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We show that diagonal subalgebras and generalized Veronese subrings of a bigraded Koszul algebra are Koszul. We give upper bounds for the regularity of side-diagonal and relative Veronese modules and apply the results to symmetric algebras and Rees rings.

INTRODUCTION

In this article we study standard bigraded algebras. Let K denote a field and let $R = S/J$, where $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a polynomial ring with standard bigrading $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$ and $J \subset S$ is a bihomogeneous ideal. For such an algebra we consider two kinds of subalgebras. Let $a, b \geq 0$ be two integers with $(a, b) \neq (0, 0)$. Then the (a, b)-diagonal subalgebra is the positively graded subring

$$R_\Delta = \bigoplus_{i \geq 0} R_{(ia, ib)},$$

where $R_{(i,j)}$ denotes the (i, j)th bigraded component of R. Moreover, for two integers $a, b \geq 0$ such that $(a, b) \neq (0, 0)$, a generalized bigraded Veronese subring of R is defined by

$$R_{\tilde{\Delta}} = \bigoplus_{i, j \geq 0} R_{(ia, jb)}.$$

Recall that a positively graded K-algebra A is called Koszul if the residue class field K, considered as a trivial A-module, has linear A-free resolution. During the past 30 years Koszul algebras have been studied in various contexts. A good survey is given by Fröberg in [9].

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In the past years, diagonal subalgebras have been studied intensively because they naturally appear in Rees algebras and symmetric algebras. In [7] Conca *et al*. discuss many algebraic properties. In particular, they prove that for an arbitrary bigraded algebra R the diagonals R_Δ are Koszul provided that one chooses a and b large enough. In that article they ask two questions, one of which is positively answered by Aramova *et al*. in [2], who showed that the defining ideal of R_Δ has a quadratic Gröbner basis for $a, b \gg 0$. It is a well-known fact that this is a stronger property than Koszulness. The main result of this article is the positive answer to the second question posed in [7]: Suppose that R is a Koszul algebra. Are all diagonal subalgebras R_Δ Koszul? Moreover, we prove that all generalized Veronese subrings $R_{\tilde{\Delta}}$ inherit the Koszul property. The algebras $R_{\tilde{\Delta}}$ appear in [15], where Römer studies the homological properties of bigraded algebras.

Note that the Castelnouvo–Mumford regularity (see Section 1 for the definition) measures the complexity of the minimal free resolution of a finitely generated R-module. Provided that R is a Koszul algebra, all finitely generated modules have finite regularity over R (see [3]).

For a finitely generated bigraded R-module and two integers $c, d \geq 0$ we define a side-diagonal module $M_\Delta^{(c,d)}$ as the R_Δ-module with graded components $(M_\Delta^{(c,d)})_i = M_{(ia+c, ib+d)}$. One similarly defines a bigraded relative Veronese module $M_{\tilde{\Delta}}^{(c,d)}$. Provided that R is Koszul, we get upper bounds for the regularity of these modules.

In particular, if the initial degree of M (see Section 1 for the definition) equals 0 and if we choose c, d such that $R_\Delta^{(c,d)}$, respectively $R_{\tilde{\Delta}}^{(c,d)}$, are generated in degree 0, then $\operatorname{reg}_{R_\Delta} M_\Delta^{(c,d)} \leq 1$ and $\operatorname{reg}_{R_{\tilde{\Delta}}} M_{\tilde{\Delta}}^{(c,d)} \leq 2$ for all Δ, respectively $\tilde{\Delta}$, with $a, b \geq \operatorname{reg}_R M$. For the proof we use techniques similar to those of Aramova *et al*. in [1], where they give upper bounds for rates of modules over arbitrary Veronese algebras. Note that our results also hold with similar proofs if one considers multigraded K-algebras and the corresponding multigraded subalgebras.

This paper is structured in the following way. In the first section we recall definitions and introduce notation.

In Section 2 we prove the main result and get the upper bounds for the regularities mentioned above.

In the third section we discuss some applications of the main result. Let A be a positively graded K-algebra and M a finitely generated A-module. Provided that the symmetric algebra $S(M)$ is Koszul, we show that all symmetric powers of M have linear resolutions. In the specific case that $M = \mathfrak{m}$ is the graded maximal ideal of A, we have a necessary condition for $S(\mathfrak{m})$ to be Koszul, that is, when the defining ideal of A has a 2-linear resolution. Under the weaker assumption that A is Koszul, we obtain that all symmetric powers of $\mathfrak{m}$ have a linear A-resolution. Let $I \subset A$ be an homogeneous ideal generated in one degree. If the Rees ring $R(I)$ is Koszul, then all powers of I have linear A-resolutions.

Moreover, we recover some well-known results about graded Koszul algebras which Backelin and Fröberg first proved in [5], saying that the Koszul property is preserved for tensor products over K, Segre products, and Veronese subrings.

In the last section we interpret the main result for semigroup rings. The Koszul property for these rings corresponds to the Cohen–Macaulay property of certain divisor posets (see [12] and [14]). Therefore, we obtain that Cohen–Macaulayness for these divisor posets is preserved under taking diagonals and generalized Veronese subrings.

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1. NOTATION

Throughout this paper R always denotes a bigraded K-algebra of the form $R = S/J$, where J is a bihomogeneous ideal of S. We recall that for two integers $a, b \geq 0$ with $(a, b) \neq (0,0)$ the (a, b)-diagonal is the set $\Delta = \{(sa, sb)\colon s \in \mathbb{Z}\}$ of $\mathbb{Z}^2$. As defined in [7], the diagonal subalgebra R_Δ of R is generated by the residue classes of all monomials which have degree (a, b) in S. Therefore, R_Δ is standard graded. Let $a, b \geq 0$ be two integers with $(a, b) \neq (0,0)$ and $\tilde{\Delta} = \{(sa, tb)\colon s, t \in \mathbb{Z}\}$. According to [15] we define the bigraded generalized Veronese subring $R_{\tilde{\Delta}}$ as the subalgebra of R with bigraded components $(R_{\tilde{\Delta}})_{(i,j)} = R_{(ia, jb)}$. The algebra $R_{\tilde{\Delta}}$ is generated by the residue classes of all monomials which have degree $(a, 0)$ or $(0, b)$ in S. Thus, $R_{\tilde{\Delta}}$ is a standard bigraded algebra. Note that $R = R_{\tilde{\Delta}}$ for $(a, b) = (1,1)$. In the case that $n = 0$ or $m = 0$, the algebra R is simply standard graded and the subrings R_Δ, resp. $R_{\tilde{\Delta}}$, are the well-known Veronese subrings of R. Observe also that the $(1,1)$-diagonal of $R_{\tilde{\Delta}}$ equals R_Δ.

