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A geometrical approach on generalized inverses by Neumann-type series

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Abstract

The convergence of the Neumann-type series to $\{1, 2\}$-inverses has been shown by K. Tanabe [Linear Algebra Appl. 10 (1975) 163]. In this paper, these results indicating conditions characterizing the convergence of this series to different generalized inverses are extended. In addition, these results for obtaining different generalized inverses from the hyperpower method are applied. Finally, generalized involutory matrices are introduced and characterized using the obtained results. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction and notation

It is well known that the inverse A^{-1} of a nonsingular matrix has a series expansion of the type

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$$A^{-1} = \sum_{i=0}^{\infty} (I - XA)^i X, \tag{1}$$

which is satisfied when $\rho(I - XA) < 1$, where $\rho(M)$ is the spectral radius of the square matrix M, that is the maximum moduli of its eigenvalues. It is also well known [2] that expansion (1) is valid for a rectangular or a singular matrix A, in the case that A^{-1} is replaced by the Moore–Penrose inverse A^+. Furthermore, a similar expression for the {1,2}-inverses was established by Tanabe [8] and by Bru and Thome [5] for the group inverse. It is very easy to see that the series

$$X + (I - XA)X + (I - XA)^2 X + \cdots \tag{2}$$

and the series

$$X + X(I - AX) + X(I - AX)^2 + \cdots$$

coincide; then it is possible to use them indistinctly. However, we will normally use the first one.

In many papers, the hyperpower iterative method has been studied. It was introduced by Ben-Israel [1] for $p = 2$. Petryshyn [7] and Zlobec [11] extended it for $p \geqslant 2$. Garnet et al. [6] applied it for computing the product A^+B, where moreover the optimum order p was found in the sense of minimizing the computational effort. In 1975, Tanabe [8] used this iterative method for generating reflexive generalized inverses. Chen and Hartwig [9] generalized the hyperpower method in 1996 by inserting an idempotent matrix into the residual of the Neumann-type expansion $I - AX$. Sufficient conditions for the convergence of the Neumann-type series to the generalized inverse $A^{(2)}_{T,S}$ has been given in terms of the V-norm by Wei [10] in 1998.

We will use the mentioned hyperpower method to give conditions that characterize the convergence of the method to different generalized inverses as $A^{\#}$, A^{D}, A^+ and $A^{(2)}_{T,S}$. Various known results are obtained as corollaries.

Given a complex matrix A, we denote by A^* the conjugate transpose of A, and by $\mathrm{Ker}(A)$ and $\mathrm{Im}(A)$, the kernel and range of A, respectively. Furthermore, the index of a square matrix A, denoted by ind(A), is the smallest nonnegative integer such that $\mathrm{Im}(A^k) = \mathrm{Im}(A^{k+1})$. The following conditions (see [4])

(gi.1) $AXA = A$,
(gi.2) $XAX = X$,
(gi.3) $AX = (AX)^*$,
(gi.4) $XA = (XA)^*$,
(gi.5) $AX = XA$,
(gi.6) $A^{k+1} = XA^k$, where $k = \mathrm{ind}(A)$,

define different generalized inverse matrices of a nonzero matrix A. In fact, if Γ is a subset of $\{1, 2, 3, 4, 5, 6\}$, then a complex matrix X is called a Γ-inverse of A if X satisfies conditions (gi.n), for each $n \in \Gamma$.

It is well known that the Moore–Penrose inverse is the unique $\{1, 2, 3, 4\}$-inverse of A which we will denote by A^+. Also, there exists a unique $\{2, 5, 6\}$-inverse of

A called the Drazin inverse denoted by A^{D}. If there exists a $\{1, 2, 5\}$-inverse of A, then it is unique, denoted by $A^{\#}$ and called the group inverse of A. Furthermore, there always exist $\{1\}$- and $\{1, 2\}$-inverses and they are not unique. Another kind of generalized inverse is the $A^{(2)}_{T,S}$ which, when it exists is unique, is defined in the following lemma.

Lemma 1 (Theorem 2.12 of [3]). *Let $A \in \mathbb{C}^{m\times n}$ be of the* rank r, *let T be a subspace of $\mathbb{C}^n$ of dimension $s \leqslant r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Then A has a $\{2\}$-inverse X such that* $\mathrm{Im}(X) = T$ *and* $\mathrm{Ker}(X) = S$ *if and only if $AT \oplus S = \mathbb{C}^n$ in which case X is unique and denoted by $A^{(2)}_{T,S}$.*

We recall the following properties for further references (see [3]).

