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Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$ ☆

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Abstract

In this paper, we deal with the existence of solutions to the nonuniformly elliptic equation of the form

$$-\mathrm{div}\big(a(x,\nabla u)\big)+V(x)|u|^{N-2}u=\frac{f(x,u)}{|x|^{\beta}}+\varepsilon h(x) \tag{0.1}$$

in $\mathbb{R}^N$ when $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ behaves like $\exp(\alpha|u|^{N/(N-1)})$ when $|u|\to\infty$ and satisfies the Ambrosetti–Rabinowitz condition. In particular, in the case of N-Laplacian, i.e., $a(x,\nabla u)=|\nabla u|^{N-2}\nabla u$, we obtain multiplicity of weak solutions of (0.1). Moreover, we can get the nontriviality of the solution in this case when $\varepsilon=0$. Finally, we show that the main results remain true if one replaces the Ambrosetti–Rabinowitz condition on the nonlinearity by weaker assumptions and thus we establish the existence and multiplicity results for a wider class of nonlinearity, see Section 7 for more details.

Keywords: Ekeland variational principle; Mountain-pass theorem; Variational methods; Critical growth; Moser–Trudinger inequality; N-Laplacian; Ambrosetti–Rabinowitz condition

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1. Introduction

In this paper, we consider the existence and multiplicity of nontrivial weak solution $u \in W^{1,N}(\mathbb{R}^N)$ ($u \geqslant 0$) for the nonuniformly elliptic equations of N-Laplacian type of the form:

$$-\operatorname{div}\big(a(x,\nabla u)\big) + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^{\beta}} + \varepsilon h(x) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where, in addition to some more assumptions on $a(x,\tau)$ and f which will be specified later in Section 2, we have

$$\big|a(x,\tau)\big| \leqslant c_0\big(h_0(x) + h_1(x)|\tau|^{N-1}\big)$$

for any τ in $\mathbb{R}^N$ and a.e. x in $\mathbb{R}^N$, $h_0 \in L^{N/(N-1)}(\mathbb{R}^N)$ and $h_1 \in L^{\infty}_{loc}(\mathbb{R}^N)$ and f satisfies critical growth of exponential type such as $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp(\alpha|u|^{N/(N-1)})$ when $|u| \to \infty$ and when f either satisfies or does not satisfy the Ambrosetti–Rabinowitz condition.

A special case of our equation in the whole Euclidean space when $a(x,\nabla u) = |\nabla u|^{N-2}\nabla u$ has been studied extensively, both in the case $N = 2$ (the prototype equation is the Laplacian in $\mathbb{R}^2$) and in the case $N > 3$ in $\mathbb{R}^N$ for the N-Laplacian, see for example [10,2,3,26,18,14–16, 5], etc. We should mention that problems involving Laplacian in bounded domains in $\mathbb{R}^2$ with critical exponential growth have been studied in [4,18,8,7,11,29], etc. and for N-Laplacian in bounded domains in $\mathbb{R}^N$ ($N > 2$) by the authors of [2,14,26].

The problems of this type are important in many fields of sciences, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. They have been extensively studied by many authors in many different cases: bounded domains and unbounded domains, different behavior of the nonlinearity, different types of boundary conditions, etc. In particular, many works focus on the subcritical and critical growth of the nonlinearity which allows us to treat the problem variationally using general critical point theory.

In the case $p < N$, by the Sobolev embedding, the subcritical and critical growth mean that the nonlinearity cannot exceed the polynomial of degree $p^* = \frac{Np}{N-p}$. The case $p = N$ is special, since the corresponding Sobolev space $W_0^{1,N}(\Omega)$ is a borderline case for Sobolev embeddings: one has $W_0^{1,N}(\Omega) \subset L^q(\Omega)$ for all $q \geqslant 1$, but $W_0^{1,N}(\Omega) \not\subseteq L^{\infty}(\Omega)$. So, one is led to ask if there is another kind of *maximal growth* in this situation. Indeed, this is the result of Pohozaev [27], Trudinger [32] and Moser [25], and is by now called the Moser–Trudinger inequality: it says that if $\Omega \subset \mathbb{R}^N$ is a bounded domain, then

$$\sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_{L^N} \leqslant 1} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx < \infty$$

where $\alpha_N = N w_{N-1}^{\frac{1}{N-1}}$ and w_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$. Moreover, the constant α_N is sharp in the sense that if we replace α_N by some $\beta > \alpha_N$, the above supremum is infinite.

This well-known Moser–Trudinger inequality has been generalized in many ways. For instance, in the case of bounded domains, Adimurthi and Sandeep proved in [3] that the following inequality

$$\sup_{u\in W_0^{1,N}(\Omega),\|\nabla u\|_{L^N}\leqslant 1}\int_\Omega \frac{e^{\alpha_N|u|^{\frac{N}{N-1}}}}{|x|^\beta}\,dx<\infty$$

holds if and only if $\frac{\alpha}{\alpha_N}+\frac{\beta}{N}\leqslant 1$ where $\alpha>0$ and $0\leqslant\beta<N$.

On the other hand, in the case of unbounded domains, B. Ruf when $N=2$ in [28] and Y.X. Li and B. Ruf when $N>2$ in [23] proved that if we replace the L^N-norm of ∇u in the supremum by the standard Sobolev norm, then this supremum can still be finite under a certain condition for α. More precisely, they have proved the following:

$$\sup_{u\in W_0^{1,N}(\mathbb{R}^N),\|u\|_{L^N}^N+\|\nabla u\|_{L^N}^N\leqslant 1}\int_{\mathbb{R}^N}\big(\exp\big(\alpha|u|^{N/(N-1)}\big)-S_{N-2}(\alpha,u)\big)\,dx\begin{cases}\leqslant\infty & \text{if } \alpha\leqslant\alpha_N,\\ =+\infty & \text{if } \alpha>\alpha_N,\end{cases}$$

where

$$S_{N-2}(\alpha,u)=\sum_{k=0}^{N-2}\frac{\alpha^k|u|^{kN/(N-1)}}{k!}.$$

We should mention that for $\alpha<\alpha_N$ when $N=2$, the above inequality was first proved by D. Cao in [10], and proved for $N>2$ by Panda [26] and J.M. do Ó [14,15] and Adachi and Tanaka [1].

Recently, Adimurthi and Yang generalized the above result of Li and Ruf [23] to get the following version of the singular Trudinger–Moser inequality (see [5]):

Lemma 1.1. *For all $0\leqslant\beta<N$, $0<\alpha$ and $u\in W^{1,N}(\mathbb{R}^N)$, there holds*

$$\int_{\mathbb{R}^N}\frac{1}{|x|^\beta}\big\{\exp\big(\alpha|u|^{N/(N-1)}\big)-S_{N-2}(\alpha,u)\big\}<\infty.$$

Furthermore, we have for all $\alpha\leqslant(1-\frac{\beta}{N})\alpha_N$ and $\tau>0$,

$$\sup_{\|u\|_{1,\tau}\leqslant 1}\int_{\mathbb{R}^N}\frac{1}{|x|^\beta}\big\{\exp\big(\alpha|u|^{N/(N-1)}\big)-S_{N-2}(\alpha,u)\big\}<\infty$$

where $\|u\|_{1,\tau}=(\int_{\mathbb{R}^N}(|\nabla u|^N+\tau|u|^N)\,dx)^{1/N}$. The inequality is sharp: for any $\alpha>(1-\frac{\beta}{N})\alpha_N$, the supremum is infinity.