Let M be a finitely generated, bigraded R-module. For two integers $c, d \geq 0$ we define $M_\Delta^{(c,d)}$ to be the finitely generated, $\mathbb{Z}$-graded R_Δ-module with components $(M_\Delta^{(c,d)})_i = M_{(ia+c, ib+d)}$. For $(c, d) = (0,0)$ we simply use M_Δ instead of $M_\Delta^{(0,0)}$. We call $M_\Delta^{(c,d)}$ the *(c, d)-side-diagonal module* of M. Similarly, we write $M_{\tilde{\Delta}}^{(c,d)}$ for the bigraded $R_{\tilde{\Delta}}$-module with components $(M_{\tilde{\Delta}}^{(c,d)})_{(i,j)} = M_{(ia+c, jb+d)}$ and call it the *relative (c, d)-Veronese module* of M. If $n = 0$ or $m = 0$, then these modules coincide with the relative Veronese modules defined in [1]. We need two index sets,

$$\mathcal{I}(a, b) = \{(c, d) \in \mathbb{N}^2\colon c < a \text{ or } d < b\}$$

and

$$\tilde{\mathscr{I}}(a,b) = \begin{cases} \{(c,d) \in \mathbb{N}^2 \colon c < a \text{ and } d < b\} & \text{if } a, b \geq 1, \\ \{(c,0) \in \mathbb{N}^2 \colon c < a\} & \text{if } a \geq 1 \text{ and } b = 0, \\ \{(0,d) \in \mathbb{N}^2 \colon d < b\} & \text{if } a = 0 \text{ and } b \geq 1. \end{cases}$$

Note that the index set $\mathscr{I}(a,b)$ is infinite while $\tilde{\mathscr{I}}(a,b)$ is a finite set. For $(c,d) \in \mathscr{I}(a,b)$ the module $R_\Delta^{(c,d)}$ is generated in degree 0 and, for arbitrary $c, d \geq 0$, it is $R_\Delta^{(c,d)} = R_\Delta^{(c',d')}(-l)$ with some integer $l \geq 0$ and some $(c',d') \in \mathscr{I}(a,b)$. An analogous fact holds for the modules $R_{\tilde{\Delta}}^{(c,d)}$. We have the decomposition

$$R = \bigoplus_{(c,d) \in \mathscr{I}(a,b)} R_\Delta^{(c,d)}.$$

Analogously, if $a, b \geq 1$, then R is the finite direct sum of the $R_{\tilde{\Delta}}^{(c,d)}$ with $(c,d) \in \tilde{\mathscr{I}}(a,b)$.

The map $M \mapsto M_\Delta^{(c,d)}$ (resp., $M \mapsto M_{\tilde{\Delta}}^{(c,d)}$) defines an exact functor from the category of bigraded finitely generated R-modules to the category of $\mathbb{Z}$-graded finitely generated R_Δ-modules (resp., bigraded $R_{\tilde{\Delta}}$-modules). In particular, consider a bigraded free resolution

$$F_\bullet \colon \; \cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where every free module F_i decomposes into a finite direct $F_i = \bigoplus_{p,q} R(-p,-q)^{b_{i,(p,q)}}$. Here, $R(-p,-q)$ denotes the bigraded R-module with components $R(-p,-q)_{(i,j)} = R_{(i-q,j-q)}$. Then we get an exact complex of R_Δ-modules

$$(F_\bullet)_\Delta^{(c,d)} \colon \; \cdots \to (F_i)_\Delta^{(c,d)} \to \cdots \to (F_1)_\Delta^{(c,d)} \to (F_0)_\Delta^{(c,d)} \to M_\Delta^{(c,d)} \to 0$$

with $(F_i)_\Delta^{(c,d)} = \bigoplus_{p,q} (R(-p,-q))_\Delta^{(c,d)})^{b_{i,(p,q)}}$. Analogous statements are true for the functor $-_{\tilde{\Delta}}^{(c,d)}$. It is important for the main result to write every module $R(-p,-q))_\Delta^{(c,d)}$ as a shifted side-diagonal module of the form $R_\Delta^{(c',d')}$ for some $(c',d') \in \mathscr{I}(a,b)$. For a real number α we use $\lceil \alpha \rceil$ for the smallest integer z such that $z \geq \alpha$. We observe the following.

Remark 1.1. Let Δ be the (a,b)-diagonal. For $z \in \mathbb{Z}$ let $\alpha(z) \in \{0, \ldots, a-1\}$ be the integer such that $\alpha(z) \equiv z \bmod a$ and $\beta(z) \in \{0, \ldots, b-1\}$ with $\beta(z) \equiv z \bmod b$.

(a) (i) Let $a > 0$, $b = 0$, and $(c,d) \in \mathscr{I}(a,b)$. Then

$$R(-p,-q)_\Delta^{(c,d)} = \begin{cases} 0, & \text{if } q > d, \\ R_\Delta^{(\alpha(c-p),d-q)}(-l), & \text{if } q \leq d, \end{cases}$$

where $l = \max\{0, \lceil \frac{p-c}{a} \rceil\}$.

(ii) Let $a = 0$, $b > 0$, and $(c, d) \in \mathscr{A}(a, b)$. Then

$$R(-p, -q)_{\Delta}^{(c,d)} = \begin{cases} 0, & \text{if } p > c, \\ R_{\Delta}^{(c-p,\, \beta(d-q))}(-l), & \text{if } p \le c, \end{cases}$$

where $l = \max\{0, \lceil \frac{q-d}{b} \rceil\}$.

(iii) Let $a, b \ge 1$ and $(c, d) \in \mathscr{A}(a, b)$. Then

$$R(-p, -q)_{\Delta}^{(c,d)} = R_{\Delta}^{(la+c-p,\, lb+d-q)}(-l),$$

where $l = \max\{0, \lceil \frac{p-c}{a} \rceil, \lceil \frac{q-d}{b} \rceil\}$.

