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Note

On the orbits of the product of two permutations

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Abstract

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We consider the following problem: given three partitions A, B, C of a finite set Ω, do there exist two permutations α and β such that A, B, C are induced by α, β and $\alpha\beta$ respectively? This problem is NP-complete. However it turns out that it can be solved by a polynomial time algorithm when some relations between the number of classes of A, B, C hold.

1. Introduction and notation

A permutation α of a set Ω *induces* a partition A of Ω defined by the orbits of α.

We are interested in the existence of permutations α, β on a finite set Ω such that α, β and $\gamma = \beta \cdot \alpha$ induce three given partitions A, B and C on Ω.

If we only take into account the length of the orbits of α, β and γ, while ignoring their elements, this problem is a classical one in symmetric group algebra theory [11]. Brenner and Lyndon [3] examined this problem in detail when γ is transitive (i.e. a circular permutation). Similar problems were studied by Bertram in [1], who characterized the integer l for which any even permutation could be represented as the product of two cycles of length l. Boccara gave a generalization to products of two cycles of different length [2].

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Here we consider the partitions defined by the orbits of α, β and γ rather than merely the types of these partitions. This is a more selective problem. This question was considered in the planar case [12].

A nondirected graph G embedded in an orientable surface can be represented by a couple (α, β) of permutations (where β is an involution without fixed point); the *faces* are defined by the orbits of the product $\beta\alpha$.

In this framework, it is possible, when all classes of B have two elements, to state our problem in terms of graph theory: Given a set E of edges, let $2E$ be the set obtained by duplicating each element of E and let A and C be two partitions of $2E$. Does there exist a graph $G=(V,E)$ and an embedding of G in an orientable manifold such that each class of A consists of the edges incident with a given vertex and each class of C consists of the edges bording a given face? Note that, since the number of vertices, edges and faces is given by A, B and C the genus of the embedding can be obtained by Euler formula.

When the classes of B have an arbitrary number of elements, a similar translation can be obtained for our problem (see below).

In the following we prove that *PPP* is NP-complete if no additional hypothesis are made for A, B and C. However a polynomial time algorithm solves the problem if $\|A\|+\|B\|+\|C\|=|\Omega|+2$ (planar case) and if $\|C\|=1$ ($\alpha\beta$-transitive case).

In Section 2 we study the general problem, and in Section 3 its computational complexity. Then we specialize to the planar case and that in which $\alpha\beta$ is transitive.

The notation we use is the following:

Ω is a finite set of elements called *darts*. $|\Omega|$ is the cardinality of Ω.

$[n]$ is the set of integers from 1 to n. $]n, m]$ represents the set of integers from $n+1$ to m; if $m \leqslant n$, this set is empty.

α, β, γ are permutations of Ω. $z(\alpha)$ is the number of cycles of α. $\beta\cdot\alpha$ or $\beta\alpha$ is the composition of the permutation α and β. Products are written from right to left: $\beta\alpha(x)=(\beta\alpha)(x)=\beta(\alpha(x))$.

A, B, C and D are partitions of Ω. $part(\alpha)$ is the partition of Ω induced by the orbits of α. a is a block or class of A, $\|A\|$ is the number of classes in A. Two partitions A and B of Ω induce a hypergraph G whose vertices and hyperedges are classes a_i and b_j respectively. A vertex a_i is *incident* to the hyperedge b_j if $a_i \cap b_j \neq \emptyset$.

In this notation our problem *PPP* is as follows: Given three partitions A, B and C of a set Ω, are there permutations α and β such that $part(\alpha)=A$, $part(\beta)=B$ and $part(\beta\alpha)=C$? Note that the answer is the same if the roles of A, B and C are exchanged because $part(\alpha)=part(\alpha^{-1})$ and $\beta\alpha=\gamma \Rightarrow \alpha=\gamma\beta^{-1}$. PPP_2 is a *PPP*-problem where C is a bipartition (i.e. it consists of two blocks c_1 and c_2).

2. General facts

Considerations of the parities of α, β and $\beta\alpha$ leads to:

PPP has a solution only if $|\Omega| \equiv \|A\|+\|B\|+\|C\| \bmod 2$.

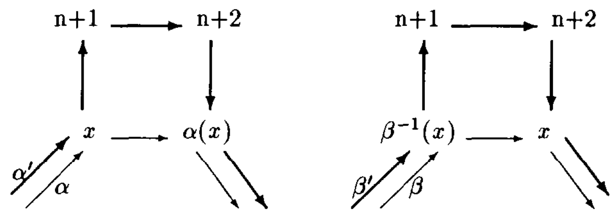

Fig. 1. Modifying α and β.

