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Regularity of Minimizers of Relaxed Problems for Harmonic Maps

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We prove that every minimizer on $H^1(\Omega; S^2)$ of the relaxed energy $\int |\nabla u|^2 + 8\pi\lambda L(u)$, where $0 \leqslant \lambda < 1$ and $L(u)$ is the length of a minimal connection connecting the singularities of u, is smooth except at a finite number of points.

INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Set

$$H^1(\Omega; S^2) = \{u \in H^1(\Omega; \mathbb{R}^3);\ u(x) \in S^2 \text{ a.e.}\},$$

where S^2 is the unit sphere in $\mathbb{R}^3$. Given a boundary data $\varphi: \partial\Omega \to S^2$ we define

$$H^1_\varphi(\Omega; S^2) = \{u \in H^1(\Omega; S^2);\ u = \varphi \text{ on } \partial\Omega\}.$$

If $u \in H^1_\varphi(\Omega; S^2)$ is smooth except at a finite number of singularities in Ω and if moreover $\deg \varphi = 0$, then the length of a minimal connection connecting the singularities has been introduced in [BCL] and is given by

$$L(u) = \text{Min} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)}),$$

where $(p_1, p_2, ..., p_k)$ are the singularities of positive degree (counted

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according to their multiplicity), $(n_1, n_2, ..., n_k)$ are the singularities of negative degree, d is the geodesic distance in Ω, and the minimum is taken over all permutations of the integers $\{1, 2, ..., k\}$. (Since $\deg \varphi_{|\partial\Omega} = 0$ the number of positive singularities is the same as the number of negative singularities.)

For any $u \in H^1(\Omega; S^2)$ the vector field $D(u)$ defined as follows

$$D(u) = (u.u_y \wedge u_z, u.u_z \wedge u_x, u.u_x \wedge u_y)$$

plays an important role (see [BCL]). Set

$$R = \{u \in H^1(\Omega; S^2); u \text{ is smooth except at a finite number of singularities}\}$$

and

$$R_\varphi = R \cap H^1_\varphi(\Omega; S^2).$$

Recall (see [BZ]) that R is dense in $H^1(\Omega; S^2)$ and R_φ is dense in $H^1_\varphi(\Omega; S^2)$. If $u \in R$ (with singularities (a_i)) then

$$\operatorname{div} D(u) = 4\pi \sum \deg(u, a_i)\delta_{a_i}.$$

If $u \in R_\varphi$ then (see [BCL])

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leq 1}} \left\{ \int_\Omega D(u).\nabla\zeta \, dx - \int_{\partial\Omega} (\text{Jac } \varphi)\zeta \, d\sigma \right\}. \tag{1}$$

Clearly this makes sense for any $u \in H^1_\varphi(\Omega; S^2)$ and we shall use formula (1) as a definition of L for a general $u \in H^1_\varphi(\Omega; S^2)$. By a result of [BBC], L is continuous (and even locally Lipschitz) on H^1_φ.

The functional

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 + 8\pi\lambda L(u), \qquad \lambda \in [0, 1] \tag{2}$$

introduced in [BBC] has some remarkable properties. In particular, it is weakly lower semicontinuous for the weak topology on H^1. Thus

$$\operatorname*{Min}_{u \in H^1_\varphi} F_\lambda(u) \text{ is achieved} \tag{3}$$

and we recall that minimizers of F_λ are (weakly) harmonic maps, i.e., they are weak solutions of

$$-\Delta u = u \, |\nabla u|^2 \qquad \text{on } \Omega.$$

For $\lambda = 0$ it is known (see [SU1, SU2]) that mimizers of (3) are smooth except at a finite number of points. Our main result asserts that this is still true for $\lambda \in [0, 1)$.

THEOREM 1. *Every minimizer of F_λ is smooth on $\bar{\Omega}$ except at isolated singularities.*

For $\lambda = 0$ it is known (see [BCL]) that the singularities of minimizers have a simple form; i.e., if x_0 is a singularity then, for some rotation R, $u(x) \simeq \pm R((x - x_0)/|x - x_0|)$ as $x \to x_0$. In particular all singularities have degree ± 1. This last property can be established for λ small, but it is an open problem for λ large.

Unfortunately, our arguments do not give any information about the nature of singularities when $\lambda = 1$. The case $\lambda = 1$ is very important (because it corresponds to the relaxed energy; see [BBC]), and it would be extremely interesting to decide whether minimizers of F_1 are smooth. Partial regularity results for minimizers of F_1 have been obtained in [GMS2].