(b) (i) Let $a > 0$, $b = 0$, and $(c, 0) \in \tilde{\mathscr{A}}(a, b)$. Then

$$R(-p, -q)_{\tilde{\Delta}}^{(c,0)} = \begin{cases} 0, & \text{if } q > 0, \\ R_{\tilde{\Delta}}^{(\alpha(c-p),\, 0)}(-k, 0), & \text{if } q = 0, \end{cases}$$

where $k = \max\{0, \lceil \frac{p-c}{a} \rceil\}$.

(ii) Let $a = 0$, $b > 0$, and $(0, d) \in \tilde{\mathscr{A}}(a, b)$. Then

$$R(-p, -q)_{\tilde{\Delta}}^{(0,d)} = \begin{cases} 0, & \text{if } p > 0, \\ R_{\tilde{\Delta}}^{(0,\, \beta(d-q))}(0, -l), & \text{if } p = 0, \end{cases}$$

where $l = \max\{0, \lceil \frac{q-d}{b} \rceil\}$.

(iii) Let $a, b \ge 1$ and $(c, d) \in \tilde{\mathscr{A}}(a, b)$. Then

$$R(-p, -q)_{\tilde{\Delta}}^{(c,d)} = R_{\tilde{\Delta}}^{(\alpha(c-p),\, \beta(d-q))}(-k, -l),$$

where $k = \max\{0, \lceil \frac{p-c}{a} \rceil\}$ and $l = \max\{0, \lceil \frac{q-d}{b} \rceil\}$

We recall some well-known definitions. For a bigraded, finitely generated R-module M, each $\operatorname{Tor}_i^R(M, K)$-group is naturally bigraded and the bigraded Poincaré series is given by

$$P_M^R(s, t, z) = \sum_{i,j,k} \dim_K \operatorname{Tor}_i^R(M, K)_{(j,k)} s^j t^k z^i.$$

Let A be a positively graded K-algebra and N a finitely generated, graded A-module. We set

$$t_s(N) = \sup\{j \colon \operatorname{Tor}_s^A(N, K)_j \ne 0\},$$

with $t_s(N) = -\infty$ if $\operatorname{Tor}_s^A(N, K) = 0$. Recall that the Castelnuovo–Mumford regularity is defined as

$$\operatorname{reg}_A N = \sup\{t_i(N) - i \colon i \ge 0\}.$$

The initial degree $\operatorname{indeg}(N)$ of N is the minimum of the i such that $N_i \neq 0$. Note that M is said to have an i-linear resolution if $\operatorname{reg}_A M = \operatorname{indeg}(M) = i$. By definition, A is Koszul if and only if $\operatorname{reg}_A K = 0$. Every bigraded K-algebra R is also naturally $\mathbb{N}$-graded with ith component $R_i = \bigoplus_{j+k=i} R_{(j,k)}$. Similarly, every bigraded R-module M can be considered as $\mathbb{Z}$-graded. We say that M has a bigraded a-linear resolution if $\operatorname{Tor}_i^R(M, K)_{(j,k)} = 0$ for all $i \geq 0$ and all $j + k \neq i + a$.

2. MAIN RESULT

In this section we prove the main result of this article.

THEOREM 2.1. *If R is a Koszul algebra, then every diagonal subalgebra R_Δ and every generalized Veronese subring $R_{\tilde{\Delta}}$ is a Koszul algebra.*

For the proof we need several lemmata. Let $\mathfrak{n}_x = (x_1, \ldots, x_n) \subset R$, resp. $\mathfrak{n}_y = (y_1, \ldots, y_m) \subset R$, be the ideal generated by the residue classes of the x_i, resp. the y_j.

LEMMA 2.2. *If R is Koszul, then the ideals $\mathfrak{n}_x$ and $\mathfrak{n}_y$ have bigraded 1-linear R-resolutions.*

Proof. By symmetry, it is enough to show that $\mathfrak{n}_y$ has a bigraded linear resolution. The residue class field K has a 0-linear minimal free R-resolution $F_\bullet$ because R is Koszul. Let Δ be the $(1,0)$-diagonal. By applying the functor $-_{\tilde{\Delta}}$ we get the exact complex $(F_\bullet)_{\tilde{\Delta}} \to K \to 0$. By Remark 1.1(b) the ith module $(F_i)_{\tilde{\Delta}}$ is a direct sum of copies of $R_{\tilde{\Delta}}$ shifted by $(-i, 0)$. Thus $R_{\tilde{\Delta}}$ is a standard bigraded Koszul algebra.

Let $p\colon R \to R_{\tilde{\Delta}}$ be the projection map and $i\colon R_{\tilde{\Delta}} \to R$ the inclusion. Note that both maps p and i are bigraded homomorphisms. Since p is a ring epimorphism and since $p \circ i = \mathrm{id}_{R_{\tilde{\Delta}}}$, the map i is a bigraded algebra retract. We may apply a result from [10] to the bigraded situation. This yields that $P_K^R = P_{R_{\tilde{\Delta}}}^R P_K^{R_{\tilde{\Delta}}}$, where $R_{\tilde{\Delta}} = R/\mathfrak{n}_y$ is considered as a bigraded R-module. Since R and $R_{\tilde{\Delta}}$ are Koszul, the equality of bigraded Poincaré series implies that $\mathfrak{n}_y$ has a bigraded 1-linear R-resolution. This concludes the proof. ∎

PROPOSITION 2.3. *Let $c, d \geq 0$ be two integers. If $\mathfrak{n}_x$ and $\mathfrak{n}_y$ have bigraded linear resolutions, then:*

(a) *The sidediagonal module $R_\Delta^{(c,d)}$ has a linear R_Δ-resolution.*

(b) *The relative Veronese module $R_{\tilde{\Delta}}^{(c,d)}$ has a bigraded linear $R_{\tilde{\Delta}}$-resolution.*

For the proof of the proposition we need a fact which is stated in [7].