Proof. α, β and γ have the same parities as $|\Omega| - \|A\|$, $|\Omega| - \|B\|$ and $|\Omega| - \|C\|$ respectively. Moreover $\gamma = \beta\alpha$ has the same parity as $2|\Omega| - \|A\| - \|B\|$ because it is the product of α by β. Thus $(|\Omega| - \|C\|) \bmod 2 \equiv (\|A\| + \|B\|) \bmod 2$ or equivalently $|\Omega| \bmod 2 \equiv (\|A\| + \|B\| + \|C\|) \bmod 2$. □

Let G be the hypergraph induced by the partitions A and B of a problem P. If P has a solution, then the partition $C = part(\beta\alpha)$ defines the faces of the combinatorial hypermap (Ω, α, β). The genus $g = 1/2(|\Omega| - (\|A\| + \|B\| + \|C\| - 2))$ of such an hypermap is proved to be a positive integer in [10]. It defines the genus of the surface in which G is embedded. Thus if P has a solution there is an embedding of G on a surface of genus g, and if G is connected then P has a solution only if $|\Omega| \geqslant \|A\| + \|B\| + \|C\| - 2$. If we have all the embeddings of G with genus g, then we can solve P checking the faces of every embedding of G.

In order to solve P we could imagine an incremental method. Let P be a problem for which we got a solution (α, β) and let P' the problem we are trying to solve. When there is some relationship between P and P' we have a solution for P'. This is the content of the following lemmas.

Lemma 2.1. *Given the PPP-problem $P = (\Omega, A, B, C)$ and blocks $a \in A$, $b \in B$ and $c \in C$ satisfying the mild condition that $a \cap b \cap c \neq \emptyset$, we can create a new PPP-problem $P' = (\Omega', A', B', C')$ where Ω' is Ω with the addition of two new elements; and A', B' and C' are partitions of Ω' formed from A, B and C by adding the two new elements to the blocks a, b and c respectively. If P has a solution (α, β), then P' has a solution (α', β').*

Proof. Without loss of generality, $\Omega = [n]$ and $\Omega' = [n+2]$. Let $x \in a \cap b \cap c$. We define α' by $\alpha'(x) = n+1$, $\alpha'(n+1) = n+2$, $\alpha'(n+2) = \alpha(x)$, $\alpha'(y) = \alpha(y)$ if $y \neq x$; and we define β' by $\beta'(\beta^{-1}(x)) = n+1$, $\beta'(n+1) = n+2$, $\beta'(n+2) = x$ and $\beta'(y) = \beta(y)$ if $y \neq \beta^{-1}(x)$ (see Fig. 1).

Thus $A' = part(\alpha)$ and $B' = part(\beta')$. The values of $\beta\alpha(y)$ induced by changing from (α, β) into (α', β') are modified only for $y = x$ and $y = \alpha^{-1}\beta^{-1}(x)$.

If $\alpha^{-1}\beta^{-1}(x) = x$ ($\beta\alpha(x)$ is the only element in c), then $\beta'\alpha'(x) = \beta'(n+1) = n+2$, $\beta'\alpha'(n+2) = \beta'(x) = n+1$, $\beta'\alpha'(n+1) = \beta'(n+2) = x$; so we inserted $n+1$ and $n+2$ in the $\beta\alpha$-orbit of x.

If $\alpha^{-1}\beta^{-1}(x)\neq x$ $(\alpha(x)\neq\beta^{-1}(x))$, then $\beta'\alpha'(\alpha^{-1}\beta^{-1}(x))=\beta'\beta^{-1}(x)=n+1$, $\beta'\alpha'(n+1)=\beta'(n+2)=x$, $\beta'\alpha'(x)=\beta'(n+1)=n+2$, $\beta'\alpha'(n+2)=\beta'(\alpha(x))=\beta\alpha(x)$ because $\alpha(x)\neq\beta^{-1}(x)$; here as well we inserted $n+1$ and $n+2$ on both sides of x in its $\beta\alpha$-orbit.

Thus, $C'=part(\gamma')$. □

Note: The converse is false. For instance, problem P defined by $\Omega=[5]$, $A=\{\{1,2,3\},\{4,5\}\}, B=\{\{1,2\},\{3,4,5\}\}$ and $C=\{\Omega\}$ does not have a solution but (α': $(1,6,2,3,7)(4,5)$, β': $(1,2)(3,5,7,4,6)$) is a solution to problem P' defined by $\Omega'=[7]$, $A'=part(\alpha')$, $B'=part(\beta')$ and $C'=\{\Omega'\}$.

Lemma 2.2. *Let $P=(\Omega,A,B,C)$ be a PPP-problem such that A contains distinct blocks a_1 and a_2, B and C contains blocks b and c, respectively, such that $a_1\cap b\cap c\neq\emptyset$ and $a_2\cap c\neq\emptyset$. Let $P'=(\Omega',A',B',C')$ where Ω' has two new elements, both of which are added to b and c and each one is added to a_1 and a_2 (with A,B and C otherwise unchanged). If there is a solution (α,β) of P, then problem P' has a solution (α',β').*

If B contains distinct blocks b_1 and b_2, A and C contains blocks a and c such that $a\cap b_1\cap c\neq\emptyset$ and $b_2\cap c\neq\emptyset$, and $\Omega=[n]$. Let $P''=(\Omega'',A'',B'',C'')$ where Ω'' has two new elements both of which are added to a and c and each one is added to b_1 and b_2 (with A,B and C otherwise unchanged). If there is a solution (α,β) of P, then there is a solution (α'',β'') of P''.

Proof. As we can exchange the role of partitions A,B and C, we shall only prove the first part of lemma.