Lemma 2. *Let $A \in \mathbb{C}^{m\times n}$ and let $P_{R,S}$ be a projector on R along S. Then*
(i) $\mathrm{Ker}(A) = [\mathrm{Im}(A^*)]^{\perp}$, $\mathrm{Ker}(A^*) = [\mathrm{Im}(A)]^{\perp}$;
(ii) *for L, M subspaces of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, $P^*_{L,M} = P_{M^{\perp},L^{\perp}}$ holds.*

The paper is organized as follows. First, in Section 2, we give the conditions to characterize the convergence of the Neumann-type series to the Moore–Penrose, the Drazin inverse and to the generalized inverse $A^{(2)}_{T,S}$. And then, in Section 3, we apply the above results. More exactly, using the hyperpower method we present an iterative scheme for obtaining the mentioned generalized inverses; and furthermore, we introduce and characterize some generalized involutory matrices.

2. Generalized inverses and Neumann-type series

In this section we establish some representations of the group inverse, the Moore–Penrose inverse, the Drazin inverse and the generalized inverse $A^{(2)}_{T,S}$ of a matrix A.

We recall that a square matrix R is convergent if there exists $\lim_{i\to\infty} R^i$ (see, for example [8]).

Throughout the paper we will consider two complex matrices (of appropriate size) A and X satisfying some of the following conditions:

(a) $I - XA$ is a convergent matrix,
(b) $\mathrm{Im}(XA) = \mathrm{Im}(X)$,
(c) $\mathrm{Ker}(XA) = \mathrm{Ker}(A)$,
(d) $AX = XA$,
(e) $\mathrm{Im}(AX) = [\mathrm{Ker}(X)]^{\perp}$,
(f) $\mathrm{Ker}(XA) = [\mathrm{Im}(X)]^{\perp}$,
(g) $A^{k+1}X = A^k$, where $k = \mathrm{ind}(A)$.

Next, we quote some known results for further references.

Theorem 1 (Theorem 2.2 of [8]). *Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$. The convergence of series* (2) *is equivalent to conditions* (a) *and* (b). *Furthermore, if G is the limit of the convergent series* (2), *then G is a* $\{2\}$*-inverse of A.*

Theorem 2 (Theorem 2.5 of [8]). *Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$. Series* (2) *converges to a* $\{1\}$*-inverse of A if and only if conditions* (a)–(c) *hold. In this case, the limit G of series* (2) *is a* $\{1, 2\}$*-inverse of A.*

In the result below, the convergence of a Neumann-type series to the group inverse was studied. More precisely, the conditions under which the group inverse of A can be expressed by means of series (2) for some matrix X were established.

Theorem 3 (Theorem 6 of [5]). *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ matrices satisfying the four conditions* (a)–(d). *Then series* (2) *converges to the group inverse of A.*

Now, necessary and sufficient geometrical conditions are given under which the Neumann-type expansion converges to the Moore–Penrose inverse. Analogously, although under more restrictive conditions we also give conditions for the convergence of this series to the Drazin inverse. We also derive a similar result for the generalized inverse $A^{(2)}_{T,S}$.

In the following theorem we find all matrices X such that the Neumann-type series converges to the Moore–Penrose inverse.

Theorem 4. *Let $A \in \mathbb{C}^{m\times n}$. The matrix $X \in \mathbb{C}^{n\times m}$ satisfies conditions* (a)–(c), (e) *and* (f) *if and only if series* (2) *converges to the Moore–Penrose inverse of A.*

Proof. By Theorem 2, conditions (a)–(c) are equivalent to the convergence of series (2) to a matrix G which is a $\{1,2\}$-inverse of A. It is necessary to be shown that conditions (e) and (f) are, respectively, equivalent to the equalities $(AG)^* = AG$ and $(GA)^* = GA$. In fact, by using that $AG = P_{\mathrm{Im}(AX),\mathrm{Ker}(X)}$ (see [8, Corollary 2.4]), together with condition (d) and Lemma 2 we have that

$$(AG)^* = [P_{\mathrm{Im}(AX),\mathrm{Ker}(X)}]^* = P_{[\mathrm{Ker}(X)]^\perp,[\mathrm{Im}(AX)]^\perp} = P_{\mathrm{Im}(AX),\mathrm{Ker}(X)} = AG.$$

