\leq (1 + O(1))\sqrt{3N}$. The following theorem provides an approximation for $s$ by utilizing the method of Diophantine approximation.

**Theorem 3.2.** For every $\varepsilon$ and $N > N_0(\varepsilon)$, there is a number $s = (1 + O(1))\sqrt{3N}$ so that $d(N; 1, s) < (1 + \varepsilon)\sqrt{3N}$ and we have to go around the circle only at most twice.

**Proof.** Let $s$ be any integer of size $(1 + \varepsilon)\sqrt{3N}$. Let $l = \lfloor N/s \rfloor + 1$. If we go around the circle $l$ times using $s$, we of course pass 0. Assume that we land in $[0, s]$ at the node $\alpha$. If $\alpha = 2s/3$, we are done. If not, replace $s$ by $s + 1$. After $\lfloor N/s \rfloor + 1$ steps, we pass the node 0 and arrive at $\alpha + \lfloor N/s \rfloor + 1$. Repeat the operation, i.e., replace $s$ successively by $s + 1, s + 2, \ldots, s + t$, we reach the nodes (mod $s$) of the form $\alpha, \alpha + \lfloor N/s \rfloor, \alpha + 2\lfloor N/s \rfloor, \ldots, \alpha + t\lfloor N/s \rfloor$. The error would be $O(s)$ as long as $s$ is small. By the way, $\alpha + t\lfloor N/s \rfloor$ is to be understood to be taken modulo $s$. 


To complete the proof, it suffices to show that the numbers \( u_{\lfloor N/s \rfloor}, 1 \leq u \leq t \), cover the interval \((0, s)\) with an error \(<\varepsilon\). That is, every interval of length \(\varepsilon\) contains one of our points \( u_{\lfloor N/s \rfloor}\). Now observe that \(N\) is large and \(s=(1+o(1))\sqrt{3N}\) or \((N/s)/s=(1+o(1))/3\). Let \( t \) be large and choose \( s \) so that \((N/s)/s=1/s+1/t+o(1/t)\), where \( t \) is large but is small compared to \( s \) and \( N \). Then the numbers \( u_{\lfloor N/s \rfloor}\) cover the interval \((0, s)\) by a mesh of length \( s/t \) for every fixed \( t \). □

In order to obtain the doubly linked loop topology which has the optimal diameter, more specific values of \( s \) would have to be properly chosen such that the node \((2s/3)\mod N\) can be carefully located.

Now suppose \( N=3t^2+kt+h, \ 3 \leq k \leq 9 \) and \( 0 \leq h < t \), as defined before. Let \( s=3t+(k-1) \) be the jump size. Hence, \( b=t \) and \( m=N-bs=t+h \) and \( s-m=2t+k-h-1 \). By Theorem 3.1, we have

\[
d(N; 1, s) \leq \max \{3t+k-h-3, 3t+h-1\}.
\]

These numbers depend on the values \( k \) and \( h \) in the representation of \( N \) as \(3t^2+kt+h\). We now calculate the diameter \( d(N; 1, s) \) with respect to the various assignments of \( k \) and \( h \). A partial list is given in Table 1. For comparison, we also calculate the lower bound \( lb(N)=\lceil (3N)^{1/2} \rceil - 2 \) for the diameters \( d(N; 1, s) \), where \( N=3t^2+kt+h \). These are listed in Table 2.

Note that not all lower bounds are optimal. For example, one can show that the lower bound \( 3t+1 \) for \( k=6, \ h=3 \) can never be achieved. In fact, \( d(N; 1, 3t+(k-1))=3t+2 \) is optimal for \( N=3t^2+6t+3 \). This (6, 3) entry in Table 1 is marked by a rectangle. Other underlined entries in Table 1 are all optimal. Summarizing, we have the following infinite classes of values of \( N \) for which optimal doubly linked loop networks can be constructed.

**Theorem 3.3.** Let \( N=3t^2+kt+h \) be the number of nodes in the doubly linked loop network \( G(N; 1, s) \). Then \( d(N; 1, 3t+(k-1))=d(N) \) for the following pairs of \( k \) and \( h \): \((k, h)=(3,0), (3,1), (4,1), (4,2), (5,1), (5,2), (6,2), (6,3), (7,2), (7,3), (8,3), (9,3) and (9,4)\). The necessary condition for the case (9,3) is that \( t \geq 3 \) and for the case (9,4) is that \( t \geq 2 \).