Motivated by this Trudinger–Moser inequality, do Ó [14,15] and do Ó, Medeiros and Severo [16] studied the quasilinear elliptic equations when $\beta=0$ and Adimurthi and Yang [5] studied the singular quasilinear elliptic equations for $0\leqslant\beta<N$, both with the maximal growth on the singular nonlinear term $\frac{f(x,u)}{|x|^\beta}$ which allows them to treat the equations variationally in a

subspace of $W^{1,N}(\mathbb{R}^N)$. More precisely, they can find a nontrivial weak solution of mountain-pass type to the equation with the perturbation

$$-\operatorname{div}\big(|\nabla u|^{N-2}\nabla u\big)+V(x)|u|^{N-2}u=\frac{f(x,u)}{|x|^\beta}+\varepsilon h(x).$$

Moreover, they proved that when the positive parameter ε is small enough, the above equation has a weak solution with negative energy. However, it was not proved in [5] if those solutions are different or not. We should also stress that they need a small nonzero perturbation $\varepsilon h(x)$ in their equation to get the nontriviality of the solutions.

In this paper, we will study further about the equation considered in the whole space [2,14–16,5]. More precisely, we consider the existence and multiplicity of nontrivial weak solution for the nonuniformly elliptic equations of N-Laplacian type of the form:

$$-\operatorname{div}\big(a(x,\nabla u)\big)+V(x)|u|^{N-2}u=\frac{f(x,u)}{|x|^\beta}+\varepsilon h(x) \tag{1.2}$$

where

$$\big|a(x,\tau)\big|\leqslant c_0\big(h_0(x)+h_1(x)|\tau|^{N-1}\big)$$

for any τ in $\mathbb{R}^N$ and a.e. x in $\mathbb{R}^N$, $h_0\in L^{N/(N-1)}(\mathbb{R}^N)$ and $h_1\in L^\infty_{loc}(\mathbb{R}^N)$. Note that the equation in [5] is a special case of our equation when $a(x,\nabla u)=|\nabla u|^{N-2}\nabla u$. In fact, the elliptic equations of nonuniform type is a natural generalization of the p-Laplacian equation and were studied by many authors, see [17,19,31,34,30]. As mentioned earlier, the main features of this class of equations are that they are defined in the whole $\mathbb{R}^N$ and with the critical growth of the singular nonlinear term $\frac{f(x,u)}{|x|^\beta}$ and the nonuniform nonlinear operator of p-Laplacian type. In spite of a possible failure of the Palais–Smale compactness condition, in this paper, we still use the mountain-pass approach for the critical growth as in [14,5,15,16] to derive a weak solution and get the nontriviality of this solution thanks to the small nonzero perturbation $\varepsilon h(x)$.

In the case of N-Laplacian, i.e.,

$$a(x,\nabla u)=|\nabla u|^{N-2}\nabla u,$$

our equation is exactly the equation studied in [5]:

$$-\operatorname{div}\big(|\nabla u|^{N-2}\nabla u\big)+V(x)|u|^{N-2}u=\frac{f(x,u)}{|x|^\beta}+\varepsilon h(x). \tag{1.3}$$

Using the Radial Lemma, Schwarz symmetrization and a modified result of Lions [24] about the singular Moser–Trudinger inequality, we will prove that two solutions derived in [5] are actually different. Thus as our second main result, we get the multiplicity of solutions to Eq. (1.3). Our existence result extends that in [5] to the nonuniformly elliptic type of equations. The multiplicity of nontrivial solutions was considered in [16] when $\beta=0$ using a rearrangement inequality which does not hold in general. We will give a substantially different proof here and establish the multiplicity in all the singular cases $0\leqslant\beta<N$. (See Remark 5.2 in Section 5 for more details.)

Our next concern is about the existence of solution of the equation without the perturbation

$$-\mathrm{div}\big(|\nabla u|^{N-2}\nabla u\big)+V(x)|u|^{N-2}u=\frac{f(x,u)}{|x|^{\beta}}. \tag{1.4}$$

Using an approach as in [14–16], we prove that we don't even require the nonzero perturbation as in [5] to get the nontriviality of the mountain-pass type weak solution.

Our main tool in this paper is critical point theory. More precisely, we will use the mountain-pass theorem that is proposed by Ambrosetti and Rabinowitz in the celebrated paper [6]. Critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. We stress that to use the mountain-pass theorem, we need to verify some types of compactness for the associated Lagrange–Euler functional, namely the Palais–Smale condition and the Cerami condition. Or at least, we must prove the boundedness of the Palais–Smale or Cerami sequence [12,13]. In almost all of works, we can easily establish this condition thanks to the Ambrosetti–Rabinowitz (AR) condition, see $(f2)$. However, there are many interesting examples of nonlinear terms f which do not satisfy the Ambrosetti–Rabinowitz condition. Thus our next result is that we will show the main results remain true when one replaces the (AR) condition by weaker assumptions (see Section 7). For the N-Laplacian equation or polyharmonic operators in a bounded domain in $\mathbb{R}^N$, such a result of existence has been established by the authors in [20] and [22].

We mention in passing that the study of the existence and multiplicity results of nonuniformly elliptic equations of N-Laplacian type are motivated by our earlier work on the Heisenberg group [21]. Our assumptions on the potential V are exactly those considered in [14–16,5], namely $V(x)\geqslant V_0>0$ in $\mathbb{R}^N$ and $V^{-1}\in L^1(\mathbb{R}^N)$. Very recently, Yang has established in [33] when $a(x,\nabla u)=|\nabla u|^{N-2}\nabla u$ the multiplicity of solutions when the nonlinear term f satisfies the Ambrosetti–Rabinowitz condition and the potential V is under a stronger assumption than ours. More precisely, it is assumed in [33] that $V^{-1}\in L^{\frac{1}{N-1}}(\mathbb{R}^N)$ which implies $V^{-1}\in L^1(\mathbb{R}^N)$ when $V(x)\geqslant V_0>0$ in $\mathbb{R}^N$. The stronger assumption of integrability on V^{-1} in [33] guarantees that the embedding $E\to L^q(\mathbb{R}^N)$ is compact for all $1\leqslant q<\infty$. The argument in [33], as pointed out by the author of [33], depends crucially on this compact embedding for all $1\leqslant q<\infty$. The assumption on the potential V in our paper only assures the compact embedding $E\to L^q(\mathbb{R}^N)$ for $q\geqslant N$. Nevertheless, this compact embedding for $q\geqslant N$ is sufficient for us to carry out the proof of the multiplicity of solutions to Eq. (1.3) and existence of solutions to Eq. (1.4) without the perturbation term. (See Proposition 5.2 and Remark 5.2 in Section 5 for more details.)

The paper is organized as follows: In the next section, we give the main assumptions which are used throughout this paper except the last section and our main results. In Section 3, we prove some preliminary results. Section 4 is devoted to study the existence of nontrivial solutions for the nonuniformly elliptic equations of N-Laplacian type (1.2). The multiplicity of nontrivial solutions to Eq. (1.3) is investigated in Section 5. Section 6 is about the existence of nontrivial solutions to the equation without the perturbation (1.4). Finally, in Section 7 we study the results in Sections 5 and 6 without the Ambrosetti–Rabinowitz (AR) condition.

2. Assumptions and main results

Motivated by the Trudinger–Moser inequality in Lemma 1.1, we consider here the maximal growth on the nonlinear term $f(x,u)$ which allows us to treat Eq. (1.2) variationally in a subspace of $W^{1,N}(\mathbb{R}^N)$. We assume that $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous, $f(x,0)=0$ and f behaves like

$\exp(\alpha|u|^{N/(N-1)})$ as $|u| \to \infty$. More precisely, we assume the following growth conditions on the nonlinearity $f(x,u)$ as in [14–16,5]:

$(f1)$ There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^+$,

$$0 < f(x,u) \leqslant b_1|u|^{N-1} + b_2\big[\exp\big(\alpha_0|u|^{N/(N-1)}\big) - S_{N-2}(\alpha_0,u)\big],$$

where

$$S_{N-2}(\alpha_0,u) = \sum_{k=0}^{N-2} \frac{\alpha_0^k}{k!}|u|^{kN/(N-1)}.$$

$(f2)$ There exists $p > N$ such that for all $x \in \mathbb{R}^N$ and $s > 0$,

$$0 < pF(x,s) = p\int_0^s f(x,\tau)\,d\tau \leqslant sf(x,s).$$

This is the well-known Ambrosetti–Rabinowitz condition.

