Domains Containing the Field of Values of a Matrix

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ABSTRACT

Two groups of estimates are given for the field of values of a general complex matrix. The first group gives a domain containing both the field of values and the convex hull of the diagonal elements, while the second group gives a domain containing both the field of values and the convex hull of the eigenvalues. The domain given by the second group reduces to the smallest domain possible if the matrix is normal. The results have useful applications in control theory and in stability theory for differential equations.

1. INTRODUCTION

The field of values of an $n\times n$ complex matrix A is defined [7] as the set

$$\mathfrak{F}_A=\{z^*Az\,|\,z\in\mathbb{C}^n,\ z^*z=1\},$$

where z^* is the complex conjugate transpose of z. It is known [4, 10] that $\mathfrak{F}_A$ is a convex set which contains the convex hull $\mathfrak{L}_A$ of the eigenvalues of A, and that $\mathfrak{F}_A=\mathfrak{L}_A$ if A is normal (i.e., $AA^*=A^*A$). If A is not normal, $\mathfrak{L}_A$ will usually be a strict subset of $\mathfrak{F}_A$ [7].

Although the problem of locating the field of values is interesting in its own right and has been widely studied [1, 4, 5, 6, 7, 10, 11], it is worth giving

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some motivation for our results, as in one case we shall assume that the eigenvalues of A are known, whereas the most common reason for studying $\mathcal{F}_A$ is to find a set that contains the eigenvalues. Our motivation comes from stability theory, but our results are, of course, applicable in general.

In stability theory for differential equations it is often necessary to be able to show that for some matrix A, $\mathcal{F}_A$ lies in a given set: typically, a half plane [8, 9]. For example, one criterion [3, 9] requires that for a given matrix function $A:\mathbb{R}_+ \to \mathbb{C}^{n\times n}$, the inequality

$$\inf_{\omega} \inf_{z^*z=1} z^*\{(1+i\alpha)A(\omega)+(1-i\alpha)A^*(\omega)\}z \geqslant 0$$

is satisfied for some $\alpha \geqslant 0$. Although this looks like a simple matter of checking, at each ω, the minimum eigenvalue of the Hermitian matrix in braces, the problem lies in finding α. Rearrangement of the above expression makes it apparent that an equivalent condition is that $\cup_\omega \mathcal{F}_{A(\omega)}$ is contained in a half plane

$$\{q \in \mathbb{C} | \operatorname{Re} q \geqslant \alpha \operatorname{Im} q\},$$

i.e., that for every ω, $\mathcal{F}_{A(\omega)}$ lies to the right of some line passing through the origin and having positive or infinite slope (the same slope for each ω). Thus the question of the existence of α can be answered *a posteriori*.

Other stability criteria [2, 8] require some field of values to lie in a different half plane or inside a circle. The present paper gives a number of results which make it easy to check whether $\mathcal{F}$ lies in any given convex set; they make use of information which is already needed for the above stability criteria, such as the values of the diagonal elements of $A(\omega)$, or perhaps its eigenvalues. The idea is to provide quick estimates of $\mathcal{F}_{A(\omega)}$, since the fact that one has to work with a range of ω values usually makes it impractical to get an accurate numerical estimate of $\mathcal{F}_{A(\omega)}$ at each ω [1].

The first quick estimate one would probably try is to use a theorem [7] closely related to one by Hirsch [7, 12]. This states that for any complex matrix A, $\mathcal{F}_A$ is contained in the rectangle

$$\mathcal{K}_A = \{q \in \mathbb{C} | \rho_1 \leqslant \operatorname{Re} q \leqslant \rho_n, \sigma_1 \leqslant \operatorname{Im} q \leqslant \sigma_n\},$$

where $A = R + iS$ and R and S are Hermitian; ρ_1, ρ_n, σ_1 and σ_n are respectively the smallest and largest eigenvalues of R and the smallest and largest eigenvalues of S.