In a similar way, it is possible to see that $(GA)^* = GA$, by using $GA = P_{\mathrm{Im}(X),\mathrm{Ker}(XA)}$, condition (e) and Lemma 2. Thus, G is the Moore–Penrose inverse of matrix A. Conversely, since $(AG)^* = AG$ we have

$$P_{[\mathrm{Ker}(X)]^\perp,[\mathrm{Im}(AX)]^\perp} = P_{\mathrm{Im}(AX),\mathrm{Ker}(X)}.$$

Then $[\mathrm{Ker}(X)]^\perp = \mathrm{Im}(AX)$. Furthermore, from $(GA)^* = GA$ we can check that condition $\mathrm{Ker}(XA) = [\mathrm{Im}(X)]^\perp$ is satisfied, for which we use that

$$P_{[\mathrm{Ker}(XA)]^\perp,[\mathrm{Im}(X)]^\perp} = P_{\mathrm{Im}(X),\mathrm{Ker}(XA)}.$$

This concludes the proof. □

Corollary 1 (Theorem 16 of [2]). *Let $A \in \mathbb{C}^{m\times n}$. Then equality*

$$A^{+} = \sum_{i=0}^{\infty}(I - A^{*}A)^{i}A^{*}$$

holds if $\rho(I - A^{*}A) < 1$.

Proof. Taking $X = A^{*}$, using Lemma 2(i) and the chain of equalities

$$\operatorname{rank}(AA^{*}) = \operatorname{rank}(A^{*}) = \operatorname{rank}(A) = \operatorname{rank}(A^{*}A),$$

the properties (a)–(c), (e) and (f) required in Theorem 4 can be shown. Then, the proof is fulfilled. $\square$

We now present a similar result related to the Drazin inverse by using the Neumann-type series (2).

Theorem 5. *Let $A \in \mathbb{C}^{n\times n}$. If matrix $X \in \mathbb{C}^{n\times n}$ satisfies conditions* (a), (b), (d) *and* (g), *then series* (2) *converges to the Drazin inverse of A.*

Proof. By Theorem 1, conditions (a) and (b) are equivalent to the convergence of series (2) to a matrix G which satisfies the equality $GAG = G$. By using the Newton binomial we can prove that condition (d) implies $AG = GA$, with the same technique employed to prove Theorem 3. We only need to prove that $A^{k+1}G = A^{k}$ holds if condition (g) is fulfilled. To do this, it is sufficient to show that

$$A^{k+1}(I - XA)^{n}X = O \quad \text{for all } n \geqslant 1 \tag{3}$$

because

$$\begin{aligned} A^{k+1}G &= A^{k+1}\left[\lim_{j\to\infty}\sum_{n=0}^{j}(I - XA)^{n}X\right] \\ &= A^{k} + \lim_{j\to\infty}\sum_{n=1}^{j}A^{k+1}(I - XA)^{n}X. \end{aligned}$$

Applying the Newton binomial formula, expression (3) can be easily obtained. $\square$

Note that Theorem 5 implies Theorem 3 because if $\operatorname{ind}(A) = 1$, the condition $\operatorname{Ker}(XA) = \operatorname{Ker}(A)$ is deduced from $A^{2}X = A$ and $AX = XA$. In fact, if $z \in \operatorname{Ker}(XA)$, commuting and premultiplying by A we have $XAz = 0$, that is $AXz = 0$ and then $A^{2}Xz = 0$. Therefore, $z \in \operatorname{Ker}(A)$, that is $\operatorname{Ker}(XA) \subseteq \operatorname{Ker}(A)$. The other inclusion can be easily shown.

We finish this section with the following general result.

Theorem 6. *Let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Let $A \in \mathbb{C}^{m\times n}$ and consider a matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$\mathrm{Im}(X) = T \quad \textit{and} \quad \mathrm{Ker}(X) = S.$$

Then, series (2) *converges to the matrix $A^{(2)}_{T,S}$ if and only if conditions* (a), (b) *and* (c) *hold.*

Proof. From Corollary 2.4 of Tanabe [8] we see that

$$GAG = G, \quad GA = P_{\mathrm{Im}(X),\mathrm{Ker}(XA)} \quad \text{and} \quad AG = P_{\mathrm{Im}(AX),\mathrm{Ker}(X)}.$$

Then, we have

$$AG = P_{\mathrm{Im}(AX),\mathrm{Ker}(X)} = P_{A\mathrm{Im}(X),\mathrm{Ker}(X)} = P_{AT,S}$$

and

$$\begin{aligned} GA &= P_{\mathrm{Im}(X),\mathrm{Ker}(XA)} \\ &= P_{\mathrm{Im}(X),[\mathrm{Im}(XA)^*]^\perp} = P_{\mathrm{Im}(X),[A^*(\mathrm{Ker}(X))^\perp]^\perp} = P_{T,(A^*S^\perp)^\perp}. \end{aligned}$$

So, by Exercise 2.5.31 of Ben-Israel and Greville [3] we conclude that $G = A^{(2)}_{T,S}$. □

As a consequence of Theorem 6 we can obtain several well-known results as corollaries. We summarize all these results in Table 1.