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</table>

Table 1

Diameter \( d(N; 1, s) \) for \( N=3t^2+kt+h \) and \( s=3t+(k-1) \)
Table 2
The lower bound \(lb(N)\) for \(d(N; 1, s), N = 3t^2 + kt + h\)

<table>
<thead>
<tr>
<th>(k)</th>
<th>(h)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3t</td>
<td>3t</td>
<td>3t</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3t</td>
<td>3t+1</td>
<td>3t+1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3t+1</td>
<td>3t+1</td>
<td>3t+1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3t+1</td>
<td>3t+1</td>
<td>3t+1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3t+2</td>
<td>3t+2</td>
<td>3t+2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3t+2</td>
<td>3t+2</td>
<td>3t+2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>3t+3</td>
<td>3t+3</td>
<td>3t+3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Moreover, in all cases \(d(N; 1, s) = lb(N)\) except for the case \((k, h) = (6, 3)\), where \(d(N; 1, s) = lb(N) + 1 = \lceil (3N)^{1/2} \rceil - 1\).

We note that the selection of \(s = 3t + (k - 1)\) in Theorem 3.3 is not a necessary condition to obtain loop networks with \(d(N; 1, s) = d(N)\). This can be seen from the following lemma and theorem.

**Lemma 3.4.** If \(N = 3t^2 + kt + h\), where \(3 \leq k \leq 9\) and \(0 \leq h < t\), then \(d(N; 1, 3t + (k - 4)) \leq \max\{3t + h - k + 5, 3t + 2k - h - 9\}\).

**Proof.** Since \(N = 3t^2 + kt + h\) and \(s = 3t + (k - 4)\), we have \(b = t + 1\) and \(m = t - k + h + 4\). Hence, by Theorem 3.1,

\[
d(N; 1, 3t + (k - 4)) \leq \max\{2b + m - 1, b + s - m - 2\} = \max\{3t + h - k + 5, 3t + 2k - h - 9\}.
\]

By Lemma 3.4, we can prove the following theorem.

**Theorem 3.5.** If \(N = 3t^2 + 9t + 4\), then the network \(G(N; 1, 3t + 3)\) is optimal. Moreover, \(d(N; 1, 3t + 3) = lb(N) = 3t + 2\). If \(N = 3t^2 + 8t + 6\), then the network \(G(N; 1, 3t + 4)\) is optimal and \(d(N; 1, 3t + 4) = lb(N) = 3t + 3\).

The selection of \(s\) also depends on the representation of \(N\). Had we started with a different representation for \(N\), we would have to choose a different value of \(s\). Theorems 3.3, 3.5 and Lemma 3.4 exhibit different patterns of selection. However, Theorem 3.2 offers an asymptotic approximation of the situation. For example, let \(N = t^2 + kt + h\), where \(2 \leq k \leq 4\) and \(0 \leq h < t\). By choosing \(s = t + k\), we have \(d(N; 1, s) \leq \max\{2t + h - 1, 2t + k - h - 2\}\). Although this selection may not give an optimal solution, it does provide nearly optimal solutions for many cases of \(k\) and \(h\).

It is natural to ask if optimal loop networks can be found for all values \(N \leq 50\). By Theorems 3.3 and 3.5, there are 10 values of \(N\) for which optimal loop topologies are
not found. These are 20, 25, 32, 35, 38, 39, 42, 45, 46 and 49. However, a computer
search by Fiol et al. [S] confirmed that some of the networks $G(N; 1, s)$ with
$(k, h) = (4, 0), (6, 1)$ and $(8, 4)$ in Table 1 are optimal for certain small values of $t$. These
include the cases $N = 20, 25$ and $32$. Hence, the only values of $N \leq 50$ not covered in
our study are $N = 35, 38, 39, 42, 45, 46$ and 49. All of these numbers when represented
as $3t^2 + kt + h$ seem to have large values of $h$.