The proof of Theorem 1 is divided into several steps.

Step 1. Minimizers of F_λ satisfy a reverse Hölder inequality.

Step 2. This is used to prove an "ε-regularity lemma."

Step 3. One concludes by a blow-up technique (similar to the one used by [SU1]) that singularities are isolated.

1. A Reverse Hölder Inequality

The usefulness of "reverse Hölder inequality" was originally discovered by Gehring. It has been extensively used to establish partial regularity (see, e.g., Giaquinta's book [G] and references therein). We shall follow a technique recently introduced by [HKL] in variational problems involving S^2-valued maps.

In what follows we fix $\lambda \in [0, 1)$ and some minimizer u of F_λ on $H^1_\varphi(\Omega; S^2)$.

THEOREM 2. *There exist constants $q > 2$ and C (depending only on λ) such that*

$$\left(\fint_{B_r} |\nabla u|^q\right)^{1/2} \leqslant C\left(\fint_{B_{2r}} |\nabla u|^2\right)^{1/2} \tag{4}$$

for every ball B_r such that $B_{2r} \subset \Omega$.

We shall use the following:

LEMMA 1. *Let B_r be a ball contained in Ω. Then*

$$\int_{B_r} |\nabla u|^2 \leqslant \frac{1+\lambda}{1-\lambda} \int_{B_{2r}} |\nabla v|^2,$$

$$\forall v \in H^1(B_r; S^2) \quad \text{such that } v = u \text{ on } \partial B_r. \tag{5}$$

Proof. Set

$$w = \begin{cases} v & \text{on} \quad Br \\ u & \text{on} \quad \Omega \setminus B_r. \end{cases}$$

Since $w \in H^1_\varphi$ we have

$$F_\lambda(u) \leqslant F_\lambda(w),$$

i.e.,

$$\int_\Omega |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int_\Omega |\nabla w|^2 + 8\pi\lambda L(w)$$

and therefore

$$\int_{B_r} |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int_{B_r} |\nabla v|^2 + 8\pi\lambda L(w). \tag{6}$$

On the other hand,

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_0 D(u).\nabla\zeta \, dx - \int_{\partial\Omega} (\text{Jac } \varphi)\zeta \, d\sigma \right\}$$

and

$$L(w) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_\Omega D(w).\nabla\zeta \, dx - \int_{\partial\Omega} (\text{Jac } \varphi)\zeta \, d\sigma \right\}.$$

It follows that

$$L(w) - L(u) \leqslant \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_\Omega (D(w) - D(u)).\nabla\zeta \, dx \right\}$$

$$= \frac{1}{4\pi} \sup_{\substack{\zeta \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_{B_r} (D(w) - D(u)).\nabla\zeta \, dx \right\}$$

$$\leqslant \frac{1}{8\pi} \left\{ \int_{B_r} (|\nabla v|^2 + |\nabla u|^2) \right\}.$$

Combining this with (6) we obtain

$$\int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla v|^2 \leqslant \lambda \left[\int_{B_r} |\nabla u|^2 + \int_{B_r} |\nabla v|^2 \right].$$

We also recall the following result of [HKL].

LEMMA 2. *There is a universal constant C such that, for every $\xi \in \mathbb{R}^3$,*

$$\operatorname*{Min}_{\substack{v \in H^1(B_r; S^2) \\ v = u \text{ on } \partial B_r}} \int_{B_r} |\nabla v|^2 \leqslant C \, \|\nabla_T u\|_{L^2(\partial B_r)} \, \|u - \xi\|_{L^2(\partial B_r)}. \tag{7}$$

For the proof we refer to [HKL, Appendix]. The idea is to consider the usual harmonic extension $\bar{u}$ of $u_{|\partial B_r}$ (with values in B^3) and then some appropriate radial projection of $\bar{u}$ on S^2.

Proof of Theorem 2. Combining Lemma 1 and Lemma 2 we obtain

$$\int_{B_r} |\nabla u|^2 \leqslant C_\lambda [\|\nabla_T u\|_{L^2(\partial B_r)} \, \|u - \xi\|_{L^2(\partial B_r)}].$$

We now follow the argument of [HKL, Theorem 4.1] to deduce that for every $\delta > 0$

$$\int_{B_r} |\nabla u|^2 \leqslant \delta \int_{B_{2r}} |\nabla u|^2 + \frac{C}{\delta r^2} \left(\int_{B_{2r}} |\nabla u|^p \right)^{2/p}$$

with $p = 6/5$.