LEMMA 2.4. *Let A be a standard graded K-algebra, M a finitely generated A-module, and*

$$\cdots \to N_r \to N_{r-1} \to \cdots \to N_1 \to N_0 \to M \to 0$$

an exact complex of finitely generated graded A-modules. Then:

(a) *Let $h \in \mathbb{N}$ and let $a \in \mathbb{Z}$ such that $t_s(N_r) \le a + r + s$ for all $0 \le r \le h$ and $0 \le s \le h - r$. Then $t_h(M) \le a + h$.*

(b) $\operatorname{reg}_A M \le \sup\{\operatorname{reg}_A N_i - i\colon i \in \mathbb{N}\}$.

Proof of Proposition 2.3. Since the proofs of (a) and (b) are similar, we only consider Part (a). Moreover, it is enough to show that all modules $R_\Delta^{(c,d)}$ with $(c,d) \in \mathcal{A}(a,b)$ have linear resolutions. Let $G_\bullet$ be the bigraded minimal free R-resolution of $\mathfrak{n}_x$. Since $\mathfrak{n}_x$ has a bigraded 1-linear resolution, every free module G_r is of the form

$$G_r = \bigoplus_{\substack{p+q=r+1\\ p\ge 1}} R(-p,-q)^{\beta_{r,(p,q)}},$$

with nonnegative integers $\beta_{r,(p,q)}$. Observe that for $c \ge 1$ and $(c,d) \in \mathcal{A}(a,b)$ we have $(\mathfrak{n}_x)^{(c,d)} = R_\Delta^{(c,d)}$. Applying the functor $-_\Delta^{(c,d)}$, we obtain an acyclic complex $(G_\bullet)_\Delta^{(c,d)} \to R_\Delta^{(c,d)} \to 0$, where

$$(G_r)_\Delta^{(c,d)} = \bigoplus_{\substack{p+q=r+1\\ p\ge 1}} \left(R_\Delta^{(c_{p,q},d_{p,q})}(-l_{p,q})\right)^{\beta_{r,(p,q)}}.$$

By Remark 1.1(a), all occurring shifts $l_{p,q}$ are at most r. Similarly, let $H_\bullet$ be the minimal free resolution of $\mathfrak{n}_y$. Then for $d \ge 1$ and $(c,d) \in \mathcal{A}(a,b)$ we observe that $(\mathfrak{n}_y)_\Delta^{(c,d)} = R_\Delta^{(c,d)}$ and that the shifts in $(H_r)_\Delta^{(c,d)}$ are bounded by r.

To conclude the proof we show by induction that $t_h(R_\Delta^{(c,d)}) \le h$ for all $h \in \mathbb{N}$ and $(c,d) \in \mathcal{A}(a,b)$. First we use induction on h. The modules $R_\Delta^{(c,d)}$ are generated in degree 0, thus $t_0(R_\Delta^{(c,d)}) = 0$. Let now $h \ge 1$. We apply induction on $c+d$ where $(c,d) \in \mathcal{A}(a,b)$. For $c+d=0$ it follows that $c=0$ and $d=0$ and therefore trivially that $t_h(R_\Delta) \le h$. Let now $c+d>0$. Then $c \ge 1$ or $d \ge 1$.

We discuss the case $c \ge 1$ first. In order to apply Lemma 2.4(a) to the exact complex $(G_\bullet)_\Delta^{(c,d)} \to R_\Delta^{(c,d)} \to 0$ we show that $t_s((G_r)_\Delta^{(c,d)}) \le r + s$ for all $0 \le r \le h$ and $0 \le s \le h - r$. Observe that $(G_0)_\Delta^{(c,d)}$ is a direct sum of n copies of $(R_\Delta^{(c-1,d)})$. Since $(c-1,d) \in \mathcal{A}(a,b)$, the induction hypothesis on $c+d$ implies that $t_s((G_0)_\Delta^{(c,d)}) \le s$ for all $0 \le s \le h$. For $1 \le r \le h$ and $0 \le s \le h - r$, we have

$$t_s\left((G_r)_\Delta^{(c,d)}\right) \le t_s\left(\bigoplus_{\substack{p+q=i+1\\ p\ge 1}} R_\Delta^{(c_{p,q},d_{p,q})}\right) + r \le s + r,$$

where the first inequality holds because $l_{p,q} \le r$ for all occurring p, q, and the second inequality holds by induction on h. Now Lemma 2.4 implies that $t_h(R_\Delta^{(c,d)}) \le h$.

If $c = 0$ and $d \ge 1$ the argument above similarly applies to the complex $(H_\bullet)_\Delta^{(c,d)} \to R_\Delta^{(c,d)} \to 0$. ∎

As a direct consequence of Lemma 2.2 and Lemma 2.3 we obtain

COROLLARY 2.5. *Let $c, d \ge 0$ be two integers. If R is Koszul, then all side-diagonal modules $R_\Delta^{(c,d)}$ have linear R_Δ-resolutions and all relative Veronese modules $R_{\tilde\Delta}^{(c,d)}$ have bigraded linear $R_{\tilde\Delta}$-resolutions.*

We use this corollary to get upper bounds for the regularity of side-diagonal and relative Veronese modules.

THEOREM 2.6. *Let R be Koszul and M be a finitely generated, bigraded R-module with $r = \operatorname{reg}_R M$ and $\operatorname{indeg}(M) = 0$.*

(a) *Let $(c, d) \in \mathscr{I}(a, b)$. Then*

$$\operatorname{reg}_{R_\Delta} M_\Delta^{(c,d)} \le \begin{cases} \max\{0, \lceil \frac{r-c}{a} \rceil\}, & \text{if } b = 0 \text{ and } a > 0, \\ \max\{0, \lceil \frac{r-d}{b} \rceil\}, & \text{if } a = 0 \text{ and } b > 0, \\ \max\{0, \lceil \frac{r-c}{a} \rceil, \lceil \frac{r-d}{b} \rceil\}, & \text{if } a, b \ge 1. \end{cases}$$

(b) *Let $(c, d) \in \tilde{\mathscr{I}}(a, b)$. Then*

$$\operatorname{reg}_{R_{\tilde\Delta}} M_{\tilde\Delta}^{(c,d)} \le \begin{cases} \max\{0, \lceil \frac{r}{a} \rceil\}, & \text{if } b = 0 \text{ and } a > 0, \\ \max\{0, \lceil \frac{r}{b} \rceil\}, & \text{if } a = 0 \text{ and } b > 0, \\ \min\{r, \lceil \frac{r-c}{a} - \frac{d}{b} + 1 \rceil\}, & \text{if } 1 \le a \le b. \end{cases}$$

In particular, if $1 \le a \le b$, then $\operatorname{reg}_{R_{\tilde\Delta}} M \le \min\{r, \lceil \frac{r}{a} + 1 \rceil\}$.