4. Multiply linked loop networks

In this section, we consider transmission delays for the cases $l \geq 3$. For the loop
network $G(N; 1, s_1, s_2, \ldots, s_l)$, Lemma 2.1 gives the lower bound $(l!N)^{1/l} - \frac{1}{2}(l + 1)$ for
any $N$ and $l$ and the upper bound $(l!N)^{1/l} - 1$ for $N = u^l$, where $u$ is an integer. The
upper bound is obtained by the network topology $G(N; 1, u, u^2, \ldots, u^{l-1})$. Instead of
searching for an optimum solution, we improve the upper bound for loop topologies
for $N$ nodes, with $N$ sufficiently large.

First, we consider the case $l = 3$. Let $G(N; 1, s_2, s_3)$ be the loop network with
$N$ nodes. The $N$ nodes are placed evenly on the circle as before. In this case, we have
$\text{lb}(N) = \lceil 6N^{1/3} \rceil - 2$ and $d(N) = \min \{d(N; 1, s_2, s_3) : 2 \leq s_2, s_3 \leq N - 1\}$. It is known
that when $N = u^3$, where $u$ is an integer, $d(N; 1, u, u^2) \leq 3N^{1/3} - 3$. Hence, we have
$d(N) \leq 3N^{1/3} - 3$ for $N = u^3$. This upper bound can be improved as follows.

**Theorem 4.1.** For $N$ sufficiently large and some constant $c$, if there exists an $s$
such that $d(cN^{2/3}; 1, s) = \text{lb}(cN^{2/3})$, then there exist $s_2, s_3$ such that
$d(N; 1, s_2, s_3) \leq (\sqrt{3c} + 1/c)N^{1/3} - 3$.

**Proof.** Let $N$ be large enough. Choose $x = cN^{2/3}$ for some constant $c$. Partition the
circle into equidistant points, i.e., $0, (N/x), 2(N/x), \ldots, (x - 1)(N/x), x(N/x) = N$. Since
there exists an $s$ such that $d(x; 1, s) = \text{lb}(x)$, we can visit all these points in $\text{lb}(x)$ steps.
Here $\text{lb}(x) = \lceil \sqrt{3x} \rceil - 2$. The remaining points between any two equidistant points can
be reached in $(N/x) - 1$ steps from the points $i(N/x), i = 0, 1, 2, \ldots, x - 1$. Hence, any
point in $[0, N - 1]$ can be reached from the point 0 in less than or equal to $t$ steps, where

\[
t = \sqrt{3x} - 2 + \left( \frac{N}{x} \right) - 1
\]

\[
= \sqrt{3cN^{2/3} - 2 + \left( \frac{N}{cN^{2/3}} \right) - 1}
\]

\[
= \left( \sqrt{3c} + \frac{1}{c} \right)N^{1/3} - 3. \quad \square
\]
The minimum of $t$ is assumed when $c=(4/3)^{1/3}$. Hence, the minimum of $t$ is $(81/4)^{1/3}N^{1/3}-3$. Note that in Theorem 4.1, $s_2=s\cdot(N/x)$, where $s$ is the jump size in the network $G(cN^{2/3},1,s)$ and $s_3=(N/x)=(1/c)N^{1/3}=(3/4)N^{1/3}$. To illustrate Theorem 4.1, we take $N=27$. For convenience, let $c=1$. Then $x=cN^{2/3}=9$. Since $d(9)=d(9,1,3)=4$, we have $s\cdot(N/x)=3\cdot(27/9)=9$ and $d(27;1,9,3)\leq 4+(N/x-1)=4+2=6$. Note that $3N^{1/3}-3=3\cdot3-3=6$. It follows that for $N>27$, the upper bound obtained in Theorem 4.1 is much better than the upper bound $3N^{1/3}-3$ obtained by Wong and Coppersmith [25].

The recursive nature of Theorem 4.1 can also be generalized to the general case $l\geq4$. We have the following theorem.

**Theorem 4.2.** Let $N$ be the number of nodes in the $l$-linked loop network $G(N;1,s_2,s_3,\ldots,s_l)$ and $c$ be a constant. If there exists an $m$-linked loop network $G(N;1,t_2,t_3,\ldots,t_m)$, where $m=l-1$, which has the lower bound $(m!N)^{1/m}-\frac{1}{2}(m+1)$ as its optimal diameter, then there exist $s_2,s_3,\ldots,s_l$ such that $d(N;1,s_2,s_3,\ldots,s_l)\leq(((l-1)!c)^{1/(l-1)}+1/c)N^{1/l}-(l+2)/2$.