This implies, using Hölder's inequality, that

$$\int_{B_r} |\nabla u|^2 \leqslant 2\delta \int_{B_{2r}} |\nabla u|^2 + \frac{C}{\delta r^3} \left(\int_{B_{2r}} |\nabla u| \right)^2$$

i.e.,

$$⨍_{B_r} |\nabla u|^2 \leqslant 16\delta ⨍_{B_{2r}} |\nabla u|^2 + \frac{C}{\delta} \left(⨍_{B_{2r}} |\nabla u| \right)^2.$$

Fixing $\delta < 1/16$ we may now apply this reverse-Hölder inequality to conclude the existence of a $q > 2$ such that

$$\left(⨍_{B_r} |\nabla u|^q \right)^{1/q} \leqslant C \left(⨍_{B_{2r}} |\nabla u|^2 \right)^{1/2}.$$

2. ε-REGULARITY

In what follows, we fix $\lambda \in [0, 1)$ and some minimizer u of F_λ on $H^1_\varphi(\Omega; S^2)$.

THEOREM 3. *There is some $\varepsilon_0 > 0$ (depending only on λ) such that if*

$$\frac{1}{r}\int_{B_r} |\nabla u|^2 < \varepsilon_0 \qquad \textit{for some ball } B_r,$$

then u is smooth on $B_{r/4}$ (and u is a minimizing harmonic map on $B_{r/4}$ in the usual sense).

The proof relies on several lemmas.
Set for $\mu \in (0, 1)$

$$W_\mu^- = \{\sigma \in S^2, |P\sigma| < \mu\}$$
$$W_\mu^+ = \{\sigma \in S^2, |P\sigma| > \mu\},$$

where $P: S^2 \to \mathbb{C}$ denotes the stereographic projection. Note that, as $\mu \to 0$, area $(W_\mu^-) \sim \pi\mu^2$.

LEMMA 3. *Let $G \subset \mathbb{R}^3$ be a smooth bounded domain. There exists $\delta > 0$ (depending only on $\lambda \in [0, 1)$ and not on G) such that if $\psi: \partial G \to S^2$ and $\psi(\partial G) \subset W_\delta^-$ then every $v \in H^1_\psi(G; S^2)$ satisfying*

$$\int_G |\nabla v|^2 \leqslant \operatorname*{Min}_{w \in H^1_\psi(G; S^2)} \int_G |\nabla w|^2 + 8\pi\lambda L(v) \tag{8}$$

is smooth on G.

Proof of Lemma 3. For $\mu \in (0, 1)$ (to be determined later) let

$$G_\mu^\pm = \{x \in G; u(x) \in W_\mu^\pm\}.$$

The proof is divided into three steps.

Step 1. We have

$$\int_{G_\mu^+} |\nabla v|^2 \geqslant 8\pi(1 - C\mu^2) L(v) \tag{9}$$

for every $v \in H^1_\psi(G; S^2)$ with $\psi(\partial G) \subset W_\mu^-$ (C is some universal constant).

Proof of Step 1. By density (see [BZ]) we may always assume that v

has just a finite number of singularities. Following [ABL], we have, using Federer's coarea formula,

$$\int_{G_\mu^+} |\nabla v|^2 \geqslant 2 \int_{G_\mu^+} |D(v)| = 2 \int_{W_\mu^+} \mathscr{H}^1(v^{-1}(\sigma))\, d\sigma.$$

For a.e. $\sigma \in W_\mu^+$, $v^{-1}(\sigma)$ consists of curves connecting the singularities (and possibly some closed loops). Note that there are no curves connecting the singularities to $\psi(\partial G) \subset W_\mu^+$. Thus

$$\mathscr{H}^1(v^{-1}(\sigma)) \geqslant L(v) \qquad \text{for} \quad \text{a.e. } \sigma \in W_\mu^+,$$

and consequently

$$\begin{aligned} \int_{G_\mu^+} |\nabla v|^2 &\geqslant 2(\text{area } W_\mu^+) L(v) \\ &\geqslant 8\pi(1 - C\mu^2) L(v). \end{aligned}$$

Step 2. Suppose as above that $\psi\colon \partial G \to W_\delta^-$. Then

$$\operatorname*{Min}_{w \in H_\psi^1(G;S^2)} \int_G |\nabla w|^2 \leqslant \mu^2 \int_{G_\mu^+} |\nabla v|^2 + \int_{G_\mu^-} |\nabla v|^2 \tag{10}$$

for every $v \in H_\varphi^1(G; S^2)$ and every $\mu > \sqrt{\delta}$.