Proof. Let $F_\bullet$ be the minimal free R-resolution of M. Since $\operatorname{reg}_R M = r$ we have $F_i = \bigoplus_{i \le p+q \le i+r} R(-p, -q)^{\beta_{i,(p,q)}}$ for some nonnegative integers $\beta_{i,(p,q)}$. For the proof of Part (a) we restrict to the case $a, b \ge 1$. The other cases follow similarly. By Remark 1.1(a) and Corollary 2.5 we observe that

$$\begin{aligned} \operatorname{reg}_{R_\Delta}(F_i)_\Delta^{(c,d)} &\le \max\{0, \lceil \tfrac{i+r-c}{a} \rceil, \lceil \tfrac{i+r-d}{b} \rceil\} \\ &\le \max\{0, \lceil \tfrac{r-c}{a} \rceil, \lceil \tfrac{r-d}{b} \rceil\} + i. \end{aligned}$$

Now the claim follows from Lemma 2.4(b). For part (b) we also restrict to the case $a \ge 1$ and $b \ge 1$. Use Remark 1.1(b) and Corollary 2.5 to observe that

$$\operatorname{reg}_{R_{\tilde\Delta}}(F_i)_{\tilde\Delta}^{(c,d)} \le \max\{\max\{0, \lceil \tfrac{p-c}{a} \rceil\} + \max\{0, \lceil \tfrac{q-d}{b} \rceil\} : i \le p + q \le i + r\}.$$

The claim follows from a case-by-case computation using $1 \le a \le b$ and Lemma 2.4(b). Then the upper bound for $\operatorname{reg}_{R_{\tilde{\Delta}}} M$ follows from the fact that M decomposes into the finite sum $M = \bigoplus_{(c,d) \in \tilde{I}(a,b)} M_{\tilde{\Delta}}^{(c,d)}$. ∎

As a direct consequence of Theorem 2.6 the modules M_Δ and $M_{\tilde{\Delta}}$ have small regularities for $a, b \gg 0$. More concretely, we have

COROLLARY 2.7. *Let M be a finitely generated, bigraded R-module.*

(a) *If* $\max\{a, b\} \ge \operatorname{reg}_R M$, *then* $\operatorname{reg}_{R_\Delta} M_\Delta \le \min\{1, \operatorname{reg}_R M\}$.

(b) *Let* $a, b \ge 1$. *If* $\min\{a, b\} \ge \operatorname{reg}_R M$, *then* $\operatorname{reg}_{R_{\tilde{\Delta}}} M_{\tilde{\Delta}} \le \min\{2, \operatorname{reg}_R M\}$ *and* $\operatorname{reg}_{R_{\tilde{\Delta}}} M \le \min\{2, \operatorname{reg}_R M\}$.

The main result follows immediately from the results above.

Proof of Theorem 2.1. Note that a graded K-algebra A is Koszul if and only if $\operatorname{reg}_A K = 0$. Since $K_\Delta = K$ and $K_{\tilde{\Delta}} = K$, the claim follows from Theorem 2.6. ∎

Note that the converse of Theorem 2.1 is false. Take, for example, the algebra $R = K[x_1, y_1]/(x_1 y_1^2)$. Since the defining ideal of R is generated in degree 3, R is not Koszul. But every diagonal R_Δ is Koszul because R_Δ is isomorphic to a polynomial ring $K[t]$, to K, or to the Koszul algebra $K[t]/(t^2)$.

3. APPLICATIONS

In this section we present some applications which arise naturally in the study of symmetric algebras and Rees algebras. In the following A will always denote a positively graded algebra; i.e., $A = K[x_1, \dots, x_n]/Q$ where $\deg(x_i) = 1$ and $Q \subset K[x_1, \dots, x_n]$ is a homogeneous ideal. Let $\mathfrak{m} \subset A$ be the graded maximal ideal.

We first consider symmetric algebras. Let M be a graded A-module with homogeneous generators $f_1, \dots, f_m$, and let $(a_{ij})_{i=1,\dots,t,\, j=1,\dots,m}$ be the corresponding relation matrix. The symmetric algebra $S(M) = \bigoplus_{j \ge 0} S^j(M)$ of M has a presentation of the form $S(M) = A[y_1, \dots, y_m]/J$, where $J = (g_1, \dots, g_t)$ and $g_i = \sum_{j=1}^m a_{ij} y_j$ for $i = 1, \dots, t$. If $f_1, \dots, f_m$ have the same degree, then $S(M)$ is standard bigraded by assigning the degree $(1, 0)$ to the residue class of x_i for $i = 1, \dots, n$ and by setting $\deg(y_i) = (0, 1)$. Note that $S^j(M)$ is a graded A-module. As an application of the main result we obtain

COROLLARY 3.1. *If* $S(M)$ *is Koszul, then* A *is Koszul and the module* $S^j(M)$ *has a linear resolution for all* $j \ge 0$.

Proof. Let Δ be the $(1,0)$-diagonal. Then $S(M)_\Delta = A$ and $S^j(M) = S(M)_\Delta^{(0,j)}$. Thus A is Koszul and, by Corollary 2.5, the module $S^j(M)$ has a linear A-resolution. ∎

As one might expect, it seems to be a strong condition that the symmetric algebra $S(M)$ is Koszul. In a more specific case, however, when $M = \mathfrak{m}$ is the graded maximal ideal of a Koszul algebra, we have a sufficient condition.

THEOREM 3.2. *Let $A = K[x_1, \dots, x_n]/Q$ and K be an infinite field with* $\operatorname{char}(K) \neq 2$. *If Q has a 2-linear resolution over $K[x_1, \dots, x_n]$, then the defining ideal of $S(\mathfrak{m})$ has a quadratic Gröbner basis with respect to a reverse lexicographic term order. In particular, $S(\mathfrak{m})$ is Koszul.*

It is well-known that the existence of a quadratic Gröbner basis for the defining ideal of an algebra implies the Koszul property. For details on Gröbner bases and generic initial ideals refer to [8].