We suppose $\Omega=[n]$ and $\Omega'=[n+2]$. Let $x\in a_1\cap b\cap c$ and $y\in a_2\cap c$. Then α' is defined as follows: $\alpha'(x)=n+1, \alpha'(n+1)=\alpha(x), \alpha'(y)=n+2$, $\alpha'(n+2)=\alpha(y)$ and $\alpha'(z)=\alpha(z)$ if $z\notin\{x,y\}$. β' is defined as follows: $\beta'(\beta^{-1}(x))=n+1$, $\beta'(n+1)=n+2$, $\beta'(n+2)=x$, and $\beta'(z)=\beta(z)$ if $z\neq\beta^{-1}(x)$. We have $A'=part(\alpha')$ and $B'=part(\beta')$ as desired. Now it suffices to show $C'=part(\beta'\alpha')$.

Note that $\alpha^{-1}\beta^{-1}(x)\neq x$ since x and y are in the same orbit of $\beta\alpha$. Moreover, if $u\notin\{x,y,\alpha^{-1}\beta^{-1}(x)\}$, we have $\beta'\alpha'(u)=\beta\alpha(u)$.

There are two cases to be considered.

Case a: If $\alpha^{-1}\beta^{-1}(x)=y$, then $\beta'\alpha'(\alpha^{-1}\beta^{-1}(x))=\beta'\alpha'(y)=\beta'(n+2)=x$; $\beta'\alpha'(x)=\beta'(n+1)=n+2$; $\beta'\alpha'(n+2)=\beta'\alpha(y)=\beta'\beta^{-1}(x)=n+1$; $\beta'\alpha'(n+1)=\beta'\alpha(x)=\beta\alpha(x)$ because $\beta\alpha(x)\neq x$.

In the $\beta'\alpha'$-orbit of x and y, the sequence $y\to x\to\beta\alpha(x)$ has become $y\to x\to n+2\to n+1\to\beta\alpha(x)$.

Case b: If $\alpha^{-1}\beta^{-1}(x)\neq y$, then $\beta'\alpha'(\alpha^{-1}\beta^{-1}(x))=\beta'\beta^{-1}(x)=n+1$; $\beta'\alpha'(n+1)=\beta'\alpha(x)=\beta\alpha(x)$; $\beta'\alpha'(x)=\beta'(n+1)=n+2$; $\beta'\alpha'(n+2)=\beta'\alpha(y)=\beta\alpha(y)$; $\beta'\alpha'(y)=\beta'(n+2)=x$.

Now we have $\alpha^{-1}\beta^{-1}(x)\to n+1\to\beta\alpha(x)$ and $y\to x\to n+2\to\beta\alpha(y)$ instead of $\alpha^{-1}\beta^{-1}(x)\to x\to\beta\alpha(x)$ and $y\to\beta\alpha(y)$.

In both cases, $n+1$ and $n+2$ are inserted in the $\beta\alpha$-orbit of x and y (see Fig. 2). Thus $C'=part(\beta'\alpha')$. □

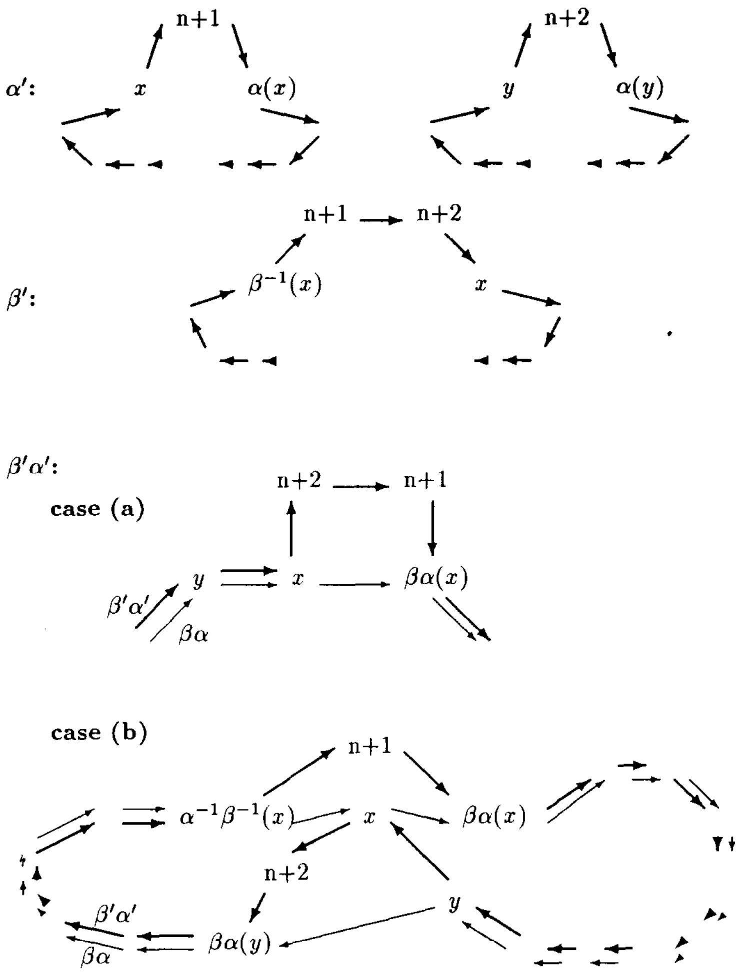

Fig. 2. Addition of two elements in a $\beta\alpha$ orbit.