Proof of Step 2. Fix a map $\Phi\colon S^2 \to S^2$ satisfying

$$\begin{aligned} &\Phi(S^2) \subset W_{\mu^2}^- \\ &\Phi = Id \qquad \text{on } W_{\mu^2}^- \\ &|\nabla\Phi| \leqslant 1 \qquad \text{on } S^2 \\ &|\nabla\Phi| \leqslant \mu^2 \qquad \text{on } W_\mu^+. \end{aligned}$$

To construct such a Φ one can, using sterographic projection, define $\Phi\colon \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} \Phi(z) &= z \qquad &&\text{for} \quad |z| < \mu^2 \\ \Phi(z) &= \frac{\mu^4}{\bar{z}} \qquad &&\text{for} \quad |z| \geqslant \mu^2. \end{aligned}$$

It is clear that $|\nabla\Phi| \leqslant 1$ everywhere and $|\nabla\Phi| \leqslant \mu^4/|z|^2$ for $|z| \geqslant \mu^2$. In particular $|\nabla\Phi| \leqslant \mu^2$ for $|z| > \mu$.

Given $v \in H_\varphi^1$, set

$$w = \Phi \circ v$$

so that $w \in H^1_\varphi(G; S^2)$ (since $\mu > \sqrt{\delta}$). We have

$$\int |\nabla w|^2 = \int_{G_\mu^+} |\nabla w|^2 + \int_{G_\mu^-} |\nabla w|^2$$
$$\leqslant \mu^2 \int_{G_\mu^+} |\nabla v|^2 + \int_{G_\mu^-} |\nabla v|^2.$$

This proves Step 2.

Step 3. *Proof of Lemma* 3 *Completed.* Let v satisfy (8). By (10) we have

$$\int_G |\nabla v|^2 \leqslant \mu^2 \int_{G_\mu^+} |\nabla v|^2 + \int_{G_\mu^-} |\nabla v|^2 + 8\pi\lambda L(v)$$

and thus

$$(1-\mu^2) \int_{G_\mu^+} |\nabla v|^2 \leqslant 8\pi L(v).$$

Combining this with Step 1 we have

$$8\pi(1-\mu^2)(1-C\mu^2) L(v) \leqslant 8\pi\lambda L(v).$$

Now choose $\mu > 0$ small enough so that

$$(1-\mu^2)(1-C\mu^2) > \lambda$$

and then choose $0 < \delta < \mu^2$. It follows that

$$L(v) = 0.$$

Going back to (8) we see that v is a minimizing harmonic map. We may now invoke [HKW] to conclude that v is smooth (since $\psi(\partial G)$ is contained in a hemisphere). Alternatively, we may also invoke [SU1] together with the fact that $L(v) = 0$.

This concludes the proof of Lemma 3. With its help we are going to prove Theorem 3.

Proof of Theorem 3. We split the proof in two steps.

Step 1. Let $B \subset \Omega$ be a ball such that

$$u_{|\partial B} \in H^1 \qquad \text{and} \qquad u(\partial B) \subset W_\delta^-, \tag{11}$$

where δ is defined in Lemma 3. Then

$$\int_B |\nabla v|^2 \leqslant \min_{\substack{w \in H^1_\psi(B;S^2) \\ v = u \text{ on } \partial B}} \int |\nabla v|^2 + 8\pi\lambda L_B(u), \tag{12}$$

where L_B denotes the length of a minimal connection connecting the singularities of u *in* B (without connections to the boundary). Moreover u is smooth in B.

Proof. Let u_0 be a minimizer for

$$\min_{\substack{v \in H^1_\psi(B;S^2) \\ v = u \text{ on } \partial B}} \int |\nabla v|^2.$$

By (11) and [HKW], u_0 is smooth inside B. Set

$$w = \begin{cases} u_0 & \text{on } B \\ u & \text{on } \Omega \setminus B. \end{cases}$$

Since u is a minimizer for F_λ on $H^1_\varphi(\Omega; S^2)$ we have

$$\int |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int |\nabla w|^2 + 8\pi\lambda L(w).$$

Thus

$$\int_B |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int_B |\nabla u_0|^2 + 8\pi\lambda L(w). \tag{13}$$