Unfortunately this estimate is usually too coarse for the kind of problems we have been describing. The present paper gives a number of results which work in a similar way to Hirsch's but may provide tighter estimates. The results divide into two groups, and within each group the differences are concerned with whether one wants to do a fair amount of work to get a good estimate or a little work to get a poorer one. The first group concerns itself with estimates using the convex hull of the diagonal elements of A and is dealt with in Sec. 2; these results are related to those of Johnson [5, 6]. Such estimates are important in their own right in control theory, but our main interest here is that the second group of results in Sec. 3 can now be obtained by performing a unitary transformation. If A is normal, then this transformation diagonalizes A, and the estimate for $\mathcal{F}_A$ becomes the set $\mathcal{L}_A$, which is in fact equal to $\mathcal{F}_A$; if A is not normal, then we obtain a convex set containing $\mathcal{L}_A$, and the departure of A from normality determines how large this set becomes.

2. ESTIMATES IN TERMS OF THE DIAGONAL ELEMENTS OF A

Throughout this paper, A is an $n\times n$ complex matrix with elements a_{uv}, and R and S are $n\times n$ Hermitian matrices such that $A=R+iS$. Thus R and S have elements

$$r_{uv}=\tfrac{1}{2}\left(a_{uv}+\bar{a}_{vu}\right)$$

and

$$s_{uv}=\frac{1}{2i}\left(a_{uv}-\bar{a}_{vu}\right).$$

We write A^0, R^0 and S^0 for the matrices A, R and S with the diagonal elements removed; for example,

$$r^0_{uv}=\begin{cases} r_{uv} & (u\neq v)\\ 0 & (u=v).\end{cases}$$

We define $\mathcal{F}_A$ and $\mathcal{L}_A$ as in Sec. 1, and $\mathcal{D}_A$ to be the convex hull of the diagonal elements of A.

THEOREM 1. *Let ρ^0_1, ρ^0_n, σ^0_1 and σ^0_n be respectively the minimum and maximum eigenvalues of R^0 and the minimum and maximum eigenvalues of S^0. Then $\mathcal{F}_A$ lies in the convex hull $\mathcal{R}_A$ of the n rectangles $\mathcal{R}_u$ $(u=1,\ldots,n)$,* *where*

$$\mathcal{R}_u = \{ q \in \mathbb{C} | \rho_1^0 \leqslant \mathrm{Re}(q - a_{uu}) \leqslant \rho_n^0,\ \sigma_1^0 \leqslant \mathrm{Im}(q - a_{uu}) \leqslant \sigma_n^0 \}.$$

REMARKS.

(1) Since R^0 is Hermitian, ρ_1^0 and ρ_n^0 are real; and since the trace of R^0 is zero, $\rho_1^0 < 0 < \rho_n^0$ unless $R^0 \equiv 0$; similarly for σ_1^0 and σ_n^0.

(2) Notice that if it is required that $\mathcal{F}_A$ should lie in some convex set, it is clearly sufficient that each of the $\mathcal{R}_u$ should do so.

Proof. Let

$$e = z^* A z - \sum_{u=1}^{n} |z_u|^2 a_{uu}.$$

Then

$$\begin{aligned} \mathrm{Re}\, e &= \mathrm{Re} \sum_{u=1}^{n} \sum_{v \neq u} \bar{z}_u a_{uv} z_v \\ &= \sum_{u,v=1}^{n} \bar{z}_u r_{uv}^0 z_v \\ &= z^* R^0 z. \end{aligned}$$

It is trivial to prove that $z^* R^0 z$ lies in $[\rho_1^0, \rho_n^0]$ if $z^* z = 1$, so

$$\rho_1^0 \leqslant \mathrm{Re}\, e \leqslant \rho_n^0,$$

and similarly $\sigma_1^0 \leqslant \mathrm{Im}\, e \leqslant \sigma_n^0$.