3. Applications

The aim of this section is to apply the above results to derive an iterative scheme to compute the generalized inverses $A^\#$, A^{D}, A^+ and $A^{(2)}_{T,S}$. First, we describe briefly the pth order hyperpower method.

Let $A \in \mathbb{C}^{n\times n}$ and let $X_0 \in \mathbb{C}^{n\times n}$ be an arbitrary initial matrix. The pth order hyperpower method

$$X_{i+1} = X_i + (I - X_iA)X_i + \cdots + (I - X_iA)^{p-1}X_i$$

Table 1

T	S	$A^{(2)}_{T,S}$	Result
$\mathrm{Im}(A^*)$	$\mathrm{Ker}(A^*)$	A^+	[8, Theorem 2.5]
$\mathrm{Im}(N^{-1}A^*M)$	$\mathrm{Ker}(N^{-1}A^*M)$	A^+_{MN}	Note before Corollary 2.6 in [8]
$\mathrm{Im}(A)$	$\mathrm{Ker}(A)$	$A^\#$	[5, Theorem 6.1]
$\mathrm{Im}(A^k)$	$\mathrm{Ker}(A^k)$	A^{D}	Theorem 5

generates the sequence $\{X_i\}_{i=0}^{\infty}$. Our purpose is to use it for obtaining different generalized inverses.

Theorem 7 (Lemma 3.1 of [8]). *The* pth *order hyperpower method generates the partial sum of series* (2) *by means of*

$$X_l = \sum_{i=0}^{p^l-1} (I - XA)^i X,$$

where p^l *indicates the* lth *power of* p.

Theorem 3.2 of Tanabe [8] shows that assumptions (a)–(c) guarantee the convergence of series (2) to a one $\{1, 2\}$-inverse of A. The following theorem states similar results for other kinds of generalized inverses.

Theorem 8. *Let* $A \in \mathbb{C}^{n\times n}$. *Then the* pth *order hyperpower method generates a convergent sequence of matrices* $\{X_i\}_{i=0}^{\infty}$ *that converge to*

(i) *the group inverse of* A, *if conditions* (a)–(d) *hold*;
(ii) *the Moore–Penrose inverse of* A, *if conditions* (a)–(c), (e) *and* (f) *hold*;
(iii) *the Drazin inverse of* A, *if conditions* (a), (b), (d) *and* (g) *hold*;
(iv) *the generalized inverse* $A_{T,S}^{(2)}$ *of* A, *if conditions* (a)–(c) *hold.*

Proof. The proof of this theorem holds from Theorems 3, 4, 5 and 6, respectively; and using Theorem 3.2 of Tanabe [8]. □

Remark 1. We can observe that Theorem 3.1 of Wei [10] related to the hyperpower method can be obtained as a corollary of Theorem 6.

Another application of the results of Section 2 will be presented below. First, we introduce the generalized involutory matrices in a similar way to how the group involutory matrix was introduced by Bru and Thome [5].

Definition 1. The matrix $A \in \mathbb{C}^{n\times n}$ is called

- Moore–Penrose involutory if $A^{+} = A$,
- Drazin involutory if $A^{\mathrm{D}} = A$,
- $A_{T,S}^{(2)}$ involutory if $A_{T,S}^{(2)} = A$.

Now, as an immediate consequence of Theorems 4, 5 and 6 we have the following result.

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$. Then A is Moore-Penrose, Drazin or $A^{(2)}_{T,S}$ involutory matrix if and only if the* pth *order hyperpower method generates a sequence of matrices $\{X_i\}_{i=0}^{\infty}$ that converges to A when conditions*:

(i) (a)–(c), (e) *and* (f) *hold*,

(ii) (a), (b), (d) and (g) *hold*, *or*

(iii) (a)–(c) *hold*,

respectively.

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