**Proof.** As in the proof of Theorem 4.1, choose $x=cN^{(l-1)/l}$ for some constant $c$. Partition the circle into equidistant points, i.e., $0,(N/x),2(N/x),\ldots,(x-1)(N/x),x(N/x)=N$. Since there exist $t_2,t_3,\ldots,t_{l-1}$ for the network $G(x;1,t_2,t_3,\ldots,t_{l-1})$ such that $d(x;1,t_2,t_3,\ldots,t_{l-1})=lb(x)=(m!x)^{1/m}-\frac{1}{2}(m+1)$, where $m=l-1$, we can visit all the points in $lb(x)$ steps. The remaining points between any two equidistant points can be reached in $(N/x)-1$ steps from the points $i(N/x)$, $i=0,1,2,\ldots,x-1$. Therefore, any point in $[0,N-1]$ can be reached in less than or equal to $t$ steps, where

\[
t=lb(x)+\left(\frac{N}{x}\right)-1
=\left((m!x)^{1/m}-\frac{1}{2}(m+1)+\left(\frac{N}{x}\right)-1.
\]

Replacing $x$ by $cN^{(l-1)/l}$, we have

\[
t=((l-1)!cN^{(l-1)/l})^{1/(l-1)}-\frac{1}{2}(l)+\left(\frac{N^{1/l}}{c}\right)-1
=\left(((l-1)!c)^{1/(l-1)}+\frac{1}{e}\right)N^{1/l}-\frac{1}{2}(l+2),
\]

and the network $G(N;1,s_2,s_3,\ldots,s_l)$ is the one with $s_l=t_l(N/x)$, $i=2,3,\ldots,l-1$ and $s_l=N/x=(1/c)N^{1/l}$. \qed
For \( l = 4 \), the minimum of \( t \) in Theorem 4.2 can be obtained when \( c = (9/2)^{1/4} \). Hence, in this case the minimum value of the leading coefficient in the expression of \( t \) is

\[
(6c)^{1/3} + \frac{1}{c} = \left( 6 \cdot \left( \frac{9}{2} \right)^{1/4} \right)^{1/3} + \left( \frac{2}{9} \right)^{1/4}
\]

\[
= \left( 5832 \right)^{1/12} + \left( \frac{2}{9} \right)^{1/4}.
\]

This is definitely an improvement over the number 4 for the leading coefficient of the upper bound when \( l = 4 \).

By Stirling’s formula, which is \( x \approx (x/e)^x \sqrt{2\pi x} \), we have that the leading coefficient of \( t \) in Theorem 4.2 is equal to

\[
\left( \frac{l-1}{e} \right)^{1/(l-1)} + \frac{1}{c} = \left( \frac{l-1}{e} \right)^{1/(l-1)} \cdot \sqrt{2\pi (l-1)} \cdot c^{-1/(l-1)} + \frac{1}{c}
\]

\[
= \left( \frac{l-1}{e} \right) (2(l-1)^{1/2}(l-1))^{1/(l-1)} \cdot c^{1/(l-1)} + \frac{1}{c}.
\]

It can be shown that by suitably choosing \( c \), this number is less than \( l \), which is the leading coefficient of the upper bound \( l \cdot N^{1/l} - l \) for \( d(N) \).

5. Routing and reliability

The number-theoretic approach used in Section 3 to characterize the optimal topologies is also useful in the study of routing and reliability problems. It enables us to design simple routing algorithms.

Since the network is node-symmetric, we have to consider the routing algorithm only for the short path from node 0 to any other node \( a \) in the network and that from \( a \) to 0. Let \( N - 3t^2 + kt + h \) as in Theorems 3.3 and 3.5, where \( 3 < k < 9 \) and \( 0 < h < t \). The idea is that there always exists a node labeled as \( js \) (mod \( N \)) on the circle between \( is \) and \( (i-1)s \), where \( t+1 \leq j \leq 2t \). More specifically, \( j = t+i+1 \). It follows that \( js - is = 2t + k - h - 1 \) and \( (i+1)s - js = t + h \) as in Fig. 1.