Now $\sum_{u=1}^{n} |z_u|^2 a_{uu}$ is a particular point in the convex hull $\mathcal{D}_A$ of the a_{uu}, so we have shown that $z^* A z$ is displaced by e from some point in $\mathcal{D}_A$, where $\mathrm{Re}\, e$ and $\mathrm{Im}\, e$ are bounded as above. Thus $z^* A z$ must lie in the convex hull $\mathcal{R}_A$ of the $\mathcal{R}_u$ as defined above, since every point outside this set is displaced by f from the nearest point in $\mathcal{D}_A$, where $\mathrm{Re} f$ or $\mathrm{Im} f$ lies outside the above bounds.

This is true for all z with $z^* z = 1$, proving the theorem. ■

COROLLARY. *$\mathcal{F}_A$ is contained in $\mathcal{R}_A'$, the convex hull of the n rectangles $\mathcal{R}_u'$ $(u=1,\dots,n)$ defined by*

$$\mathcal{R}_u' = \{ q \in \mathbb{C} \,|\, |\mathrm{Re}(q-a_{uu})| \leqslant \rho',\ |\mathrm{Im}(q-a_{uu})| \leqslant \sigma' \},$$

where

$$\rho' = \max_u \sum_v |r_{uv}^0|$$

and

$$\sigma' = \max_u \sum_v |s_{uv}^0|.$$

Proof. By Gershgorin's theorem,

$$|\rho_1^0| \leqslant \rho' \quad \text{and} \quad |\rho_n^0| \leqslant \rho',$$

which gives the result directly from Theorem 1. ∎

REMARKS.

(1) If $n=2$, the corollary gives the same result as the theorem, because R^0 is of the form

$$\begin{pmatrix} 0 & r_{12} \\ \bar{r}_{12} & 0 \end{pmatrix}.$$

If $n>2$ the corollary will usually give weaker results, but of course it gives them with less effort.

(2) Johnson [5, 6] shows that $\mathcal{F}_A$ is contained in the convex hull of the n discs $\{ q \in \mathbb{C} \,|\, |q-a_{uu}| \leqslant g_u \}$, where $g_u = \frac{1}{2}\sum_{v \neq u}(|a_{uv}|+|a_{vu}|)$. It is easy to find examples where this set is either larger or smaller than that given by the theorem or the corollary. Usually our set contains points not in Johnson's, and vice versa.

3. ESTIMATES IN TERMS OF THE EIGENVALUES OF A

It is clear that we should be able to improve on the results of Sec. 2 by transforming coordinates. For example, if we could diagonalize A, the ρ and

σ values would be zero. Unfortunately we cannot usually diagonalize A by using a unitary transformation, but since we have to preserve the condition $z^*z=1$ in a reasonably simple form, a unitary transformation is the most desirable one.

For any matrix A, there is a unitary transformation U that will reduce A to upper triangular form T: U can be obtained by the Schmidt orthonormalization process applied to the generalized eigenvectors of A [13]. The eigenvalues of A appear as the diagonal elements of T, and the other nonzero elements of T may be regarded as a measure of the departure of A from normality, since for a normal matrix T will be diagonal. We can now use the results of Sec. 2 with T in place of A.

To simplify the notation, let us write $\tilde{A}$ for T, i.e.,

$$\tilde{A}=U^*AU,$$

where U is unitary and $\tilde{A}$ is upper triangular. Similarly we write $\tilde{R}$, $\tilde{S}^0$, $\tilde{\rho}_n^0$ and so on, with obvious meanings.

THEOREM 2. *$\mathcal{F}_A$ is contained in the convex hull $\tilde{\mathcal{R}}_A$ of the n rectangles $\tilde{R}_u$ $(u=1,\ldots,n)$, where*

$$\tilde{R}_u=\left\{q\in\mathbb{C}\,|\,\tilde{\rho}_1^0\leqslant \operatorname{Re}(q-\lambda_u)\leqslant\tilde{\rho}_n^0,\ \tilde{\sigma}_1^0\leqslant \operatorname{Im}(q-\lambda_u)\leqslant\tilde{\sigma}_n^0\right\}$$

and λ_u $(u=1,\ldots,n)$ are the eigenvalues of A.