On the other hand (using (1)),

$$L(w) - L(u) \leqslant L(w, u), \tag{14}$$

where

$$L(w, u) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \int_\Omega (D(w) - D(u)).\nabla\zeta$$

$$= \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \int_B [D(u_0) - D(u)].\nabla\zeta.$$

But

$$\int_B D(u_0).\nabla\zeta = \int_{\partial B} (D(u_0).n)\zeta.$$

(Since $u_0(B) \subset W_\delta^-$ by [HKW] and so we may approximate u_0 by a sequence of smooth maps converging to u_0 in $H^1(B; S^2)$ (and also in $H^1(\partial B; S^2)$).) Therefore

$$\begin{aligned} L(w, u) &= \frac{1}{4\pi} \sup_{\substack{\zeta : \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_{\partial B} (D(u_0).n)\zeta - \int D(u).\nabla\zeta \right\} \\ &= L_B(u) \end{aligned} \tag{15}$$

(since $u = u_0$ on ∂B).

Combining (13), (14), (15) we find

$$\int_B |\nabla u|^2 \leqslant \int_B |\nabla u_0|^2 + 8\pi\lambda L_B(u).$$

This completes the proof of Step 1.

Step 2. *Proof of Theorem* 3 *Completed.* Suppose

$$\frac{1}{r} \int_{B_r} |\nabla u|^2 < \varepsilon_0$$

for some ball $B_r \subset \Omega$ and some ε_0 (to be determined later). By Theorem 2 we have

$$\frac{1}{r^3} \int_{B_{r/2}} |\nabla u|^q \leqslant C \left(\frac{1}{r^2} \varepsilon_0 \right)^{q/2}.$$

Thus

$$\int_{r/4}^{r/2} d\rho \int_{S_\rho} |\nabla u|^q \, d\sigma \leqslant C r^{3-q} \varepsilon_0^{q/2}$$

and hence there is some $r_0 \in [r/4, r/2]$ such that

$$\int_{S_{r_0}} |\nabla u|^q \, d\sigma \leqslant C r^{2-q} \varepsilon_0^{q/2},$$

that is,

$$\left(\int_{S_{r_0}} |\nabla u|^q \, d\sigma \right)^{1/q} \leqslant C r^{(2/q)-1} \varepsilon_0^{1/2}.$$

By the Sobolev imbedding we conclude that

$$u(S_{r_0}) \subset W_\mu^-,$$

where

$$\mu = C\varepsilon_0^{1/2}.$$

We now choose ε_0 such that $C\varepsilon_0^{1/2} < \delta$ (δ given by Lemma 3). By Step 1 above (with $B = B_{r_0}$) we conclude that u is smooth in B_{r_0} and thus in $B_{r/4}$. This concludes the proof of Theorem 3.

Remark 1. We may now assert that $\mathscr{H}^1_{\text{loc}}(Z) = 0$, where Z is the singular set of u (as above u is a minimizer of F_λ on H_φ). Let

$$\theta(a) = \lim_{r \to 0} \sup \frac{1}{r} \int_{B_r(a)} |\nabla u|^2$$

and

$$\tilde{Z} = \{a \in \Omega; \theta(a) > 0\}.$$

Note that if $a \in \Omega \backslash \tilde{Z}$ then u is smooth in a neighborhood of a (by Theorem 3). Thus $Z \subset \tilde{Z}$. Since $\tilde{Z}$ is obviously contained in Z we conclude that $Z = \tilde{Z}$. Since $u \in H^1(\Omega; S^2)$ it follows by a standard covering argument that $\mathscr{H}^1_{\text{loc}}(Z) = 0$. In fact, since $u \in W^{1,q}_{\text{loc}}$ we conclude that $\mathscr{H}^{3-q}_{\text{loc}}(Z) = 0$ (see, e.g., [HKL]).

3. The Singular Set Consists of Isolated Points

We prove here that u has only isolated singularities by a variant of the blow-up technique of [SU1, SU2]. Here we rely on a new monotonicity formula of [GMS2]. Let Z be the complement of the largest open set on which u is smooth.

THEOREM 4. *Z consists of isolated points (in Ω).*

Step 1. *A Monotonicity Formula*

Recall that for a (standard) minimizing harmonic map u we have the well-known monotonicity formula

$$\frac{\partial}{\partial r}\left(\frac{1}{r} \int_{B_r} |\nabla u|^2\right) \geq 0.$$

This is not true any more for minimizers of F_λ but we have a variant of this formula due to Giaquinta, Modica, and Souček [GMS2].