$(f3)$ There exist constants $R_0, M_0 > 0$ such that for all $x \in \mathbb{R}^N$ and $s \geqslant R_0$,

$$F(x,s) \leqslant M_0 f(x,s).$$

Since we are interested in nonnegative weak solutions, it is convenient to define

$$f(x,u) = 0 \quad \text{for all } (x,u) \in \mathbb{R}^N \times (-\infty,0]. \tag{2.1}$$

Let A be a measurable function on $\mathbb{R}^N \times \mathbb{R}$ such that $A(x,0) = 0$ and $a(x,\tau) = \frac{\partial A(x,\tau)}{\partial \tau}$ is a Caratheodory function on $\mathbb{R}^N \times \mathbb{R}$. Assume that there are positive real numbers c_0, c_1, k_1 and two nonnegative measurable functions h_0, h_1 on $\mathbb{R}^N$ such that $h_1 \in L^\infty_{loc}(\mathbb{R}^N)$, $h_0 \in L^{N/(N-1)}(\mathbb{R}^N)$, $h_1(x) \geqslant 1$ for a.e. x in $\mathbb{R}^N$ and the following conditions hold:

$(A1)$ $|a(x,\tau)| \leqslant c_0(h_0(x) + h_1(x)|\tau|^{N-1})$, $\forall \tau \in \mathbb{R}^N$, a.e. $x \in \mathbb{R}^N$,
$(A2)$ $c_1|\tau - \tau_1|^N \leqslant \langle a(x,\tau) - a(x,\tau_1), \tau - \tau_1\rangle$ $\forall \tau, \tau_1 \in \mathbb{R}^N$, a.e. $x \in \mathbb{R}^N$,
$(A3)$ $0 \leqslant a(x,\tau).\tau \leqslant NA(x,\tau)$ $\forall \tau \in \mathbb{R}^N$, a.e. $x \in \mathbb{R}^N$,
$(A4)$ $A(x,\tau) \geqslant k_0 h_1(x)|\tau|^N$ $\forall \tau \in \mathbb{R}^N$, a.e. $x \in \mathbb{R}^N$.

Then A verifies the growth condition:

$$\big|A(x,\tau)\big| \leqslant c_0\big(h_0(x)|\tau| + h_1(x)|\tau|^N\big) \quad \forall \tau \in \mathbb{R}^N,\ \text{a.e. } x \in \mathbb{R}^N. \tag{2.2}$$

Next, we introduce some notations:

$$E = \left\{ u \in W_0^{1,N}\left(\mathbb{R}^N\right): \int_{\mathbb{R}^N} h_1(x)|\nabla u|^N \, dx + \int_{\mathbb{R}^N} V(x)|u|^N < \infty \right\},$$

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^N + \frac{1}{k_0 N} V(x)|u|^N \right) dx \right)^{1/N}, \quad u \in E,$$

$$\lambda_1(N) = \inf\left\{ \frac{\|u\|_E^N}{\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\beta} \, dx} : \ u \in E \setminus \{0\} \right\}.$$

We also assume the following conditions on the potential as in [14–16,5]:

$(V1)$ V is a continuous function such that $V(x) \geqslant V_0 > 0$ for all $x \in \mathbb{R}^N$, we can see that E is a reflexive Banach space when endowed with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \left(h_1(x)|\nabla u|^N + \frac{1}{k_0 N} V(x)|u|^N \right) dx \right)^{1/N}$$

and for all $N \leqslant q < \infty$,

$$E \hookrightarrow W^{1,N}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$$

with continuous embedding. Furthermore,

$$\lambda_1(N) = \inf\left\{ \frac{\|u\|_E^N}{\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\beta} \, dx} : \ u \in E \setminus \{0\} \right\} > 0 \quad \text{for any } 0 \leqslant \beta < N. \tag{2.3}$$

In order to get the compactness of the embedding

$$E \hookrightarrow L^p\left(\mathbb{R}^N\right) \quad \text{for all } p \geqslant N$$

we also assume the following conditions on the potential V:

$(V2)$ $V(x) \to \infty$ as $|x| \to \infty$; or more generally, for every $M > 0$,

$$\mu\left(\left\{x \in \mathbb{R}^N: \ V(x) \leqslant M\right\}\right) < \infty,$$

or

$(V3)$ The function $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^N)$.

Now, from $(f1)$, we obtain for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$,

$$\left|F(x,u)\right| \leqslant b_3\left[\exp\left(\alpha_1 |u|^{N/(N-1)}\right) - S_{N-2}(\alpha_1, u)\right]$$

for some constants α_1, $b_3 > 0$. Thus, by Lemma 1.1, we have $F(x,u) \in L^1(\mathbb{R}^N)$ for all $u \in W^{1,N}(\mathbb{R}^N)$. Define the functionals $J, J_\varepsilon : E \to \mathbb{R}$ by

$$J_\varepsilon(u) = \int_{\mathbb{R}^N} A(x,\nabla u)\,dx + \frac{1}{N}\int_{\mathbb{R}^N} V(x)|u|^N\,dx - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta}\,dx - \varepsilon\int_{\mathbb{R}^N} hu\,dx,$$
$$J(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N\,dx + \frac{1}{N}\int_{\mathbb{R}^N} V(x)|u|^N\,dx - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta}\,dx$$

then the functionals J, J_ε are well defined by Lemma 1.1. Moreover, J, J_ε are the C^1 functional on E and $\forall u, v \in E$,

$$DJ_\varepsilon(u)v = \int_{\mathbb{R}^N} a(x,\nabla u)\nabla v\,dx + \int_{\mathbb{R}^N} V(x)|u|^{N-2}v\,dx - \int_{\mathbb{R}^N} \frac{f(x,u)v}{|x|^\beta}\,dx - \varepsilon\int_{\mathbb{R}^N} hv\,dx,$$
$$DJ(u)v = \int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u\nabla v\,dx + \int_{\mathbb{R}^N} V(x)|u|^{N-2}v\,dx - \int_{\mathbb{R}^N} \frac{f(x,u)v}{|x|^\beta}\,dx.$$

Note that in the case of N-Laplacian: $A(x,\tau) = \frac{1}{N}|\tau|^N$, we choose

$$a(x,\tau) = |\tau|^{N-2}\tau, \qquad k_0 = \frac{1}{N}, \qquad h_1(x) = 1.$$

We next state our main results.

Theorem 2.1. *Suppose that* $(V1)$ *and* $(V2)$ (*or* $(V3)$) *and* $(f1)$–$(f2)$ *are satisfied. Furthermore, assume that*

$$(f4) \qquad \limsup_{s\to 0+} \frac{F(x,s)}{k_0|s|^N} < \lambda_1(N) \quad \textit{uniformly in } x \in \mathbb{R}^N.$$

Then there exists $\varepsilon_1 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_1$, *problem* (1.2) *has a nontrivial weak solution of mountain-pass type.*

Theorem 2.2. *Suppose that* $(V1)$ *and* $(V2)$ (*or* $(V3)$) *and* $(f1)$–$(f3)$ *are satisfied. Furthermore, assume that*

$$(f4) \qquad \limsup_{s\to 0+} \frac{NF(x,s)}{|s|^N} < \lambda_1(N) \quad \textit{uniformly in } x \in \mathbb{R}^N.$$

and there exists $r > 0$ *such that*

$$(f5) \qquad \lim_{s\to\infty} sf(x,s)\exp\bigl(-\alpha_0|s|^{N/(N-1)}\bigr)$$
$$> \frac{1}{[\frac{r^{N-\beta}}{N-\beta}e^{(\alpha_N d(N-\beta)/N)} + \mathcal{C}r^{N-\beta} - \frac{r^{N-\beta}}{N-\beta}]}\left(\frac{N-\beta}{\alpha_0}\right)^{N-1} > 0$$

uniformly on compact subsets of $\mathbb{R}^N$ *where* d *and* $\mathcal{C}$ *will be defined in Section* 3*. Then there exists* $\varepsilon_2 > 0$*, such that for each* $0 < \varepsilon < \varepsilon_2$*, problem* (1.3) *has at least two nontrivial weak solutions and one of them has a negative energy.*

Theorem 2.3. *Under the same hypotheses in Theorem* 2.2*, the problem without the perturbation* (1.4) *has a nontrivial weak solution.*

As we remarked earlier in the introduction, the main theorems above remain to hold when the nonlinear term f satisfies weaker assumptions than the Ambrosetti–Rabinowitz condition. As a result, we then establish the existence and multiplicity of solutions when the nonlinear term is in a wider class. See Section 7 for more details.