Proof. This follows at once by applying Theorem 1 to $\tilde{A}$ and noting that $\tilde{\mathcal{R}}_A=\mathcal{R}_{\tilde{A}}$.

COROLLARY. *$\mathcal{F}_A$ is contained in $\tilde{\mathcal{R}}'_A$, the convex hull of the n rectangles $\tilde{\mathcal{R}}'_u$ $(u=1,\ldots,n)$ defined by*

$$\tilde{\mathcal{R}}'_u=\left\{q\in\mathbb{C}\,\big|\,|\operatorname{Re}(q-\lambda_u)|\leqslant\tilde{\rho}',\ |\operatorname{Im}(q-\lambda_u)|\leqslant\tilde{\sigma}'\right\},$$

where

$$\tilde{\rho}'=\max_u\sum_v|\tilde{r}_{uv}^0|$$

and

$$\tilde{\sigma}'=\max_u\sum_v|\tilde{s}_{uv}^0|.$$

REMARKS.

(1) If one uses the QR algorithm [13, 14] to obtain the eigenvalues of A, the matrix $\tilde{A}$ is the end result and there is no need to go through the large additional amount of computation involved in finding the eigenvectors, then U, then $\tilde{A}$.

(2) Note that $\mathcal{R}_A = \mathcal{L}_A$ if A is normal, so in this case the theorem gives the optimal result.

(3) Moyls and Marcus [10] have shown that if λ_u is on the boundary of $\mathcal{F}_A$, every element in row u and column u of T vanishes. This means that if A is not normal, yet has an eigenvalue λ_u on $\partial\mathcal{F}_A$, the presence of the zero elements will tend to reduce the size of the rectangles, and hence the difference between $\mathcal{R}_A$ and $\mathcal{F}_A$. However, it is clear that the approximation will be less than perfect, because λ_u will not lie in $\partial\mathcal{R}_A$.

It is reasonable to conjecture that $\tilde{\mathcal{R}}_A \subseteq \mathcal{R}_A \subseteq \mathcal{H}_A$, but unfortunately this is false. In fact, none of these sets need be a subset of any other, and in particular those points of the $\mathcal{R}$ sets with greatest real, greatest imaginary, least real and least imaginary coordinates all lie outside $\mathcal{H}_A$ or on its boundary. For

$$\rho_n = \max_{z^*z=1} z^*Rz$$

$$= \max_{z^*z=1} (z^*Dz + z^*R^0z),$$

where $D = R - R^0$. Thus

$$\rho_n \leqslant \max_{z^*z=1} z^*Dz + \max_{z^*z=1} z^*R^0z$$

$$= r_{uu} + \rho_n^0,$$

where r_{uu} is the maximum element of D. This (together with the obvious three other cases) proves that $\mathcal{R}_A$ cannot be contained in $\mathcal{H}_A$, and replacing $\mathcal{R}_A$ by $\tilde{\mathcal{R}}_A$ shows that $\tilde{\mathcal{R}}_A$ cannot be contained in $\mathcal{H}_A$.

However, we can easily write down normal matrices for which $\tilde{\mathcal{R}}_A$ is a strict subset of $\mathcal{H}_A$ and of $\mathcal{R}_A$, so there can be no nesting of the sets in general. (The only remaining possibility would be that $\tilde{\mathcal{R}}_A \subset \mathcal{R}_A$, but this need not be true: consider, for example,

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

where $\mathcal{H}_A = \mathcal{R}_A \subset \tilde{\mathcal{R}}_A$.) We can always take the intersection of various estimates, but this will usually be inconvenient.

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