Let $u \in R_\varphi$ and let μ be the one-dimensional Hausdorff measure uniformly distributed over a minimal connection (warning: the minimal connection need not be unique, and so μ is not uniquely determined by u). Set

$$E(r)=\int_{B_r(x_0)}|\nabla u|^2+8\pi\lambda\int_{B_r(x_0)}d\mu$$

$$\alpha(r)=\frac{d}{dr}\left[2\int_{B_r(x_0)}\left|\frac{\partial u}{\partial r}\right|^2\right]+8\pi\lambda\int_{B_r(x_0)}[1-(\zeta.\nu)^2]\,d\mu,$$

where ζ is the unit vector tangent to the minimal connection at $\nu=(x-x_0)/|x-x_0|$.

Using the formalism of currents (see [GMS1]) these expressions make sense for every $u\in H^1_\varphi$. We have

LEMMA 4. *For every minimizer of F_λ we have*

$$\frac{d}{dr}\left(\frac{1}{r}E(r)\right)\doteq\frac{1}{r}\alpha(r).$$

In particular $(1/r)E(r)$ is nondecreasing in r.

This formula has been established by [GMS2] when $\lambda=1$, but the same argument holds for any $\lambda\in(0,1)$.

Step 2. The Blow-Up

Let $x_0\in\Omega$ be any point in Ω and let u be a minmizer of F_λ on H^1_φ. For simplicty we take $x_0=0$ and we write $B_R=B_R(0)$. By Lemma 4, $(1/r)E(r)$ remains bounded as $r\to0$ and so does $(1/r)\int_{B_r}|\nabla u|^2$. Set

$$u_\sigma(x)=u(\sigma x),\qquad \text{for}\quad x\in B^3.$$

Then

$$\int_{B_1}|\nabla u_\sigma|^2=\frac{1}{\sigma}\int_{B_\sigma}|\nabla u_\sigma|^2\leqslant C\qquad\text{as}\quad\sigma\to0.\tag{16}$$

Therefore

$$u_{\sigma_n}\rightharpoonup v\qquad\text{weakly in } H^1(B_1).$$

We claim that

$$\frac{\partial v}{\partial r}=0.$$

Indeed we have, by Lemma 4

$$\int_0^{\sigma_n} \alpha(r)\frac{dr}{r} = \int_0^{\sigma_n} \frac{d}{dr}\left(\frac{1}{r}E(r)\right) dr$$

$$\leqslant \frac{1}{\sigma_n} E(\sigma_n) - \lim_{r\to 0} \frac{E(r)}{r} \to 0$$

as $\sigma_n \to 0$.

On the other hand

$$\int_0^{\sigma_n} \frac{\alpha(r)}{r} dr \geqslant 2\int_0^{\sigma_n} \frac{1}{r}\frac{d}{dr}\int_{B_r}\left|\frac{\partial u}{\partial r}\right|^2$$

$$= 2\int_0^{\sigma_n} \frac{1}{r}\int_{S_r}\left|\frac{\partial u}{\partial r}\right|^2$$

$$= 2\int_{B_{\sigma_n}} \frac{1}{r}\left|\frac{\partial u}{\partial r}\right|^2 dr.$$

It follows that $\int_{B_{\sigma_n}} (1/r)\,|\partial u/\partial r|^2 \to 0$. But

$$\int_{B_1} \frac{1}{r}\left|\frac{\partial}{\partial r} u_{\sigma_n}\right|^2 = \int_{B_{\sigma_n}} \frac{1}{r}\left|\frac{\partial u}{\partial r}\right|^2$$

and the claim follows. Therefore we have

$$v(x) = \psi\left(\frac{x}{|x|}\right) \tag{17}$$

for some $\psi \in H^1(S^2, S^2)$. Following the strategy of [SU1, Part 4, Proposition 4.6] we now prove that $u_{\delta_n} \to v$ strongly in $H^1(B_1)$. Since we have

$$-\Delta u_\sigma = u_\sigma\,|\nabla u_\sigma|^2 \qquad \text{on } B_1$$

and

$$\int_{B_1} |\nabla u_\sigma|^2 \leqslant C$$

we deduce from standard elliptic estimates that ∇u_σ is relatively compact in $L^1_{\text{loc}}(B_1)$ and therefore for an appropriate subsequence we may assume that $\nabla u_{\sigma_n} \to \nabla v$ a.e. on B_1. In order to conclude that $\nabla u_{\sigma_n} \to \nabla v$ in L^2 it suffices to know that