3. Preliminary results

First, we recall what we call the Radial Lemma (see [9,16]) which asserts:

$$\left|u(x)\right|^N \leqslant \frac{N}{\omega_{N-1}} \frac{\|u\|_N^N}{|x|^N}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}$$

for all $u \in W^{1,N}(\mathbb{R}^N)$ radially symmetric. Using this Radial Lemma, we can prove the following two lemmas with an easy adaptation from Lemma 2.2 and Lemma 2.3 in [16] for $\beta = 0$ and Lemma 4.2 in [5].

Lemma 3.1. *For* $\kappa > 0$*,* $0 \leqslant \beta < N$ *and* $\|u\|_E \leqslant M$ *with* M *sufficiently small and* $q > N$*, we have*

$$\int_{\mathbb{R}^N} \frac{[\exp(\kappa|u|^{N/(N-1)}) - S_{N-2}(\kappa, u)]|u|^q}{|x|^\beta}\, dx \leqslant C(N, \kappa)\|u\|_E^q.$$

Lemma 3.2. *Let* $\kappa > 0$*,* $0 \leqslant \beta < N$*,* $u \in E$ *and* $\|u\|_E \leqslant M$ *such that* $M^{N/(N-1)} < (1 - \frac{\beta}{N})\frac{\alpha_N}{\kappa}$*, then*

$$\int_{\mathbb{R}^N} \frac{[\exp(\kappa|u|^{N/(N-1)}) - S_{N-2}(\kappa, u)]|u|}{|x|^\beta}\, dx \leqslant C(N, M, \kappa)\|u\|_{p'}$$

for some $p' > N$*.*

Next, we have the following

Lemma 3.3. *Let* $\{w_k\} \subset W^{1,N}(\Omega)$ *where* Ω *is a bounded open set in* $\mathbb{R}^N$*,* $\|\nabla w_k\|_{L^N(\Omega)} \leqslant 1$*. If* $w_k \to w \neq 0$ *weakly and almost everywhere,* $\nabla w_k \to \nabla w$ *almost everywhere, then* $\frac{\exp\{\alpha|w_k|^{N/(N-1)}\}}{|x|^\beta}$ *is bounded in* $L^1(\Omega)$ *for* $0 < \alpha < (1 - \frac{\beta}{N})\alpha_N(1 - \|\nabla w\|_{L^N(\Omega)}^N)^{-1/(N-1)}$*.*

Proof. Using the Brezis–Lieb Lemma in [9], we deduce that

$$\|\nabla w_k\|_{L^N(\Omega)}^N - \|\nabla w_k - \nabla w\|_{L^N(\Omega)}^N \to \|\nabla w\|_{L^N(\Omega)}^N.$$

Thus for k large enough and $\delta > 0$ small enough:

$$0 < \alpha(1+\delta)\|\nabla w_k - \nabla w\|_{L^N(\Omega)}^{N/(N-1)} < \alpha_N\left(1 - \frac{\beta}{N}\right).$$

By the singular Trudinger–Moser inequality on bounded domains [3], we get the conclusion. □

In the next two lemmas we check that the functional J_ε satisfies the geometric conditions of the mountain-pass theorem. Then, we are going to use a mountain-pass theorem without a compactness condition such as the one of the (PS) type to prove the existence of the solution. This version of the mountain-pass theorem is a consequence of Ekeland's variational principle.

Lemma 3.4. *Suppose that $(V1)$, $(f1)$ and $(f4)$ hold. Then there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$, there exists $\rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $\|u\|_E = \rho_\varepsilon$. Furthermore, ρ_ε can be chosen such that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. From $(f4)$, there exist $\tau, \delta > 0$ such that $|u| \leqslant \delta$ implies

$$F(x,u) \leqslant k_0\big(\lambda_1(N) - \tau\big)|u|^N \tag{3.1}$$

for all $x \in \mathbb{R}^N$. Moreover, using $(f1)$ for each $q > N$, we can find a constant $C = C(q, \delta)$ such that

$$F(x,u) \leqslant C|u|^q\big[\exp\big(\kappa|u|^{N/(N-1)}\big) - S_{N-2}(\kappa, u)\big] \tag{3.2}$$

for $|u| \geqslant \delta$ and $x \in \mathbb{R}^N$. From (3.1) and (3.2) we have

$$F(x,u) \leqslant k_0\big(\lambda_1(N) - \tau\big)|u|^N + C|u|^q\big[\exp\big(\kappa|u|^{N/(N-1)}\big) - S_{N-2}(\kappa, u)\big]$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Now, by $(A4)$, Lemma 3.2, (2.3) and the continuous embedding $E \hookrightarrow L^N(\mathbb{R}^N)$, we obtain

$$\begin{aligned} J_\varepsilon(u) &\geqslant k_0\|u\|_E^N - k_0\big(\lambda_1(N) - \tau\big)\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\beta}\,dx - C\|u\|_E^q - \varepsilon\|h\|_*\|u\|_E \\ &\geqslant k_0\left(1 - \frac{(\lambda_1(N) - \tau)}{\lambda_1(N)}\right)\|u\|_E^N - C\|u\|_E^q - \varepsilon\|h\|_*\|u\|_E. \end{aligned}$$

Thus

$$J_\varepsilon(u) \geqslant \|u\|_E\left[k_0\left(1 - \frac{(\lambda_1(N) - \tau)}{\lambda_1(N)}\right)\|u\|_E^{N-1} - C\|u\|_E^{q-1} - \varepsilon\|h\|_*\right]. \tag{3.3}$$

Since $\tau > 0$ and $q > N$, we may choose $\rho > 0$ such that $k_0(1 - \frac{(\lambda_1(N)-\tau)}{\lambda_1(N)})\rho^{N-1} - C\rho^{q-1} > 0$. Thus, if ε is sufficiently small then we can find some $\rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $\|u\| = \rho_\varepsilon$ and even $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. □

Lemma 3.5. *There exists $e \in E$ with $\|e\|_E > \rho_\varepsilon$ such that $J_\varepsilon(e) < \inf_{\|u\|=\rho_\varepsilon} J_\varepsilon(u)$.*

Proof. Let $u \in E \setminus \{0\}$, $u \geqslant 0$ with compact support $\Omega = supp(u)$. By $(f2)$, we have that for $p > N$, there exists a positive constant $C > 0$ such that

$$\forall s \geqslant 0,\ \forall x \in \Omega: \quad F(x,s) \geqslant cs^p - d. \tag{3.4}$$

Then by (2.2), we get

$$J_\varepsilon(tu) \leqslant Ct \int_\Omega h_0(x)|\nabla u|\,dx + Ct^N \|u\|_E^N - Ct^p \int_\Omega \frac{|u|^p}{|x|^\beta}\,dx + C + \varepsilon t \left| \int_\Omega hu\,dx \right|.$$

Since $p > N$, we have $J_\varepsilon(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t sufficiently large, we get the conclusion. $\square$