If \( a \) is located between \( is \) and \( js \), then the shortest path from node 0 to node \( a \) is to traverse at steps of size \( s \) until we reach \( is \) and then go on the ring to reach \( a \). However, if \( a \) is between \( js \) and \( (i+1)s \), then we traverse at steps of size \( s \) to \( ts \). Continue at steps of size \( s \), passing the node 0 to \( (t+1)s \), ... until we reach \( js \) and then go on the ring to reach \( a \). The routing algorithm from 0 to \( a \) is, therefore, defined as in Fig. 1.

![Fig. 1](image-url)
Distributed loop network with minimum transmission delay

The resulting situation from node $a$ to node 0 can be simulated by the routing algorithms (i) and (ii). However, we include it here for the sake of completeness and easy implementation. The situation in Fig. 1 is replaced by that of Fig. 2, where the node $b = is + (t + h)$ is a crucial node. The node $b$ enables us to reach $N = 0$ after many steps of jumps at size $s$. Summarizing, we have the following routing rules:

(i) $0, s, 2s, \ldots, is, is + 1, is + 2, \ldots, a - 1, a$ if $is \leq a < js$;
(ii) $0, s, 2s, \ldots, ts, (t + 1)s, (t + 2)s, \ldots, js, js + 1, \ldots, a$ if $js \leq a < (i + 1)s$;
(iii) $is, (i + 1)s, \ldots, ts, ts + 1, \ldots, N$ if $a = is$, $1 \leq i \leq t$;
(iv) $js, (j + 1)s, \ldots, 2ts, 2ts + 1, \ldots, N$ if $a = js$, $1 \leq j \leq 2t$;
(v) $a, a + 1, a + 2, \ldots, b, b + s, \ldots, (t - 1)s + (t + h), N$ if $is < a \leq b = is + (t + h)$;
(vi) $a, a + s, a + 2s, \ldots, x, x + s, x + 2s, \ldots, y, y + s, y + s + 1, \ldots, N$ if $b < a < js$; and
(vii) $a, a + s, a + 2s, \ldots, w, w + 1, w + 2, \ldots, N$ if $js < a < (i + 1)s$,

where $N = ts + (t + h)$ in (v); $x = a + (t - i - 1)s$, $(t - 1)s + (t + h) < x < 2ts$, $(t - 1)s < y < (t - 1)s + (t + h)$, and $ts < y + s < N$ in (vi); and $2ts < w < ts$ and $w = a + (t - i - 1)s$ in (vii). The distribution of the nodes involved in the routing algorithm is shown on the circle of length $3t^2 + kt + h$ in Fig. 3.
To illustrate the two relatively complicated cases (vi) and (vii) in the routing algorithm, we give the following examples. Let \( N = 3t^2 + 3t + 1 \) and \( t = 5 \). By Theorem 3.3, \( N = 91, \ s = 3t + 2 = 17 \) and \( d(91;1,17)=15 \) is optimal among all networks \( G(91;1,s) \). In the first example, we take the node \( a = 42 \) and give a routing from \( a \) to \( 0 \). Since 42 is between \( 34 = 2 \cdot s \) and \( 51 = 3 \cdot s \), we have \( b = 34 + (t + h) = 34 + (5 + 1) = 40 \) and \( js = 8s = 8 \cdot 17 = 136 = 45 \) (mod 91). Clearly, the node \( a = 42 \) falls into the category in (vi). It follows that the routing path is:

42, 59, 76, 2, 19, 35, 70, 87, 88, 89, 90, 91,

which is of length 12. In this case \( x = 76 \) and \( y = 70 \).

The second example deals with \( a = 49 \). The node 49 lies between \( 45 = js \) and \( 51 = 3 \cdot s \). It belongs to the case (viii) in the routing procedure. Therefore, we have the routing path:

49, 66, 83, 84, 85, 86, 87, 88, 89, 90, 91,

which is of length 10.

We now turn to the issue of reliability for the optimal doubly linked loop networks designed in Section 3. We will show that the network has node connectivity = 2. That is, it can tolerate failure of any one node in the network. Moreover, we are also concerned with the transmission delay after a node fails. Does it increase the diameter of the network? If it does, how much is the increase?