$$\int_{B_1} |\nabla u_\sigma|^q \leqslant C$$

for some $q>2$. This is a consequence of Theorem 2. Indeed, we have by (4)

$$\left[\frac{1}{\sigma^3}\int_{B_\sigma}|\nabla u|^q\right]^{1/q}\leqslant C\left(\frac{1}{\sigma^3}\int_{B_{2\sigma}}|\nabla u|^2\right)^{1/2}\leqslant\frac{C}{\sigma},$$

i.e.,

$$\frac{1}{\sigma^{3-q}}\int_{B_\sigma}|\nabla u|^q\leqslant\int_{B_1}|\nabla u_\sigma|^q\leqslant C.$$

Thus we have established that $u_{\sigma_n}\to\psi$ strongly in H^1.

Finally we show that the singularities of u are isolated (in any compact subset of Ω). As in [SU1], we follow the dimension reduction argument of Federer. Let (x_n) be a subsequence of singularities such that $x_n\to x_0=0$, $x_n/|x_n|\to l\in S^2$. We choose $\sigma_n=2\,|x_n|$. Note that u_{σ_n} has a singularity at the point $x_n/(2\,|x_n|)\to l/2$. By Theorem 3, there is some ε_0 such that

$$\forall r,\ \frac{1}{r}\int_{B_r(x_n/(2|x_n|))}|\nabla u_{\sigma_n}|^2\geqslant\varepsilon_0 \tag{18}$$

(otherwise u_{σ_n} would be regular at $x_n/(2\,|x_n|)$).

Since $u_{\sigma_n}\to\psi$ strongly in H^1, we may pass to the limit in (18) and conclude that

$$\frac{1}{r}\int_{B_r(l/2)}|\nabla v|^2\geqslant\varepsilon_0,\qquad\text{for every } r. \tag{19}$$

Since $v(x)=\psi(x/|x_n|)$ the left-hand side in (16) is of the order of $\int_{S^2\cap B_{2r}(l)}|\nabla_T\psi|^2$. This is impossible since $\psi\in H^1(S^2)$. This completes the proof of Theorem 4.

4. Boundary Regularity

Here we complete the proof of Theorem 1 by showing that every minimizer of F_λ is smooth in some neighborhood of $\partial\Omega$. This follows essentially the same pattern as above with the following modifications:

(a) *Reverse Hölder Inequality near $\partial\Omega$*

THEOREM 2'. *There exist constants $q>2$ and C_1, C_2 (depending only on ψ) such that*

$$\left(⨍_{B_r(x_0)\cap\Omega}|\nabla u|^q\right)^{1/q}\leqslant C_1\left(⨍_{B_{2r}(x_0)\cap\Omega}|\nabla u|^2\right)^{1/2}+C_2\,\|\nabla\varphi\|_{L^\infty(\partial\Omega)}$$

for any $x_0\in\partial\Omega$.

To prove Theorem 2′ we adapt an idea of Jost and Meier [JM], namely we use as testing function in the inequality

$$F_\lambda(u) \leqslant F_\lambda(w)$$

the map

$$w = \Pi_a(u - \eta(x)(u - \bar{\varphi})),$$

where $\bar{\varphi}$ is the usual harmonic extension of φ, η is some appropriate cut-off function with support in B_s (as in [JM, Lemma 1]), and Π_a is the radial projection with vertex at some appropriate a (as in [HKL]). As in Lemma 1 of Section 1 we have

$$\int_{B_s} |\nabla u|^2 \leqslant \frac{1+\lambda}{1-\lambda} \int_{B_s} |\nabla w|^2 \leqslant C\left(\frac{1+\lambda}{1-\lambda}\right) \int_{B_s} |\nabla(u - \eta(u - \bar{\varphi}))|^2.$$

We then proceed as in [JM] to derive the conclusion of Theorem 2′.

(b) *ε-Regularity*

The counterpart of Theorem 3 is

THEOREM 3′. *There is some $\varepsilon_1 > 0$ (depending only on λ) and r_0 (depending only on φ) such that if*

$$\frac{1}{r} \int_{B_r(x_0) \cap \Omega} |\nabla u|^2 < \varepsilon_1 \qquad \textit{for some} \quad r < r_0 \quad \textit{and} \quad x_0 \in \partial\Omega,$$

then u is smooth on $B_{r/4}(x_0) \cap \Omega$ (and u is a minimizing harmonic map on $B_{r/4}(x_0) \cap \Omega$ in the usual sense).