LEMMA 3.3. *Let $Q \subset K[x_1, \dots, x_n]$ be an homogeneous ideal, K infinite, and* $\operatorname{char}(K) \neq 2$. *Moreover, let* $\operatorname{Gin}(Q)$ *denote the generic initial ideal with respect to the reverse lexicographic term order induced by $x_1 > \cdots > x_n$. If Q has a 2-linear resolution, then* $\operatorname{Gin}(Q)$ *is quadratic and satisfies the following condition*:

$(*)$ *If $x_i x_j \in \operatorname{Gin}(Q)$ and $i \leq j$, then $x_i x_k \in \operatorname{Gin}(Q)$ for all $k < j$.*

Proof. We use some results about $\operatorname{Gin}(Q)$. It is known that $\operatorname{Gin}(Q)$ is a Borel-fixed ideal and that $\operatorname{reg}\operatorname{Gin}(Q) = \operatorname{reg}(Q) = 2$ (see [8, 20.21]). Thus $\operatorname{Gin}(Q)$ is quadratic. Since $\operatorname{char}(K) \neq 2$ we obtain that $\operatorname{Gin}(Q)_2$ is stable (see [8, 15.23b]) and therefore satisfies the condition $(*)$. ∎

We need some notation taken from [11]. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the standard bigraded polynomial ring and

$$f = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n} a_{i_1 i_2 \cdots i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$$

a form of degree $(d, 0)$. We set

$$f^{(k)} = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n} a_{i_1 i_2 \cdots i_d} x_{i_1} x_{i_2} \cdots x_{i_{d-k}} y_{i_{d-k+1}} \cdots y_{i_d}$$

for $k = 0, \dots, d$. Note that $f^{(k)}$ is bihomogeneous and of degree $(d-k, k)$. Moreover, let $\delta_{ij} = x_i y_j - x_j y_i$ for $i \neq j$ and $L = \{\delta_{ij} \colon i \neq j\}$. We need the following lemma.

LEMMA 3.4. *Let $>$ denote the reverse lexicographic term order on S induced by $x_1 > x_2 > \cdots > x_n > y_1 > \cdots > y_n$ and $\varphi \colon S \to S$ be the homo-*

morphism with $\varphi(x_i) = x_i$ *and* $\varphi(y_i) = x_i$ *for* $i = 1, \dots, n$. *Assume that* $f \in S$ *is a bihomogeneous polynomial of degree* (s, t) *such that* $\operatorname{in}(f) = x_{i_1} x_{i_2} \cdots x_{i_s} y_{j_1} \cdots y_{j_t}$ *satisfies* $i_1 \le i_2 \le \cdots \le i_s \le j_1 \le \cdots \le j_t$. *Then* $\operatorname{in}(\varphi(f)) = \varphi(\operatorname{in}(f))$ *and* $\operatorname{in}(f) = \varphi(\operatorname{in}(f))^{(t)}$.

Proof. With the condition $i_1 \le i_2 \le \cdots \le i_s \le j_1 \le \cdots \le j_t$ it is easy to see that $\varphi(\operatorname{in}(f)) > \varphi(v)$ for all monomials v of f with $v < \operatorname{in}(f)$. ∎

Proof of Theorem 3.2. By Lemma 3.3, we may assume that the defining ideal Q of A has a quadratic Gröbner basis $g_1, \dots, g_t$ with respect to the reverse lexicographic term order induced by $x_1 > x_2 > \cdots > x_n$ and that $\operatorname{in}(Q)$ satisfies the $(*)$ condition of Lemma 3.3. It is easy to see that $S(\mathfrak{m})$ has a presentation $S(\mathfrak{m}) = S/J$, where $J = (g_1, \dots, g_t, g_1^{(1)}, \dots, g_t^{(1)}, L)$.

Let $>$ denote the reverse lexicographic term order on S induced by $x_1 > x_2 > \cdots > x_n > y_1 > \cdots > y_n$. We will show that the set $G = \{g_1, \dots, g_t, g_1^{(1)}, \dots, g_t^{(1)}\} \cup L$ is a Gröbner basis for J with respect to $>$, which concludes the proof of the theorem.

Let $f \in J$ be a bihomogeneous polynomial of degree (s, t). Then $s \ge 1$. We show that $\operatorname{in}(f)$ is divided by some $\operatorname{in}(g)$ with $g \in G$. Let $\operatorname{in}(f) = x_{i_1} x_{i_2} \cdots x_{i_s} y_{j_1} \cdots y_{j_t}$, where $i_1 \le i_2 \le \cdots \le i_s$ and $j_1 \le j_2 \le \cdots \le j_t$. If there exist indices p, q such that $i_p > j_q$ then $\operatorname{in}(\delta_{i_p j_q})$ divides $\operatorname{in}(f)$, which is the claim.

Otherwise we have $i_1 \le i_2 \le \cdots \le i_s \le j_1 \le j_2 \le \cdots \le j_t$. Let φ denote the homomorphism from Lemma 3.4. Since $f \in J$, it follows that $\varphi(f) \in Q$. By Lemma 3.4, we have $\operatorname{in}(f) = \operatorname{in}(\varphi(f))^{(t)}$, where $\operatorname{in}(\varphi(f)) = x_{i_1} x_{i_2} \cdots x_{i_s} x_{j_1} \cdots x_{j_t}$. Since $g_1, \dots, g_t$ is a Gröbner basis for Q, there exists a $g \in \{g_1, \dots, g_t\}$ such that $\operatorname{in}(g)$ divides $\operatorname{in}(\varphi(f))$. By the condition $(*)$ of Lemma 3.3, we may assume that $\operatorname{in}(g) = x_{i_1} x_{i_2}$, if $s > 1$, or $\operatorname{in}(g) = x_{i_1} x_{j_1}$, if $s = 1$. Now $\operatorname{in}(g)$ or $\operatorname{in}(g^{(1)})$ divides in (f). ∎

Under the strong assumption of Theorem 3.2 it follows from Corollary 3.1 that $S^j(\mathfrak{m})$ has linear resolution for all $j \ge 1$. Actually, we have

PROPOSITION 3.5. *Let* $j \ge 1$. *If* A *is Koszul, then* $S^j(\mathfrak{m})$ *has a linear* A*-resolution*.