3. Computational complexity of problem *PPP*

It is obvious that $PPP \in \text{NP}$: a guess (α, β) can be checked in polynomial time and space. The PPP_2-problem is shown to be NP-complete by reduction of the classical problem of existence of a hamiltonian circuit in directed graphs (*DHC*). This problem can be stated as follows: given a directed graph G, is there a simple directed circuit in G which passes through all the vertices?

3.1. Construction of the PPP_2-problem associated with a given DHC-problem

A *directed graph* G is definite by a quadruple (V, E, out, in) where V is the set of vertices, E is the set of edges; and *out* and *in* are functions that associate with each vertex the set of edges leaving and entering it. Let s be a vertex of a graph, and let $deg(s)=|out(s)|$ be the number of edges leaving s. The edge e is linking vertex s_i to vertex s_j if and only if $e \in out(s_i)$ and $e \in in(s_j)$.

Let $G(V, E, out, in)$ be a graph with n vertices $V=\{s_1, s_2, \ldots, s_n\}$ and m edges. To each vertex s define $dl(s)=deg(s)-1$ if $deg(s) \geqslant 1$, and $dl(s)=0$ if $deg(s)=0$. $dl(s)$ represents the number of choices at s when building a hamiltonian circuit. For example, if $dl(s)=1$, there is a single choice between two possible edges to explore from s.

Let $d=\sum_{i=1}^{n} dl(s_i)$, $s_i \in V$. Usually $d=|E|-|V|$ except if there is a vertex s of G without any outgoing edges, in which case there are no hamiltonian circuits.

Let $D=\{d_1, d_2, \ldots, d_n\}$ be a partition of $[d]$ into n blocks given by $d_1=[dl(s_1)]$, $d_i=]\sum_{j=1}^{i-1} dl(s_j), \sum_{j=1}^{i} dl(s_j)]$ if $i \neq 1$. In this way, we have $dl(s_i)$ darts in a set d_i for each vertex s_i.

Now we can associate with a DHC-problem H a corresponding PPP_2-problem $P=(\Omega, A, B, C)$:

$\Omega=V \cup E \cup [d]$, $A=\{a_1, a_2, \ldots, a_n\}$ where $a_i=\{s_i\} \cup out(s_i) \cup d_i$, $B=\{b_1, b_2, \ldots, b_n\}$ where $b_i=\{s_i\} \cup in(s_i) \cup d_i$, $C=\{c_1, c_2\}$ where $c_1=V$ and $c_2=E \cup [d]$.

Note that if the original graph G has n vertices and m edges, the size of the new problem is of $O(n, m)$.

Lemma 3.1. *If the PPP-problem P associated with a DHC-problem H has a solution, then H has a solution.*

Proof. Clearly, $\beta\alpha$ maps vertices to vertices, all of them being in the orbit $c_1$of $\beta\alpha$. Let s_i and s_j be vertices such that $\beta\alpha(s_i)=s_j$. Then $\alpha(s_i) \in out(s_i)$ because the other possibility ($\alpha(s_i) \in d_i$) is incompatible with $\beta(\alpha(s_i))=s_j$ ($s_j \notin \beta(d_i)$). Thus $\alpha(s_i)$ is an edge from vertex s_i to vertex s_j. Then $(\alpha(s_1), \alpha\beta\alpha(s_1), \alpha(\beta\alpha)^2(s_1), \ldots, \alpha(\beta\alpha)^{n-1}(s_1))$ gives the sequence of edges of a hamiltonian circuit in G. □

Lemma 3.2. *If a DHC-problem H has a solution, then the associated PPP-problem P has a solution.*

Proof. We shall proceed by successive additions of darts in order to build a solution (α, β). Starting with a solution (α^0, β^0) of a problem P^0; we add edges two at a time (using the lemmas of Section 2) to have a solution of problem $P^d=P$ after d steps.

Let CH be the set of edges of some hamiltonian circuit in G. We consider the problem $P^0=(\Omega^0, A^0, B^0, C^0)$ where $\Omega^0=V \cup CH$, $A^0=\{a_1^0, a_2^0, \ldots\}$, $B^0=\{b_1^0, b_2^0, \ldots\}$, $C^0=\{c_1^0, c_2^0\}$ with $c_1^0=V$, and $c_2^0=CH$, while $a_i^0=\{s_i, e_{ij}\}$ and $b_j^0=\{s_j, e_{ij}\}$ for all $e_{ij} \in CH$.

The remainder of the proof of this lemma is based on the following three propositions whose proofs are immediate.