The proof is essentially the same as the proof of Theorem 3, except that r_0 is chosen so small that $\varphi(\partial\Omega \cap B_r(x_0)) \subset W_{\delta_1}^-$ for some suitable δ_1.

(c) *Monotonicity Formula*

Set $E(r)$ and $\alpha(r)$ as in Section 4 (Step 1) except that B_r is replaced by $B_r(x_0) \cap \Omega$ with $x_0 \in \partial\Omega$. The counterpart of Lemma 4 is

LEMMA 4′.

$$\frac{d}{dr}\left(\frac{1}{r} E(r)\right) \geqslant \frac{1}{r} \alpha(r) - C$$

for some constant C depending only on $\|\nabla\varphi\|_{L^\infty(\partial\Omega)}$.

The proof has the same ingredients as in [GMS2] together with Lemma 1.3 of [SU2].

(d) *Blow-Up and Conclusion*

Let $x_0 \in \partial\Omega$ and let u be a minimizer of F_λ on H^1_φ. For simplicity assume that $\partial\Omega$ is flat near x_0 with outward normal $(0, 0, -1)$. Set

$$u_\sigma(x) = u(\sigma x)$$

for $x \in B^3_+ = \{(x_1, x_2, x_3) \in B^3;\ x_3 \geqslant 0\}$. As in Step 2 of Section 3, $u_{\sigma_n} \rightharpoonup v$ weakly in $H^1(B^3_+)$ and $\partial v/\partial r = 0$. Therefore $v(x) = \psi(x/|x|)$ for some $\psi \in H^1(S^2_+; S^2)$. As above u_{σ_n} is bounded in $L^q(B^3_+)$ for some $q > 2$ and $u_{\sigma_n} \to v$ strongly in $H^1(B^3_+)$. In particular $v \in W^{1,q}(B^3_+; S^2)$ and so $\psi \in W^{1,q}(S^2_+; S^2)$. On the other hand v is (weakly) harmonic and constant on $\partial B^3_+ \cap [x_3 = 0]$. Hence ψ is weakly harmonic from from S^2_+ into S^2 and ψ is constant on ∂S^2_+. Since $\psi \in W^{1,q}(S^2_+)$ with $q > 2$, it follows (by bootstrap) that ψ is smooth on S^2_+. Using a result of [L] we deduce that ψ is constant on S^2_+. Hence

$$\int_{B_1^+} |\nabla u_\sigma|^2 < \varepsilon_1 \qquad \text{for } \sigma \text{ small enough,}$$

i.e.,

$$\frac{1}{r}\int_{B_r(x_0)\cap\Omega} |\nabla u|^2 < \varepsilon_1 \qquad \text{for } r \text{ small enough.}$$

By Theorem 3′ we conclude that u is smooth on $B_{r/4}(x_0) \cap \Omega$.

5. A Variant of the Relaxed Energy

Here we assume that $\varphi: \partial\Omega \to S^2$ is given and smooth but deg φ need not be zero. We *fix* $v \in H^1_\varphi(\Omega; S^2)$ and we set

$$L(u, v) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{\int_\Omega D(u) - D(v)).\nabla\zeta\, dx\right\}.$$

(Note that $L(u, v) = L(u)$ if deg $\varphi = 0$ and v is smooth.) The functional

$$\Phi_\lambda(u) = \int |\nabla u|^2 + 8\pi\lambda L(u, v), \qquad \lambda \in [0, 1]$$

introduced in [BBC] is also weakly lower semicontinuous for the weak topology on H^1 and minimizers of Φ_λ on H^1_φ are weakly harmonic maps.

THEOREM 5. *Assume v is smooth on $\bar{\Omega}$ except at isolated singularities then any minimizer u of Φ_λ is smooth except at isolated singularities.*

Sketch of Proof. Let S be the singular set of v. Using the same arguments as in the proof of Theorem 1 it is easy to see that on every compact subset of $\bar{\Omega}\setminus S$, u has only isolated singularities. It could still happen that singularities of u accumulate on S. This is excluded by a blow-up analysis centered at a point on S.

Remark 2. We recall that in [BBC] we have proved that if $\deg \varphi \neq 0$ the minimizers of $\Phi_\lambda(u)$ are distinct for a sequence (λ_n). By Theorem 5 these minimizers are smooth except at isolated points. For example, if $\varphi(x) = x$ we find infinitely many distinct harmonic maps with isolated singularities and such that $u = \varphi$ on $\partial\Omega$.

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