Now, we define the Moser Functions which have been frequently used in the literature (see, for example, [14,16,5]):

$$\widetilde{m}_l(x,r) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log l)^{(N-1)/N} & \text{if } |x| \leqslant \frac{r}{l}, \\ \frac{\log \frac{r}{|x|}}{(\log l)^{1/N}} & \text{if } \frac{r}{l} \leqslant |x| \leqslant r, \\ 0 & \text{if } |x| \geqslant r. \end{cases}$$

We then immediately have $\widetilde{m}_l(.,r) \in W^{1,N}(\mathbb{R}^N)$, the support of $\widetilde{m}_l(x,r)$ is the ball B_r, and

$$\int_{\mathbb{R}^N} \left|\nabla \widetilde{m}_l(x,r)\right|^N dx = 1, \quad \text{and} \quad \|\widetilde{m}_l\|_{W^{1,N}(\mathbb{R}^N)}^N = 1 + \frac{1}{\log l}\left(\frac{(N-1)!}{N^N} r^N + o_l(1)\right). \tag{3.5}$$

Then

$$\|\widetilde{m}_l\|_E^N \leqslant 1 + \frac{\max_{|x|\leqslant r} V(x)}{\log l}\left(\frac{(N-1)!}{N^N} r^N + o_l(1)\right).$$

Consider $m_l(x,r) = \widetilde{m}_l(x,r)/\|\widetilde{m}_l\|_E$, then we can write

$$m_l^{N/(N-1)}(x,r) = \omega_{N-1}^{-1/(N-1)} \log l + d_l \quad \text{for } |x| \leqslant r/l. \tag{3.6}$$

Using (3.5), we conclude that $\|\widetilde{m}_l\| \to 1$ as $l \to \infty$. Consequently,

$$\begin{aligned} \frac{d_l}{\log l} &\to 0 \quad \text{as } l \to \infty, \\ d &= \liminf_{l\to\infty} d_l, \\ d &\geqslant -\max_{|x|\leqslant r} V(x) \omega_{N-1}^{-1/(N-1)} \frac{(N-2)!}{N^N} r^N. \end{aligned} \tag{3.7}$$

Next we will adapt the idea from J.M. do Ó's works [14,16] when no singular term is present to establish the minimax level in our case. See also [21] for a similar result on the Heisenberg group.

Lemma 3.6. *Suppose that* $(V1)$ *and* $(f1)$–$(f5)$ *hold. Then there exists* $k \in \mathbb{N}$ *such that*

$$\max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tm_k)}{|x|^\beta}\,dx\right\} < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Choose $r > 0$ as in the assumption $(f5)$ and $\beta_0 > 0$ such that

$$\lim_{s\to\infty} sf(x,s)\exp\bigl(-\alpha_0|s|^{N/(N-1)}\bigr) \geqslant \beta_0 > \frac{1}{[\frac{r^{N-\beta}}{N-\beta}e^{(\alpha_N d(N-\beta)/N)} + Cr^{N-\beta} - \frac{r^{N-\beta}}{N-\beta}]}\left(\frac{N-\beta}{\alpha_0}\right)^{N-1}, \tag{3.8}$$

where

$$\mathcal{C} = \lim_{k\to\infty} \zeta_k \log k \int_0^{\zeta_k^{-1}} \exp\bigl[(N-\beta)\log k\bigl(s^{N/(N-1)} - \zeta_k s\bigr)\bigr]\,ds > 0, \qquad \zeta_k = \|\widetilde{m}_k\|,$$

$$\mathcal{C} \geqslant \frac{1 - e^{-(N-\beta)\log n}}{N-\beta}.$$

Suppose, by contradiction, that for all k we get

$$\max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tm_k)}{|x|^\beta}\,dx\right\} \geqslant \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

where $m_k(x) = m_k(x, r)$. By (3.4), for each k there exists $t_k > 0$ such that

$$\frac{t_k^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t_k m_k)}{|x|^\beta}\,dx = \max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tm_k)}{|x|^\beta}\,dx\right\}.$$

Thus

$$\frac{t_k^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t_k m_k)}{|x|^\beta}\,dx \geqslant \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

From $F(x, u) \geqslant 0$, we obtain

$$t_k^N \geqslant \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.9}$$

Since at $t = t_k$ we have

$$\frac{d}{dt}\left(\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tm_k)}{|x|^\beta}\,dx\right) = 0$$

it follows that

$$t_k^N = \int_{\mathbb{R}^N} t_k m_k \frac{f(x, t_k m_k)}{|x|^\beta}\,dx = \int_{|x|\leqslant r} t_k m_k \frac{f(x, t_k m_k)}{|x|^\beta}\,dx. \tag{3.10}$$

Using hypothesis $(f5)$, given $\tau > 0$ there exists $R_\tau > 0$ such that for all $u \geqslant R_\tau$ and $|x| \leqslant r$, we have

$$uf(x, u) \geqslant (\beta_0 - \tau)\exp\bigl(\alpha_0 |u|^{N/(N-1)}\bigr). \tag{3.11}$$

From (3.10) and (3.11), for large k, we obtain

$$\begin{aligned} t_k^N &\geqslant (\beta_0 - \tau) \int_{|x|\leqslant \frac{r}{k}} \frac{\exp(\alpha_0 |t_k m_k|^{N/(N-1)})}{|x|^\beta}\,dx \\ &= (\beta_0 - \tau)\frac{\omega_{N-1}}{N-\beta}\left(\frac{r}{k}\right)^{N-\beta} \exp\bigl(\alpha_0 t_k^{N/(N-1)} \omega_{N-1}^{-1/(N-1)} \log k + \alpha_0 t_k^{N/(N-1)} d_k\bigr). \end{aligned}$$

Thus, setting

$$L_k = \frac{\alpha_0 N \log k}{\alpha_N} t_k^{N/(N-1)} + \alpha_0 t_k^{N/(N-1)} d_k - N \log t_k - (N-\beta)\log k$$

we have

$$1 \geqslant (\beta_0 - \tau)\frac{\omega_{N-1}}{N-\beta} r^{N-\beta} \exp L_k.$$

Consequently, the sequence (t_k) is bounded. Otherwise, up to subsequences, we would have $\lim_{k\to\infty} L_k = \infty$ which leads to a contradiction. Moreover, by (3.7), (3.9) and

$$t_k^N \geqslant (\beta_0 - \tau)\frac{\omega_{N-1}}{N-\beta} r^{N-\beta} \exp\left[\left(N \frac{\alpha_0 t_k^{N/(N-1)}}{\alpha_N} - (N-\beta)\right)\log k + \alpha_0 t_k^{N/(N-1)} d_k\right]$$

it follows that

$$t_k^N \xrightarrow{k\to\infty} \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.12}$$

Set

$$A_k = \{x \in B_r \colon t_k m_k \geqslant R_\tau\} \quad \text{and} \quad B_k = B_r \setminus A_k.$$

From (3.10) and (3.11) we have

$$t_k^N \geqslant (\beta_0 - \tau) \int\limits_{|x| \leqslant r} \frac{\exp(\alpha_0 |t_k m_k|^{N/(N-1)})}{|x|^\beta}\,dx + \int\limits_{B_k} \frac{t_k m_k f(x, t_k m_k)}{|x|^\beta}\,dx$$
$$- (\beta_0 - \tau) \int\limits_{B_k} \frac{\exp(\alpha_0 |t_k m_k|^{N/(N-1)})}{|x|^\beta}\,dx. \tag{3.13}$$

Notice that $m_k(x) \to 0$ and the characteristic functions $\chi_{B_k} \to 1$ for almost everywhere x in B_r. Therefore the Lebesgue dominated convergence theorem implies

$$\int\limits_{B_k} \frac{t_k m_k f(x, t_k m_k)}{|x|^\beta}\,dx \to 0$$

and

$$\int\limits_{B_k} \frac{\exp(\alpha_0 |t_k m_k|^{N/(N-1)})}{|x|^\beta}\,dx \to \frac{\omega_{N-1}}{N-\beta} r^{N-\beta}.$$