Proposition 3.3. *Define two permutations α^0, β^0 of Ω^0 by $\alpha^0(s_i)=e_{ij}$ and $\alpha^0(e_{ij})=s_i$, $\beta^0(s_i)=e_{ki}$ and $\beta^0(e_{ki})=s_i$. Then (α^0, β^0) is a solution of problem P^0. Moreover $a_i^0 \subseteq a_i$, $b_i^0 \subseteq b_i$ and $c_i^0 \subseteq c_i$ for all i.*

For every vertex $s_i, dl(s_i)$ edges and $dl(s_i)$ elements of d_i not in Ω^0 remain to be added. With each edge $e_{ij} \notin CH$ we associate a dart $u_{ij} \in d_i$. We shall add both e_{ij} and u_{ij}. Problem P^{p+1} is defined in terms of problem P^p: $\Omega^{p+1}=\Omega^p \cup \{e_{ij}, u_{ij}\}$; $a_i^{p+1}=a_i^p \cup \{e_{ij}, u_{ij}\}$, $a_j^{p+1}=a_j^p$ if $j \neq i$; $b_i^{p+1}=b_i^p \cup \{u_{ij}\}$, $b_j^{p+1}=b_j^p \cup \{e_{ij}\}$, $b_k^{p+1}=b_k^p$ if $k \notin \{i,j\}$; $c_1^{p+1}=c_1^p$, $c_2^{p+1}=c_2^p \cup \{e_{ij}, u_{ij}\}$.

Proposition 3.4. *If P^p has a solution, then P^{p+1} has a solution.*

This is immediate using Lemma 2.2.

Proposition 3.5. *The property $a_i^{p+1} \subseteq a_i$, $b_i^{p+1} \subseteq b_i$, $c_i^{p+1} \subseteq c_i$ (of Proposition 3.3), is invariant under the addition of $\{e_{ij}, u_{ij}\}$.*

Thus we have (α^0, β^0) a solution of problem P^0, then a sequence of (α^i, β^i), solutions of a sequence of problems P^i. After d additions we get (α^d, β^d), a solution of problem $P^d=P$, derived problem from H. This proves Lemma 3.2. □

Theorem 3.6. *PPP_2 is* NP-*complete.*

Proof. Lemmas 3.1 and 3.2 show that PPP_2 is equivalent to DHC. □

4. Solving the problem when $\|C\|=1$

A pair of partitions (A, B) defines a *bipartite graph* $G_{A,B}$, it has the blocks of A and B as vertices, and an edge between $a \in A$, $b \in B$ if $a \cap b \neq \emptyset$.

A block x of a partition A, B or C of a problem P will be called a *vertex*, *hyperedge* or *face* of P respectively. A face c is said to be *incident* to a block x of A or B if $x \cap c \neq \emptyset$.

If P (with $\|C\|=1$) has a solution, $\beta\alpha$ has one orbit. This translates in the fact that the graph $G_{A,B}$ can be embedded with one face in a surface of maximum genus. Xuong gives in [14] a criterion for the existence of such an embedding. Moreover, a polynomial time algorithm using this criterion was recently published [6].

Let $P=(\Omega, A, B, C)$ be a PPP-problem with $\Omega=[n]$ and $C=\{\Omega\}$.

Let *decn* be the function $decn(S)=\{s+n: s \in S\}$ on sets of integers.

Let $P'=(\Omega', A', B', C')$ the PPP-problem defined by $\Omega'=[2n]$, $A'=A \cup \{decn(b): b \in B\}$, $B'=\{\{1,n+1\}, \{2,n+2\}, \{3,n+3\}, \dots, \{n,2n\}\}$, and $C'=\{\Omega'\}$.

Lemma 4.1. *P accepts a solution if and only if P′ has a solution.*

Proof. (Only if) Let (α, β) be a solution of P. We define $\alpha'(x) = \alpha(x)$ if $x \leqslant n$, $\alpha'(x) = \beta(x-n)+n$ otherwise. Note that $\beta'(x) = n+x$ if $x \leqslant n$ and $\beta'(x) = x-n$ if $x > n$. Let us compute the orbits of $\beta'\alpha'$. If $x \leqslant n, \beta'\alpha'(x) = \beta'(\alpha(x)) = \alpha(x)+n$, $\beta'\alpha'(\alpha(x)+n) = \beta'(\beta\alpha(x)+n) = \beta\alpha(x)$, therefore $(\beta'\alpha')^2(x) = \beta\alpha(x)$. We have a $\beta'\alpha'$-orbit which contains $[n]$ and such that the element following an $x \leqslant n$ is greater than n. This orbit passes through $[2n]$, thus (α', β') is a solution of P'.

(If) Let (α', β') be a solution of P'. Since β' is necessarily given by $\beta'(x) = n+x$ if $x \leqslant n$ and $\beta'(x) = x-n$ if $x > n$, the $\beta'\alpha'$-orbit alternately meets an element of $[n]$ and an element of $]n, 2n]$. For $x \leqslant n$ we set: $\alpha(x) = \beta'\alpha'(x) - n$ and $\beta(x) = \beta'\alpha'(x+n)$. Thus $\beta\alpha(x) = \beta(\beta'\alpha'(x)-n) = (\beta'\alpha')^2(x)$. So we have in the $\alpha\beta$-orbit one element out of two which were in the $\alpha'\beta'$-orbit. These elements are less or equal to n. Thus (α, β) is a solution of P. □

Theorem 4.2. *If $\|C\| = 1$ then we can solve P while using polynomial time and space.*

Proof. Lemma 4.1 says in this situation that every hypergraph is equivalent to a bipartite graph and vice versa. So we can find a solution using Furst, Gross and McGeoch's algorithm [6] for maximum genus embedding of $G'_{A',B'}$, the bipartite graph associated with P'. P has a solution if and only if there is such an embedding with one face of $G'_{A',B'}$. Thus we can answer in polynomial time (and space) when $\|C\| = 1$. □

We saw in Theorem 3.6 that the *PPP*-problem is NP-complete, in particular when $\|C\| = 2$. But embedding a graph in a maximum genus surface (even with two faces) is polynomial. There is no contradiction, since we impose elements of faces in our problem; if $\|C\| = 1$, both are equivalent, because darts are necessarily all in the same face.