Instead of putting the \( N \) nodes on a circle, we now put these \( N \) nodes at lattice point locations \((x, y)\) in the first quadrant of the Euclidean plane. The values for \( x \) and \( y \) are nonnegative integers. Let the node \( n \) be associated with the location \((x, y)\) such that \( y \cdot s + x \cdot 1 = n \) (mod \( N \)) as in the routing algorithms (i) and (ii). It is easily verified that the \( N \) nodes thus located constitute an L-shaped pattern in the plane as described in Wong and Coppersmith [25]. In Fig. 4, we show the pattern for \( N = 3t^2 + 3t, \ t = 3, \ N = 36, \ s = 3t + 2 = 11 \) and \( d = 3t = 9 \).

In general, let \( N = 3t^2 + kt + h, \ 3 \leq k \leq 9 \) and \( 0 \leq h < t \). By Theorem 3.3, \( s = 3t + (k - 1) \) and \( d = \max \{3t + k - h - 3, 3t + h - 1\} \). The pattern is shown in Fig. 5, where
Distributed loop network with minimum transmission delay

Fig. 5. \( N = 3t^2 + kt + h, s = 3t + (k - 1). \)

\[
X = (t + 1)s - 1 = 2t + k - h - 2, \quad Y = 2ts, \quad Z = Y - 1 = 2ts - 1 \quad \text{and} \quad W = ts - 1.
\]
Clearly, we have

\[
0 < X < (t + 1)s < s < (t - 1)s < Z < Y < ts < N.
\]

The pattern in Fig. 5 represents the ways nodes in the networks can be reached by the node 0. This representation gives rise to short paths between 0 and any other node. It is obvious that any node at location \((x, y), x \neq 0, y \neq 0,\) can be reached from the node in two disjoint directed paths. The routing algorithms (i) and (ii) obtained earlier in this section give one path. Hence, failure of any node in one of the two paths would not disconnect the connection from 0 to the node at \((x, y)\) in the L-shaped pattern. The situation is quite different for nodes at location \((x, y)\) where \(x = 0\) or \(y = 0.\) The pattern in Fig. 5 shows only one path from the node 0 to the nodes with \(x = 0\) or \(y = 0.\) Rerouting procedure has to be designed when a node on this path fails. Without loss of generality, we can assume that \(y = 0\) and a node \(q\) located at \((q, 0)\) fails on the path \(P\) from the node 0 to the node \(X,\) where \(0 < q < X.\) The original routing path from 0 to \(X\) before the failure of the node \(q\) is \(0, 1, 2, ..., q, ..., X.\) The new routing path from 0 to \(X\) consists of 0, s, ..., X without passing through the node \(q.\) The subpath from \(s\) to \(W + s\) does not exist in the L-shaped pattern depicted in Fig. 5. However, we have the following network rerouting path:

\[
0, s, 2s, ..., ts, (t + 1)s, s, Y, Y + 1, Y + 2, ..., W, W + s,
\]
which extends the pattern of Fig. 5 and avoids passing the node \(q, 0 < q < X.\) Note that \(W + s = (ts - 1) + s = (t + 1)s - 1 = X.\) Since the path from \(s\) to \(W + s\) is of the same length as that from 0 to \(W,\) its length is at most the diameter \(d(N; 1, s)\) of the network. Hence, the resulting path has length at most \(d(N; 1, s) + 1.\) Summarizing, we have the following theorem.

**Theorem 5.1.** The doubly linked loop networks designed in Theorems 2.2, 3.2, 3.3 and 3.5 have node connectivity = 2. They can always tolerate at least one faulty node. Moreover,
the rerouting path of each network after a node fails is of length at most one greater than
the diameter of the network.

As an example to illustrate Theorem 5.1, let \( N = 36 \) be as in Fig. 4. Then \( s = 11 \)
and \( d = 9 \). Take the routing path from 0 to 30, i.e., 0, 11, 22, 33, 8, 19, 30. Suppose
node \( q \) is faulty, where \( q \in \{11, 22, 33, 8, 19\} \). The rerouting path is as follows:
0, 1, 2, 3, 4, 5, 6, 7, 18, 29, 30, which is of length \( 10 = d + 1 \).