In the proof we use results from [13] and some basic facts about the Koszul complex (see [6, Sect. I.1.6] for details).

Proof. Let $S_x = K[x_1, \dots, x_n]$ be the standard graded polynomial ring and $A = S_x/Q$. We may assume that the defining ideal Q of A does not contain linear forms. Then Q is generated in degree 2. Let $\mathfrak{m} = (x_1, \dots, x_n) \subset A$ be the graded maximal ideal of A. We denote by $K_\bullet$ the

Koszul complex of the sequence $x_1, \dots, x_n \in A$. Let $H_1(K_\bullet)$ be the first homology group of this complex. Recall that $S(\mathfrak{m}) = A[y_1, \dots, y_n]/J$ for some bihomogeneous ideal J and that $S^j(\mathfrak{m})$ is generated by the residue classes of all monomials in degree $(0, j)$. We consider $S^j(\mathfrak{m})$ as an A-module generated in degree j. For $j \geq 1$, there exists the downgrading homomorphism α_j: $S^j(\mathfrak{m}) \to \mathfrak{m}S^{j-1}(\mathfrak{m})$ mapping a residue class of $y_{i_1} y_{i_2} \cdots y_{i_j}$ to the residue class of $x_{i_1} y_{i_2} \cdots y_{i_j}$ (see [13, Sect. 2]). Note that it does not matter which of the factors y_{i_l} is replaced by x_{i_l}.

To show that $S^j(\mathfrak{m})$ has a linear R-resolution for all $j \geq 1$ we use induction on j. For $j = 1$, we have $S^1(\mathfrak{m}) = \mathfrak{m}$, which has a linear resolution because A is Koszul. Let now $j > 1$. We have the short exact sequence

$$0 \to U \to S^j(\mathfrak{m}) \xrightarrow{\alpha_j} \mathfrak{m}S^{j-1}(\mathfrak{m}) \to 0, \tag{1}$$

where $U = \ker \alpha_j$. By [13, Lemma 2.2], U is a subquotient of the module $N = H_1(K_\bullet) \otimes_{A/\mathfrak{m}} (A/\mathfrak{m})(-j+2)^s$ for some integer $s \geq 1$. The module N is annihilated by $\mathfrak{m}$. Since Q is generated in degree 2, it follows that $H_1(K_\bullet) \cong \operatorname{Tor}_1^{S_x}(A, K)$ is generated in degree 2. Therefore, $U \cong K(-j)^t$ for some integer $t \geq 0$ and U has a j-linear A-resolution because A is Koszul. By the induction hypothesis, $S^{j-1}(\mathfrak{m})$ has $(j-1)$-linear A-resolution. Thus, by [7, Lemma 6.4], the module $\mathfrak{m}S^{j-1}(\mathfrak{m})$ has a j-linear A-resolution. The assertion follows when we apply the long exact sequence of the functor $\operatorname{Tor}^A(\cdot, K)$ to the sequence (1). ∎

The hypothesis of Theorem 3.2 cannot be weakened to the assumption that A is only Koszul. A counterexample is the algebra $A = K[x_1, x_2]/(x_1^2, x_2^2)$. As a complete intersection. A is Koszul, but with the help of the program Macaulay we find that $S(\mathfrak{m})$ is not Koszul. This example shows also that the converse of Corollary 3.1 is false because, by Proposition 3.5, all symmetric powers $S^j(\mathfrak{m})$ have a linear resolution.

A further intensively studied class of bigraded algebras is the Rees algebras. Let $I \subset A$ be a homogeneous ideal which is minimality generated by homogeneous elements $f_1, f_2, \dots, f_m$ of the same degree d. Recall that the Rees ring $R(I) = A[It]$ of I admits a standard bigrading assigning the degree $(1, 0)$ to the generators of $\mathfrak{m} \subset A$ and setting $\deg(f_i t) = (0, 1)$ for $i = 1, \dots, m$. As a consequence of Theorem 2.1 we observe

COROLLARY 3.6. *If $R(I)$ is Koszul, then A is Koszul and the ideal I^j has a linear A-resolution for all $j \geq 0$.*

Proof. Let Δ be the $(1, 0)$-diagonal. Then $R(I)_\Delta = A$ and $I^j = R(I)_\Delta^{(0,j)}(-dj)$. Thus, by Corollary 2.2, the ideal I^j has a linear A resolution. ∎

Note that the Rees algebra $R(\mathfrak{m})$ is always Koszul because it is a Segre product of two Koszul algebras. In [11] Herzog *et al*. prove that the defining ideal of $R(\mathfrak{m})$ has a quadratic Gröbner basis provided Q has a quadratic Gröbner basis.

As direct consequences of Theorem 2.1 we get some well-known facts about positively graded Koszul algebras. Let $d \geq 1$ be an integer. Recall that the dth Veronese subring $A^{(d)}$ of A is the positively graded algebra $A^{(d)} = \bigoplus_{i \geq 0} A_{id}$. For two standard graded K-algebras A and B the tensor product $A \otimes_K B = \bigoplus_{i,j \geq 0} A_i \otimes_K B_j$ is a standard bigraded algebra. The Segre product of A and B, denoted by $A * B$, is the $(1,1)$-diagonal of $A \otimes_K B$. We recover some well-known results (see [9]).

COROLLARY 3.7. *Tensor products, Segre products, and Veronese subrings of Koszul algebras are Koszul.*

Proof. Let $F_\bullet$ (resp. $G_\bullet$) be the minimal free resolution of K over A (resp. B). Then the tensor product $G_\bullet \otimes_K F_\bullet$ gives a minimal free resolution of K over $A \otimes_K B$. Thus if A and B are Koszul, $A \otimes_K B$ is Koszul. Now the Segre product $A * B$ is the $(1,1)$-diagonal $A \otimes_K B$, which is Koszul by Theorem 2.1.