Moreover, using that

$$t_k^N \xrightarrow[\geqslant]{k\to\infty} \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

we have

$$\int\limits_{|x| \leqslant r} \frac{\exp(\alpha_0 |t_k m_k|^{N/(N-1)})}{|x|^\beta}\,dx$$
$$\geqslant \int\limits_{|x| \leqslant r} \frac{\exp(\alpha_N |m_k|^{N/(N-1)} (N-\beta)/N)}{|x|^\beta}\,dx$$
$$= \int\limits_{|x| \leqslant r/k} \frac{\exp(\alpha_N |m_k|^{N/(N-1)} (N-\beta)/N)}{|x|^\beta}\,dx$$
$$+ \int\limits_{r/k \leqslant |x| \leqslant r} \frac{\exp(\alpha_N |m_k|^{N/(N-1)} (N-\beta)/N)}{|x|^\beta}\,dx$$

and

$$\begin{aligned}
&\int\limits_{|x|\leqslant r/k} \frac{\exp(\alpha_N |m_k|^{N/(N-1)}(N-\beta)/N)}{|x|^\beta}\,dx \\
&= \int\limits_{|x|\leqslant r/k} \frac{\exp[\alpha_N \omega_{N-1}^{-1/(N-1)} \log k(N-\beta)/N + d_k\alpha_N(N-\beta)/N]}{|x|^\beta}\,dx \\
&= \frac{\omega_{N-1}}{N-\beta}\left(\frac{r}{k}\right)^{N-\beta} k^{(N-\beta+\alpha_N \frac{d_k}{\log k}(N-\beta)/N)} \\
&= \frac{\omega_{N-1}}{N-\beta} r^{N-\beta} k^{(\alpha_N \frac{d_k}{\log k}(N-\beta)/N)}.
\end{aligned}$$

Now, using the change of variable

$$x = \frac{\log(\frac{r}{s})}{\zeta_k \log k} \quad \text{with } \zeta_k = \|\widetilde{m}_k\|$$

by straightforward computation, we have

$$\begin{aligned}
&\int\limits_{r/k\leqslant|x|\leqslant r} \frac{\exp(\alpha_N |m_k|^{N/(N-1)}(N-\beta)/N)}{|x|^\beta}\,dx \\
&= \omega_{N-1} r^{N-\beta} \zeta_k \log k \int_0^{\zeta_k^{-1}} \exp\bigl[(N-\beta)\log k\bigl(s^{N/(N-1)} - \zeta_k s\bigr)\bigr]\,ds
\end{aligned}$$

which converges to $C\omega_{N-1} r^{N-\beta}$ as $k \to \infty$ where

$$C = \lim_{k\to\infty} \zeta_k \log k \int_0^{\zeta_k^{-1}} \exp\bigl[(N-\beta)\log k\bigl(s^{N/(N-1)} - \zeta_k s\bigr)\bigr]\,ds > 0.$$

Finally, taking $k \to \infty$ in (3.13), using (3.12) and using (3.7) (see [14,16]), we obtain

$$\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \geqslant (\beta_0 - \tau)\left[\frac{\omega_{N-1}}{N-\beta} r^{N-\beta} e^{(\alpha_N d(N-\beta)/N)} + C\omega_{N-1} r^{N-\beta} - \frac{\omega_{N-1}}{N-\beta} r^{N-\beta}\right]$$

which implies that

$$\beta_0 \leqslant \frac{1}{[\frac{r^{N-\beta}}{N-\beta} e^{(\alpha_N d(N-\beta)/N)} + C r^{N-\beta} - \frac{r^{N-\beta}}{N-\beta}]}\left(\frac{N-\beta}{\alpha_0}\right)^{N-1}.$$

This contradicts to (3.8), and the proof is complete. □

4. The existence of solution for the problem (1.2)

It is well known that the failure of the (PS) compactness condition creates difficulties in studying this class of elliptic problems involving critical growth and unbounded domains. In next several lemmas, instead of (PS) sequence, we will use and analyze the compactness of Cerami sequences of J_ε.

Lemma 4.1. *Let $(u_k) \subset E$ be an arbitrary Cerami sequence of J_ε, i.e.,*

$$J_\varepsilon(u_k) \to c, \qquad \big(1 + \|u_k\|_E\big)\big\|DJ_\varepsilon(u_k)\big\|_{E'} \to 0 \quad \text{as } k \to \infty.$$

Then there exists a subsequence of (u_k) (still denoted by (u_k)) and $u \in E$ such that

$$\begin{cases} \dfrac{f(x,u_k)}{|x|^\beta} \to \dfrac{f(x,u)}{|x|^\beta} & \text{strongly in } L^1_{loc}\big(\mathbb{R}^N\big), \\ \nabla u_k(x) \to \nabla u(x) & \text{almost everywhere in } \mathbb{R}^N, \\ a(x,\nabla u_k) \rightharpoonup a(x,\nabla u) & \text{weakly in } \big(L^{N/(N-1)}_{loc}\big(\mathbb{R}^N\big)\big)^N, \\ u_k \rightharpoonup u & \text{weakly in } E. \end{cases}$$

Furthermore u is a weak solution of (1.2).

For simplicity, we will only sketch the proof where includes the nonuniform terms $a(x,\nabla u)$ and $A(x,\nabla u)$.

Proof of Lemma 4.1. Let $v \in E$, then we have

$$\int_{\mathbb{R}^N} A(x,\nabla u_k)\,dx + \frac{1}{N}\int_{\mathbb{R}^N} V(x)|u_k|^N\,dx - \int_{\mathbb{R}^N} \frac{F(x,u_k)}{|x|^\beta}\,dx - \varepsilon\int_{\mathbb{R}^N} hu_k\,dx \xrightarrow{k\to\infty} c \tag{4.1}$$

and

$$\begin{aligned} \big|DJ_\varepsilon(u_k)v\big| &= \left|\int_{\mathbb{R}^N} a(x,\nabla u_k)\nabla v\,dx + \int_{\mathbb{R}^N} V(x)|u_k|^{N-2}u_k v\,dx - \int_{\mathbb{R}^N} \frac{f(x,u_k)v}{|x|^\beta}\,dx - \varepsilon\int_{\mathbb{R}^N} hv\,dx\right| \\ &\leqslant \frac{\tau_k\|v\|_E}{(1+\|u_k\|_E)} \end{aligned} \tag{4.2}$$

where $\tau_k \to 0$ as $k \to \infty$. Choosing $v = u_k$ in (4.2) and by $(A3)$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{f(x,u_k)u_k}{|x|^\beta}\,dx + \varepsilon\int_{\mathbb{R}^N} hu_k\,dx - N\int_{\mathbb{R}^N} A(x,\nabla u_k) - \int_{\mathbb{R}^N} V(x)|u_k|^{N-2}u_k\,dx \\ &\quad \leqslant \tau_k\frac{\|u_k\|_E}{(1+\|u_k\|_E)} \to 0. \end{aligned}$$

This together with (4.1), $(f2)$ and $(A4)$ leads to

$$\left(\frac{p}{N}-1\right)\|u_k\|_E^N \leqslant C\big(1+\|u_k\|_E\big)$$

and hence $\|u_k\|_E$ is bounded and thus

$$\int\limits_{\mathbb{R}^N} \frac{f(x,u_k)u_k}{|x|^\beta}\,dx \leqslant C, \qquad \int\limits_{\mathbb{R}^N} \frac{F(x,u_k)}{|x|^\beta}\,dx \leqslant C. \tag{4.3}$$

Thanks to the assumptions on the potential V, the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \geqslant N$, by extracting a subsequence, we can assume that

$$u_k \to u \quad \text{weakly in } E \text{ and for almost all } x \in \mathbb{R}^N.$$