A theorem of Xuong says that if the graph G has an embedding of maximum genus, then $G' = G \cup \{u, v\}$ (where u and v are new edges satisfying some technical property) has also an embedding of maximum genus. Using Lemma 4.1, adding two new adjacent edges in a bipartite graph is like adding two darts in the associated hypermap. Lemmas 2.1 and 2.2 (which allow us to add two darts to a problem P) are weaker than Xuong's theorem; but Xuong's theorem does not take into account the membership of edges to blocks C_i as we do. As above, this is not important.

5. The planar case

The problem $P = (A, B, C)$ is said to be planar if $\|A\| + \|B\| + \|C\| = |\Omega| + 2$. In the following we consider a planar problem P.

A *circuit* on a class c is an order $(x_1, x_2, \ldots, x_k)$ on the k darts of c such for all i there is a dart x_i' such x_i and x_i' are in the same block of A and x_i' and x_{i+1} (where x_{k+1} is to be interpreted as x_1) are in the same class of B.

A necessary condition for P to have a solution is that we can find a circuit for every block c. If all classes of B have two darts (G is a graph) and when moreover the problem P is planar, this condition becomes sufficient [12]. Let us recall the proof Machí gave. A permutation Π such $part(\Pi)=C$ is defined by circuits on blocks of C. Let β be the involution defined by partition B. $\Pi\beta$ defines a permutation whose cycles are included in the blocks of A. Let $(\beta(l), k_l, \ldots)$ be a cycle of Π where $l \in \Omega$. Thus $\Pi\beta(l)=\sigma(l)=k_l$ and the definition of a circuit implies that k_l and l are in the same block a_l of A. So we have $z(\Pi\beta) \geqslant \|A\|$. As G is connected we also have $z(\beta)+z(\Pi)+z(\Pi\beta) \leqslant n+2$. Thus $n+2=\|A\|+\|B\|+\|C\| \leqslant z(\Pi\beta)+z(\beta)+z(\Pi) \leqslant n+2$. Hence $(\Pi\beta, \beta)$ is a solution of P.

This is false when there is no condition on B. We can easily check that the problem defined by:

$$A=\{\{1,2,3,4\}, \{5,6,7\}, \{8,9,10\}\},$$

$$B=\{\{1,6,9\}, \{3,7\}, \{2,8\}, \{4,5,10\}\},$$

$$C=\{\{6,10\}, \{4,7\}, \{1,8\}, \{2,9\}, \{3,5\}\},$$

has no solution while we can find a circuit on elements for each cycle of C.

Theorem 5.1. *If P is planar and each class of B contains two elements then we can solve P in polynomial time and space.*

Proof. First we note that the permutation β is determined by partition B. By the remark above, we only have to find a circuit on darts for each class c_i to find a solution to P.

Let G be the directed graph such that each vertex s_i of G is associated with a block a_i of A and such that there is an edge e from s_i to s_j if and only if there is a dart $x \in a_i$ and $\beta(x) \in a_j$. Thus, each edge $e \in G$ is associated with a dart x of P.

Now let G_i be the graph G restricted to the edges associated with darts of c_i. We have a circuit on darts of c_i if and only if the graph G_i is eulerian. We can build graphs G and G_i in polynomial time and space (there are at most $n/2$ graphs G_i).

For each G_i, we count the edges leaving and entering each vertex s in G_i, check that $deg(s)=|in(s)|$ for all s, and that G_i is connected. Obviously this is done in polynomial time and space. □

Recall that a graph is 3-connected if it is connected and it remains connected after any deletion of two vertices.

Lemma 5.2. *If P is planar and the graph $G_{A,B}$ is 3-connected, then there is a linear algorithm which solves P.*

Proof (*sketch*). A theorem of Whitney [8] states that there are only two ways (one inverse of the other) to embed a 3-connected graph in the plane. If such an embedding exists and if its faces induce partition C, problem P has a solution.

In this case, we only need to find the embedding of $G_{A,B}$ in the plane using the algorithm of Hopcroft and Tarjan [9] (which is of $O(V)$) and then to check that the faces satisfy $C = part(\beta\alpha)$. Thus we can answer in $O(|\Omega|)$. □

To prove the next lemma we use the decomposition tree $\mathcal{T}$ of a graph G used in [5] in order to test planarity dynamically. This tree can be built in $O(n\log(n))$ and reflects the decomposition of G into its 3-connected components. Let us recall this technique.

With each node v of $\mathcal{T}$ there is associated a subgraph G_v of G and a graph μ_v, the *skeleton* of v. μ_v is a *planar st-graph*, that is a planar acyclic directed graph with exactly one source s and exactly one sink t. Each son of v is associated with an edge of μ_v. There are four types of nodes in $\mathcal{T}$.