Let A be a positively graded Koszul algebra. Consider A as a standard bigraded algebra where all generators have degree $(1,0)$. Then $A^{(d)}$ is a diagonal of A and, by Theorem 2.1, $A^{(d)}$ is Koszul. ∎

Let M be a graded A-module. Recall from [1] that the rate of M is given by $\operatorname{rate}_A M = \sup\{t_i(M)/i \colon i \geq 0\}$. A similar definition is given in [4], where Backelin proves that $A^{(d)}$ is Koszul for $d \gg 0$. Note an A-module M is naturally an $A^{(d)}$-module. Aramova *et al*. have proved in [1] that

$$\operatorname{rate}_{A^{(d)}} M \leq \lceil \operatorname{rate}_A M/d \rceil \tag{2}$$

for an arbitrary K-algebra A and all $d \geq c$ where c is a constant depending on A. Moreover, they showed that $c = 1$, if A is a polynomial ring. For this, they used that the relative Veronese modules $A^{(d,j)} = \bigoplus_{i \geq 0} A_{id+j}$ for $j = 0, \ldots, d-1$ have linear A-resolutions. Since the relative Veronese modules coincide with side-diagonal modules, it follows from Corollary 2.3 that (2) is valid for $c \geq 1$ provided A is Koszul. We get similar upper bounds for the regularity over Koszul algebras.

COROLLARY 3.8. *Let A be Koszul and M be a finitely generated graded A-module. Then* $\operatorname{reg}_{A^{(d)}} M \leq \lceil \operatorname{reg}_A M/d \rceil$ *for all* $d \geq 1$. *In particular,* $\operatorname{reg}_{A^{(d)}} M \leq 1$ *if* $d \geq \operatorname{reg}_A M$.

Proof. Consider A as a bigraded algebra generated in degree $(1,0)$. Let Δ be the $(d,0)$-diagonal of A. Then $A^{(d)} = A_\Delta$ and, as an $A^{(d)}$-module, we have $M = \bigoplus_{c=0}^{d-1} M_\Delta^{(c,0)}$. By Theorem 2.6, the claim follows. ∎

4. SEMIGROUP RINGS

Finally, we study the consequences of the main result for bihomogeneous semigroup rings. Let $\Lambda \subset \mathbb{N}^d$ be a finitely generated semigroup. We call Λ standard bigraded if

(a) Λ is the disjoint union $\bigcup_{i,j\geq 0} \Lambda_{(i,j)}$,

(b) $\Lambda_{(0,0)} = 0$, $\Lambda_{(i,j)} + \Lambda_{(k,l)} \subset \Lambda_{(i+k,j+l)}$ for all integers $i, j, k, l \geq 0$, and

(c) Λ is generated by elements of $\Lambda_{(1,0)}$ and $\Lambda_{(0,1)}$.

We call the elements of $\Lambda_{(i,j)}$ bihomogeneous of degree (i, j). Similarly, one defines a graded semigroup. Let Λ be a standard bigraded semigroup which is minimally generated by $\alpha_1, \dots, \alpha_n \in \Lambda_{(1,0)}$ and $\beta_1, \dots, \beta_m \in \Lambda_{(0,1)}$, and let $K[t_1, \dots, t_d]$ denote the polynomial ring. To a semigroup element $\lambda = (a_1, \dots, a_d) \in \Lambda$ we assign the monomial $t^\lambda = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$. Recall that the semigroup ring $K[\Lambda]$ is the K-algebra generated by the monomials $t^{\alpha_i}, t^{\beta_j}$, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Let $\varphi: S \to K[\Lambda]$ be the epimorphism with $\varphi(x_i) = t^{\alpha_i}$ and $\varphi(y_j) = t^{\beta_j}$. Then $J = \ker(\varphi)$ is called the toric ideal of the semigroup ring $K[\Lambda]$. If Λ is bigraded then $K[\Lambda] = S/J$ is a standard bigraded algebra.

The divisibility relation of the monomials in $K[\Lambda]$ defines a partial order $\preceq$ on Λ. For $\mu, \lambda \in \Lambda$ we set $\mu \preceq \lambda$ if $\lambda = \sigma + \mu$ for some $\sigma \in \Lambda$. Then the open intervals $(\mu, \lambda) = \{\sigma \in \Lambda: \mu \prec \sigma \prec \lambda\}$ are partially ordered with the induced ordering.

Let $(P, \preceq)$ be a finite poset. Recall that the boundary complex $\Gamma(P)$ is the simplicial complex whose faces are the totally ordered subsets of P. For $\lambda \in \Lambda$ we denote the boundary complex of the interval $(0, \lambda)$ by Γ_λ. The following is stated in [12; 14, Corollary 2.2].

PROPOSITION 4.1. *$K[\Lambda]$ is Koszul if and only if Γ_λ is Cohen–Macaulay for all $\lambda \in \Lambda$.*

Let Λ be a bigraded semigroup. In analogy to the definition for K-algebras we set

$$\Lambda_\Delta = \bigcup_{i\geq 0} \Lambda_{(ia,ib)} \quad \text{and} \quad \Lambda_{\tilde{\Delta}} = \bigcup_{i,j\geq 0} \Lambda_{(ia,jb)}$$

for the (a, b)-diagonal Δ. Note that Λ_Δ is graded and partially ordered by the induced ordering. If $\lambda \in \Lambda_{(ia,ib)}$, then we use $(\Gamma_\lambda)_\Delta$ for the boundary complex of the induced open interval $(0, \lambda) \subset \Lambda_\Delta$. Similarly, we define $(\Gamma_\lambda)_{\tilde{\Delta}}$ for $\lambda \in \Lambda_{(ia,jb)}$. Finally, we reformulate our main result for semigroup rings.

COROLLARY 4.2. *Let $\Lambda \subset \mathbb{N}^d$ be a bigraded semigroup and Δ a diagonal. If Γ_λ is Cohen–Macaulay for all $\lambda \in \Lambda$, then*:

(a) $(\Gamma_\lambda)_\Delta$ *is Cohen–Macaulay for all* $\lambda \in \Lambda_\Delta$.

(b) $(\Gamma_\lambda)_{\tilde{\Delta}}$ *is Cohen–Macaulay for all* $\lambda \in \Lambda_{\tilde{\Delta}}$.

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