Thanks to Lemma 2.1 in [18], we have

$$\frac{f(x,u_n)}{|x|^\beta} \to \frac{f(x,u)}{|x|^\beta} \quad \text{in } L^1_{loc}\big(\mathbb{R}^N\big). \tag{4.4}$$

Next, up to a subsequence, we can define an energy concentration set for any fixed $\delta > 0$,

$$\Sigma_\delta = \left\{x \in \mathbb{R}^N\colon \lim_{r\to 0}\lim_{k\to\infty} \int\limits_{B_r(x)} \big(|u_k|^N + |\nabla u_k|^N\big)\,dx' \geqslant \delta\right\}.$$

Since (u_k) is bounded, Σ_δ must be a finite set. Adapting an argument similar to [5] (we omit the details here), we can prove that for any compact set $K \Subset \mathbb{R}^N \setminus \Sigma_\delta$,

$$\lim_{k\to\infty} \int\limits_K \frac{|f(x,u_k)u_k - f(x,u)u|}{|x|^\beta}\,dx = 0. \tag{4.5}$$

Next we will prove that for any compact set $K \Subset \mathbb{R}^N \setminus \Sigma_\delta$,

$$\lim_{k\to\infty} \int\limits_K |\nabla u_k - \nabla u|^N\,dx = 0. \tag{4.6}$$

It is enough to prove for any $x^* \in \mathbb{R}^N \setminus \Sigma_\delta$, and $B_r(x^*, r) \subset \mathbb{R}^N \setminus \Sigma_\delta$, there holds

$$\lim_{k\to\infty} \int\limits_{B_{r/2}(x^*)} |\nabla u_k - \nabla u|^N\,dx = 0. \tag{4.7}$$

For this purpose, we take $\phi \in C_0^\infty(B_r(x^*))$ with $0 \leqslant \phi \leqslant 1$ and $\phi = 1$ on $B_{r/2}(x^*)$. Obviously ϕu_k is a bounded sequence. Choose $h = \phi u_k$ and $h = \phi u$ in (4.2), then we have:

$$\int\limits_{B_r(x^*)} \phi\big(a(x,\nabla u_k)-a(x,\nabla u)\big)(\nabla u_k-\nabla u)\,dx$$
$$\leqslant \int\limits_{B_r(x^*)} a(x,\nabla u_k)\nabla\phi(u-u_k)\,dx$$
$$+\int\limits_{B_r(x^*)} \phi a(x,\nabla u)(\nabla u-\nabla u_k)\,dx+\int\limits_{B_r(x^*)} \phi(u_k-u)\frac{f(x,u_k)}{|x|^\beta}\,dx$$
$$+\tau_k\|\phi u_k\|_E+\tau_k\|\phi u\|_E-\varepsilon\int\limits_{B_r(x^*)} \phi h(u_k-u)\,dx.$$

Note that by Holder's inequality and the compact embedding of $E\hookrightarrow L^N(\Omega)$, we get

$$\lim_{k\to\infty}\int\limits_{B_r(x^*)} a(x,\nabla u_k)\nabla\phi(u-u_k)\,dx=0. \tag{4.8}$$

Since $\nabla u_k \rightharpoonup \nabla u$ and $u_k \rightharpoonup u$, there holds

$$\lim_{k\to\infty}\int\limits_{B_r(x^*)} \phi a(x,\nabla u)(\nabla u-\nabla u_k)\,dx=0 \quad\text{and}\quad \lim_{k\to\infty}\int\limits_{B_r(x^*)} \phi h(u_k-u)\,dx=0. \tag{4.9}$$

This implies that

$$\lim_{k\to\infty}\int\limits_{B_r(x^*)} \phi(u_k-u)f(x,u_k)\,dx=0.$$

So we can conclude that

$$\lim_{k\to\infty}\int\limits_{B_r(x^*)} \phi\big(a(x,\nabla u_k)-a(x,\nabla u)\big)(\nabla u_k-\nabla u)\,dx=0$$

and hence we get (4.7) by $(A2)$. Thus we have (4.6) by a covering argument. Since Σ_δ is finite, it follows that ∇u_k converges to ∇u almost everywhere. This immediately implies, up to a subsequence, $a(x,\nabla u_k)\rightharpoonup a(x,\nabla u)$ weakly in $(L^{N/(N-1)}_{loc}(\mathbb{R}^N))^{N-2}$. Using all these facts, letting k tend to infinity in (4.2) and combining with (4.4), we obtain

$$\big\langle DJ_\varepsilon(u),v\big\rangle=0 \quad \forall v\in C_0^\infty\big(\mathbb{R}^N\big).$$

This completes the proof of the lemma. □

Now, we are ready to prove Theorem 2.1. The existence of the solution of (1.2) follows by a standard "mountain-pass" procedure.

4.1. The proof of Theorem 2.1

Proposition 4.1. *Under the assumptions* $(V1)$ *and* $(V2)$ (*or* $(V3)$), *and* $(f1)$–$(f4)$, *there exists* $\varepsilon_1 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_1$, *the problem* (1.2) *has a solution* u_M *via mountain-pass theorem.*

Proof. For ε sufficiently small, by Lemmas 3.4 and 3.5, J_ε satisfies the hypotheses of the mountain-pass theorem except possibly for the (PS) condition. Thus, using the mountain-pass theorem without the (PS) condition, we can find a sequence (u_k) in E such that

$$J_\varepsilon(u_k) \to c_M > 0 \quad \text{and} \quad \big(1 + \|u_k\|_E\big)\big\|DJ_\varepsilon(u_k)\big\| \to 0$$

where c_M is the mountain-pass level of J_ε. Now, by Lemma 4.1, the sequence (u_k) converges weakly to a weak solution u_M of (1.2) in E. Moreover, $u_M \neq 0$ since $h \neq 0$. $\square$

5. The multiplicity results of the problem (1.3)

In this section, we deal with the problem (1.3). Note that this is the special case of the problem (1.2) with $A(x,\tau) = \frac{|\tau|^N}{N}$. Some preliminary lemmas in the case $\beta = 0$ were treated in [14,16]. We have included details here. The key ingredient of this section is the proof of Proposition 5.2 which is substantially different from those in [14,16].

Lemma 5.1. *There exist* $\eta > 0$ *and* $v \in E$ *with* $\|v\|_E = 1$ *such that* $J_\varepsilon(tv) < 0$ *for all* $0 < t < \eta$. *In particular,* $\inf_{\|u\|_E \leqslant \eta} J_\varepsilon(u) < 0$.

Corollary 5.1. *Under the hypotheses* $(V1)$ *and* $(f1)$–$(f5)$, *if* ε *is sufficiently small then*

$$\max_{t \geqslant 0} J_\varepsilon(tm_k) = \max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int\limits_{\mathbb{R}^N} \frac{F(x, tm_k)}{|x|^\beta}\,dx - t\int\limits_{\mathbb{R}^N} \varepsilon h m_k\,dx\right\} < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Note that we can conclude by inequality (3.3) and Lemma 5.1 that

$$-\infty < c_0 = \inf_{\|u\|_E \leqslant \rho_\varepsilon} J_\varepsilon(u) < 0. \tag{5.1}$$

Next, we will prove that this infimum is achieved and generate a solution. In order to obtain convergence results, we need to improve the estimate of Lemma 3.6.