- If G_v is a single edge from s to t: v is a Q-node (without sons) whose skeleton μ_v is G_v.
- If G_v is 1-connected with cut-vertices (hyperedges) $s_1, s_2, \ldots, s_{k-1}$ from s to t: v is a S-node whose skeleton μ_v is a chain of k edges from s to t. v has k sons i whose associated hypergraphs G_i are 2-connected.
- If s and t is a separation pair of G_v with split components $G_1, G_2, \ldots, G_k$: then v is a P-node whose skeleton consists of k parallel edges from s to t. v has k sons i (whose associated hypergraphs are split components G_i).
- If none of the above cases applies: let the k maximal split pairs (s_i, t_i) with split components G_i; v is an R-node whose skeleton is obtained from G_v by replacing each subgraph G_i with an edge e_i. v has k sons i which are not Q-node with associated hypergraphs G_i. For any node of $\mathcal{T}$, s and t must lie on the same face, so we can consider skeleton μ of an R-node as a 3-connected graph, adding an edge from s to t.

Lemma 5.3. *If P is planar and the graph $G_{A,B}$ defined by A and B is 2-connected then we can solve P in polynomial time and space.*

Proof (*sketch*). In order to obtain $\mathcal{T}$, and to check $G_{A,B}$ for planarity, we use the algorithm presented in [5].

First we shall associate a *PPP*-problem P_v to each node v of $\mathcal{T}$. Let $P = (\Omega, A, B, C)$ and $\Omega' \subset \Omega$; we define $P' = (\Omega', A', B', C')$ the subproblem of P where $A'_i = \{b \in \Omega' \cap A_i\}$, $B'_i = \{b \in \Omega' \cap B_i\}$ and $C'_i = \{b \in \Omega' \cap C_i\}$. G_v, P_v and c_v refer to a current node v. P_r is an extra subproblem associated to v if v is a R-node. $G_{i(j)}$, $P_{i(j)}$ and $c_{i(j)}$ refer to a son i (j) of v. P is associated with the root of $\mathcal{T}$.

- If v is a S-node we define the subproblems $P_i = (\Omega_i, A_i, B_i, C_i)$ of P_v where Ω_i is the set of darts of G_i. A face $c_i \neq \emptyset$ of P_i which is not equal to the corresponding face c_v of P_v is called *external*.

- If v is a P-node, let the intermediate subproblems P'_i be defined as above. If i is not a Q-node, then we get the corresponding subproblem P_i by merging all the faces of P'_i that are incident to s and t or that are not equal to the corresponding face c_v; this new merged face is called *external.*
- If v is an R-node with k sons we define the subproblems P_i in the same way as for a P-node.

We define an extra subproblem $P_r = (\Omega_r, A_r, B_r, C_r)$ where Ω_r is the set of darts of Q-nodes; a_k and b_k are A_v and B_v restricted to the darts of Ω_r; c_k is C_v restricted to Ω_r where we merge the faces c_v that are incident to one and the same split pair (s_i, t_i) or whose darts of the corresponding face c_v are in a subproblem P_i and in another subproblem P_j. Thus we get at most k new external faces of P_r.

Let (A_μ, B_μ) be the representation of the skeleton μ_r obtained from (A_r, B_r) by adding: a dart b_i to the classes associated with a separation pair (s_i, t_i) when s_i and t_i are one vertex and one hyperedge; a dart b_s and b_t to vertices (hyperedges) s_i and t_i respectively, while creating a new hyperedge (vertex) $\{b_s, b_t\}$ if s_i and t_i are both vertices or hyperedges. We do this also for the source s_v and the sink t_v of the hypergraph G_v. Each additional dart (or couple of darts) is associated with one virtual edge of the skeleton μ_r. In this way the graph G_r defined by (A_r, B_r) is a subgraph of the skeleton μ_r.

Now we have decomposed the original problem P into subproblems. The following propositions (whose proofs are technical but straightforward) will be helpful in the sequel.

Proposition 5.4. *If v is a S-node then P_v has a solution if:*

(1) *each subproblem P_i has a solution;*

(2) *for all i there is exactly one external face of P_i, and this face is incident to cut-vertices (hyperedges) (s_{i-1}, s_i);*

(3) *there is exactly one face c_v of P_v whose darts are in the external faces c_i of problems P_i.*

Proposition 5.5. *If v is P-node with k sons, then P_v has a solution if:*

(1) *each subproblem P_i has a solution;*

(2) *for all i which is not a Q-node, the external face c_i results in the merging of exactly two faces of the intermediate subproblem P'_i;*

(3) *there are at most k faces of P whose darts are in different faces c_i of problems P_i.*

Proposition 5.6. *If v is an R-node with k sons, then P_v has a solution if:*

(1) *the extra subproblem P_r and each subproblem P_i have a solution;*

(2) *for all i which is not a Q-node, the external face c_i results in the merging of at most two faces of the intermediate subproblem P'_i;*

(3) *there are exactly two faces of P_v incident to a split pair (s_i, t_i) which gives one external face in P_i and one merged face in P_r.*