6. Extensions and conclusions

This paper gives the first number-theoretic and combinatorial method for solutions
in the design, evaluation and implementation of distributed-loop networks with
minimum transmission delay and maximum reliability. We show that doubly linked
loop networks with transmission delay close to the lower bound \( \lceil (3N)^{1/2} \rceil - 2 \) can be
constructed asymptotically for \( N \) sufficiently large. We demonstrate the developed
systematic and unified method for giving several classes of \( N \) for which optimal
doubly linked loop networks can be designed.

For multiply linked loop networks with \( l \geq 3 \), the analogous problem of designing
optimal loop networks remains largely open. In this paper, we take the first step to
improve the upper bounds. In fact, a new upper bound is obtained for each \( l \geq 3 \). In the
special case when \( l = 3 \), Morillo et al. [16] propose a triply linked loop network called
\( TLD(1, b, c) \) in which node \( i \) is adjacent to nodes \( i + 1, i + b, \) and \( i + c \), where \( c = b + 1 \).
Although these loop networks have as good routing and rerouting procedures, their
diameters are \( \lceil (N - 1)^{1/2} \rceil \), which is larger than what we get in Theorem 4.1, which is
of order \( O(N^{1/3}) \). Other variations have also been considered by Morillo et al.
[16,17]. However, their diameters are all of the order \( O(N^{1/2}) \).

As noted before, the existence of at least one ring in a loop network is important
both in the local area networks and in supercomputer architecture. By taking \( s_1 = 1 \) in
the network \( G(N; s_1, s_2, \ldots, s_l) \) where node \( i \) is adjacent to nodes \( i + s_j \pmod{N} \),
\( j = 1, 2, \ldots, l \), we are guaranteed to have a ring. If \( \gcd(s_j, N) = 1 \) for some \( j \), then the
jump size \( s_j \) would give rise to another ring in the network. However, it may not be the
best choice since it might lead to a loop network with diameter much larger than the
lower bound. The discrete nature of the problem makes it difficult to obtain optimal
results in any closed form. Moreover, the function \( d(N) \) does not increase monotoni-
cally with \( N \). For example, \( d(64; 1, 12) = 13 = d(64) \), but \( d(65; 1, 15) = 12 = d(65) \). How-
ever, we have successfully proved that doubly linked loop networks with optimal or
nearly optimal diameters can always be constructed asymptotically as long as the
number of nodes \( N \) is large enough.

Fiol et al. [8] report that when \( l = 2 \) and \( N = 450 \), \( d(N; 2, 185) = 35 \) is optimal among
all \( G(N; s_1, s_2) \) networks with 450 nodes, while \( d(N; 1, 59) = 36 \) is the minimum dia-
meter among all loop networks \( G(N; 1, s) \) with 450 nodes. This is the first example
known so far that networks with \( s_1 \neq 1 \) have slightly better diameter than that of loop
networks with $s_1 = 1$. However, we note that the networks with $s_1 \neq 1$ in general do not contain a ring. In fact, if we do not require the ring property in the network, the diameter can be much better. Take the class of networks with each node of indegree = outdegree = 1. It can be shown that the lower bound for the diameters of these networks is \( \lfloor \log_2 N \rfloor \), where \( N \) is the number of nodes. A special class of these networks, called generalized deBruijn networks \( G_b(N, d) \), was proposed independently by Imase and Itoh [15] and Reddy et al. [23]. These are networks of indegree = outdegree = \( d \) with \( N \) nodes and the set of \( Nd \) links \( \{i \rightarrow di + r \pmod{N} \mid 0 \leq i \leq N - 1, 0 \leq r \leq d - 1 \} \). The well-known deBruijn network is a special case of \( G_b(N, d) \) where \( N \) is a power of \( d \). The deBruijn networks have been used for constructing multistage networks such as Stone's shuffle-exchange network and Laurie's \( \Omega \)-network. The network \( G_b(N, d) \), in general, may not be a very suitable topology for multiple processor systems of local networks because of the lack of simple routing and rerouting algorithms. Moreover, a general \( G_b(N, d) \) does not contain a ring. Du et al. [5] propose a class of networks, called \( H_a \), by modifying the network \( G_b(n, 2) \) and show that it has minimum diameter \( \lfloor \log_2 n \rfloor \), maximum connectivity 2 and a ring structure. But \( H_a \) is not as practical as those designed in this paper for implementation and routing. Recently, Du et al. [6] show that if \( \gcd(N, d) = 1 \) and \( d > 2 \), then \( G_b(N, d) \) has a ring. However, it is easy to see that if \( N \) is odd, \( G_b(N, 2) \) does not have the ring property.