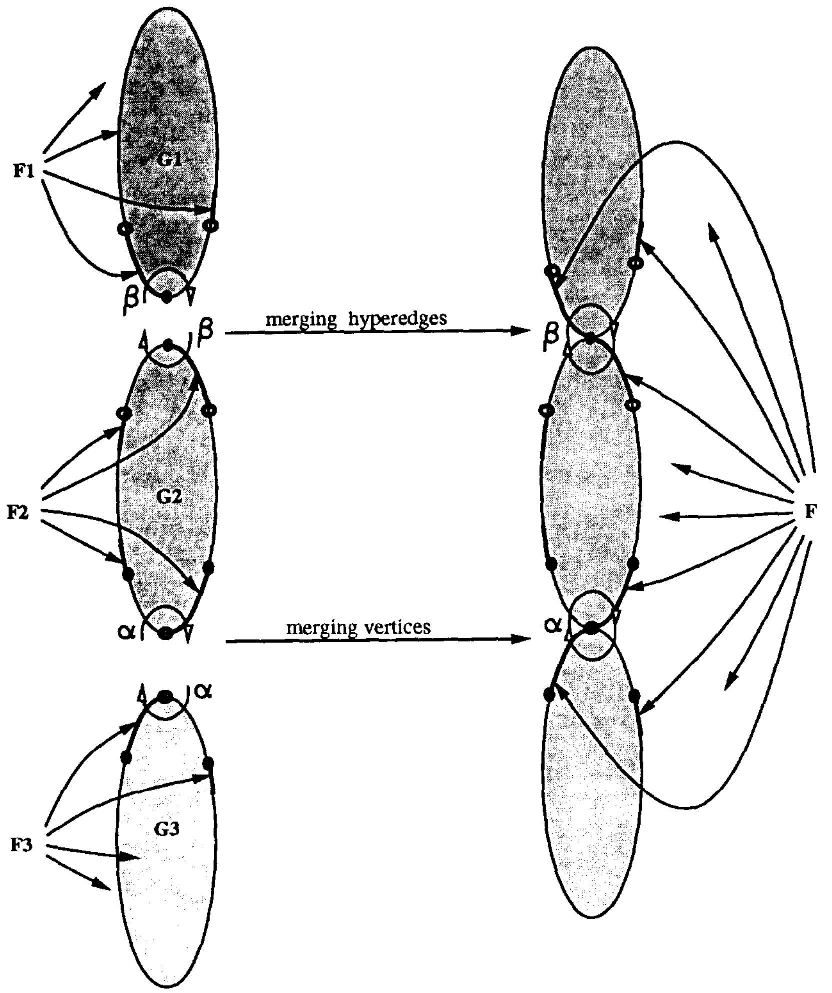

Fig. 3. Merging hyperedges and vertices of subgraphs G_i of a S-node while merging external faces of P_i.

Now we scan the tree $\mathcal{T}$ from leaves to root. We solve P_v of a node v when we have a solution for all P_i of sons i of v. We obtain (α_p, β_p) merging vertices and hyperedges as in Fig. 3. A solution of P_v only depends on a solution of P_i (and also on a solution of P_r if v is an R-node). In this way we obtain a solution of P, the original problem associated to the root of $\mathcal{T}$.

All the subproblems P_v associated to Q-nodes which are leaves of $\mathcal{T}$ obviously have a solution.

- For a S-node v we solve P_v by merging vertices and hyperedges (of the k subproblems P_i) associated to cut-vertices (hyperedges) $s_0, s_1, \ldots, s_k$ of G_v, and by merging external faces c_i into the corresponding face c_v. If this is not a solution of P_v then stop.

- For a P-node v we shall merge vertices and hyperedges associated to a split pair (s, t) of a subproblem P_i and a subproblem P_j if their external faces share darts with the same face c_v (as we did for a S-node). We do this until we have merged all the subp5roblems P_k. If this is not a solution of P_v then stop.
- For a R-node v, first we try to solve the extra subproblem P_r of v, (finding a planar embedding of the 3-connected skeleton μ_r and then removing darts associated with virtuals edges gives a guess to P_r). Then we merge the vertices and the hyperedges (as for a S-node) associated with split pair (s_i, t_i) of a subproblem P_i and of extra subproblem P_r of node v. If this is not a solution of P_v then stop.

The tree $\mathscr{T}$ is built in polynomial time and space. We can easily label each node v with his subproblem or extra subproblem v in linear time and space. Each condition (1), (2) and (3) of Propositions 5.4, 5.5 and 5.6 can be checked in polynomial time. Subproblems for a Q-node are obvious. For a node of another type, merging the subproblems P_i and checking if it is a solution takes polynomial time and space. □

Theorem 5.7. *If P is planar then there is a polynomial algorithm which solves P.*

Proof (*sketch*). Let $G_{A,B}$ the graph defined by (A, B) with k 2-connected components. We can obtain the cut-vertices (hyperedges) s_i of G in linear time and space [13]. We define k subproblems P_i and subgraphs G_i in the same way as we defined P_i and G_i for a S-node. Now we solve k subproblems P_i (where G_i is biconnected), merge them into the problem P as in the proof of the previous theorem. □

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