A variation on the problem studied in this paper is the network \( G(N; \pm 1, \pm s_2, \pm s_3, ..., \pm s_l) \) with bidirectional links between adjacent nodes. In this network, node \( i \) is adjacent to \( 2l \) other nodes \( i + 1, i - 1, i + s_2, i - s_2, ..., i + s_l, i - s_l \). Hence, the network is treated as an undirected graph of degree \( 2l \). The analogous minimization and routing problems can be similarly defined and studied. Lower bounds and upper bounds are given by Wong and Coppersmith [25]. For \( l = 2 \), i.e. the degree-4 case, Bocsh and Wang [3] and Bermond et al. [2] give the minimum diameter \( d_0 = \lfloor (1 + g)/2 \rfloor \), where \( g = (2N - 3)^{1/2} \) for the network \( (\pm s_1, \pm s_2) \) with \( N \) nodes. This diameter is obtained by taking \( s_1 = d_0 \) and \( s_2 = d_0 + 1 \). Although this network has maximum connectivity 4, the general design does not contain a ring. Recently, Du et al. [7] have successfully obtained new classes of values of \( N \) for which loop topologies \( G(N; \pm 1, \pm s) \) can be found that achieve the lower bound \( \text{lb} = \lfloor (\sqrt{2N - 1} - 1)/2 \rfloor \) for the minimum diameter. More results on this problem and other variations have been obtained recently by Du and Hsu [4], and by Hsu and Shapiro [11, 12].

Finally, we offer some questions for further investigation:

(a) Classify those \( N \)'s for which \( d(N) = \text{lb}(N) \) and those \( N \)'s for which optimal (but \( d(N) \neq \text{lb}(N) \)) loop networks can be found.

(b) We know that given a number \( N \), \( d(N) - \text{lb}(N) \geq 0 \). How big can \( d(N) - \text{lb}(N) \) be?

(c) Can a better upper bound for \( d(N) \) in terms of \( N \) be found?

(d) Extend the current study on \( G(N; 1, s) \) to the general case \( G(N; 1, s_2, s_3, ..., s_l) \) where each node \( i \) is adjacent to \( i + 1, i + s_2, i + s_3, ..., \) and \( i + s_l \).
For the cases when \( l \geq 3 \), we have successfully improved the upper bound for \( d(N) \). However, no infinite classes of \( N \) has been found for which there exist loop topologies \( G(N; 1, s_2, s_3, \ldots, s_l) \) with \( d(N; 1, s_2, s_3, \ldots, s_l) = d(N) \) or \( l\beta(N) \), where \( l \geq 3 \). In the particular case when \( l = 3 \), Hsu and Jia [10] have recently shown that \((14 - 3\sqrt{3})N^{1/3} \leq d(N) \leq (16N)^{1/3}\). We also note that with regard to (b), \( d(N) - l\beta(N) \) can be as large as a function of \( N \). It has been communicated to the authors by D. Coppersmith that there is a constant \( c \) such that there exists an infinite class of \( N \) for which the minimum diameter \( d(N) \geq \sqrt{3}N + c(\log N)^{1/4} \). Moreover, a recent computer search by Y. Cheng shows that for \( 1 \leq N \leq 75,000 \), there exist only 3 \( N \)'s which have the largest gap between \( d(N) \) and \( l\beta(N) \). In fact, \( d(N) - l\beta(N) = 4 \) for \( N = 53,749, \ s = 985, \ d(N) = 404; \ N = 64,729, \ s = 394, \ d(N) = 443; \) and \( N = 69,783, \ s = 1764, \ d(N) = 458 \). However, for \( 1 \leq N \leq 30,000 \), there are several \( N \)'s for which \( d(N) - l\beta(N) = 3 \). These numerical data might be helpful for the study of the problems mentioned.

References