Corollary 5.2. *Under the hypotheses* $(V1)$ *and* $(f1)$–$(f5)$, *there exist* $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $u \in W^{1,N}(\mathbb{R}^N)$ *with compact support such that for all* $0 < \varepsilon < \varepsilon_2$,

$$J_\varepsilon(tu) < c_0 + \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \quad \textit{for all } t \geqslant 0.$$

Proof. It is possible to increase the infimum c_0 by reducing ε. By Lemma 3.4, $\rho_\varepsilon \xrightarrow{\varepsilon\to 0} 0$. Consequently, $c_0 \xrightarrow{\varepsilon\to 0} 0$. Thus there exists $\varepsilon_2 > 0$ such that if $0 < \varepsilon < \varepsilon_2$ then, by Corollary 5.1, we have

$$\max_{t\geqslant 0} J_\varepsilon(tm_k) < c_0 + \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Taking $u = m_k \in W^{1,N}(\mathbb{R}^N)$, the result follows. □

Lemma 5.2. *If (u_k) is a Cerami sequence for J_ε at any level with*

$$\liminf_{k\to\infty} \|u_k\|_E < \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N} \tag{5.2}$$

then (u_k) possesses a subsequence which converges strongly to a solution u_0 of (1.3).

Proof. See Lemma 4.6 in [5]. □

5.1. Proof of Theorem 2.2

The proof of the existence of the second solution of (1.3) follows by a minimization argument and Ekeland's variational principle.

Proposition 5.1. *There exists $\varepsilon_2 > 0$ such that for each ε with $0 < \varepsilon < \varepsilon_2$, Eq.* (1.3) *has a minimum type solution u_0 with $J_\varepsilon(u_0) = c_0 < 0$, where c_0 is defined in* (5.1).

Proof. Let ρ_ε be as in Lemma 3.4. We can choose $\varepsilon_2 > 0$ sufficiently small such that

$$\rho_\varepsilon < \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N}.$$

Since $\overline{B}_{\rho_\varepsilon}$ is a complete metric space with the metric given by the norm of E, convex and the functional J_ε is of class C^1 and bounded below on $\overline{B}_{\rho_\varepsilon}$, by Ekeland's variational principle there exists a sequence (u_k) in $\overline{B}_{\rho_\varepsilon}$ such that

$$J_\varepsilon(u_k) \to c_0 = \inf_{\|u\|_E\leqslant\rho_\varepsilon} J_\varepsilon(u) \quad \text{and} \quad \left\|DJ_\varepsilon(u_k)\right\| \to 0.$$

Observing that

$$\|u_k\|_E \leqslant \rho_\varepsilon < \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N}$$

by Lemma 5.2 it follows that there exists a subsequence of (u_k) which converges to a solution u_0 of (1.3). Therefore, $J_\varepsilon(u_0) = c_0 < 0$. □

Remark 5.1. By Corollary 5.2, we can conclude that

$$0 < c_M < c_0 + \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proposition 5.2. *If $\varepsilon_2 > 0$ is sufficiently small, then the solutions of* (1.4) *obtained in Propositions* 4.1 *and* 5.1 *are distinct.*

Remark 5.2. Before we give a proof of the proposition, we like to make some remarks. We note the following Hardy–Littlewood inequality holds for nonnegative functions f and g in $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} f(x)g(x)\,dx \leqslant \int_{\mathbb{R}^N} f^*(x)g^*(x)\,dx$$

where f^* and g^* are symmetric and decreasing rearrangement of f and g respectively. However, the following inequality, which has been used in [16] to derive the multiplicity of nontrivial solutions in the case of $\beta = 0$,

$$\int_{|x|>R} f(x)g(x)\,dx \leqslant \int_{|x|>R} f^*(x)g^*(x)\,dx,$$

does not hold for all $R > 0$ in general. Therefore, we will avoid using the symmetrization argument when we prove

$$\int_{\mathbb{R}^N} \frac{F(x,v_k)}{|x|^\beta}\,dx \to \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^\beta}\,dx.$$

Nevertheless, this can be taken care by a "double truncation" argument in both cases of $\beta = 0$ and $0 < \beta < N$. This argument differs from those given in [14–16,33]. Using this argument, the compact embedding $E \to L^q(\mathbb{R}^N)$ for $q \geqslant N$ is sufficient and thus establish the multiplicity of nontrivial solutions under our assumptions on the potential V.

Proof. By Propositions 4.1 and 5.1, there exist sequences (u_k), (v_k) in E such that

$$u_k \to u_0, \qquad J_\varepsilon(u_k) \to c_0 < 0, \qquad DJ_\varepsilon(u_k)u_k \to 0$$

and

$$v_k \rightharpoonup u_M, \qquad J_\varepsilon(v_k) \to c_M > 0, \qquad DJ_\varepsilon(v_k)v_k \to 0,$$
$$\nabla v_k(x) \to \nabla u_M(x) \quad \text{almost everywhere in } \mathbb{R}^N.$$

Now, suppose by contradiction that $u_0 = u_M$. As in the proof of Lemma 4.1 we obtain

$$\frac{f(x,v_k)}{|x|^\beta} \to \frac{f(x,u_0)}{|x|^\beta} \quad \text{in } L^1(B_R) \text{ for all } R > 0. \tag{5.3}$$

Moreover, by $(f2)$, $(f3)$

$$\frac{F(x,v_k)}{|x|^\beta} \leqslant \frac{R_0 f(x,v_k)}{|x|^\beta} + \frac{M_0 f(x,v_k)}{|x|^\beta}$$

so by the Generalized Lebesgue's Dominated Convergence Theorem,

$$\frac{F(x,v_k)}{|x|^\beta} \to \frac{F(x,u_0)}{|x|^\beta} \quad \text{in } L^1(B_R).$$

We will prove that

$$\int_{\mathbb{R}^N} \frac{F(x,v_k)}{|x|^\beta}\,dx \to \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^\beta}\,dx.$$

It's sufficient to prove that given $\delta > 0$, there exists $R > 0$ such that

$$\int_{|x|>R} \frac{F(x,v_k)}{|x|^\beta}\,dx \leqslant 3\delta \quad \text{and} \quad \int_{|x|>R} \frac{F(x,u_0)}{|x|^\beta}\,dx \leqslant 3\delta.$$

To prove it, we recall the following facts from our assumptions on nonlinearity: there exists $c > 0$ such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}^+$:

$$\begin{gathered} F(x,s) \leqslant c|s|^N + cf(x,s), \\ F(x,s) \leqslant c|s|^N + cR(\alpha_0,s)s, \\ \int_{\mathbb{R}^N} \frac{f(x,v_k)v_k}{|x|^\beta}\,dx \leqslant C, \qquad \int_{\mathbb{R}^N} \frac{F(x,v_k)}{|x|^\beta}\,dx \leqslant C. \end{gathered} \tag{5.4}$$

First, we will prove it for the case $\beta > 0$.

We have that

$$\begin{aligned} \int_{\substack{|x|>R \\ |v_k|>A}} \frac{F(x,v_k)}{|x|^\beta}\,dx &\leqslant c \int_{|x|>R} \frac{|v_k|^N}{|x|^\beta}\,dx + c \int_{\substack{|x|>R \\ |v_k|>A}} \frac{f(x,v_k)}{|x|^\beta}\,dx \\ &\leqslant \frac{c}{R^\beta}\|v_k\|_E^N + c\frac{1}{A}\int_{\mathbb{R}^N} \frac{f(x,v_k)v_k}{|x|^\beta}\,dx. \end{aligned}$$

Since $\|v_k\|_E$ is bounded and using (5.4), we can first choose A such that

$$c\frac{1}{A}\int_{\mathbb{R}^N} \frac{f(x,v_k)v_k}{|x|^\beta}\,dx < \delta \quad \text{for all } k$$

and then choose R such that

$$\frac{c}{R^{\beta}}\|v_k\|_E^N < \delta$$

which thus

$$\int\limits_{\substack{|x|>R\\|v_k|>A}} \frac{F(x,v_k)}{|x|^{\beta}}\,dx \leqslant 2\delta.$$

Now, note that with such A, we have for $|s| \leqslant A$:

$$\begin{aligned}
F(x,s) &\leqslant c|s|^N + cR(\alpha_0,s)s \\
&\leqslant c|s|^N + c\sum_{j=N-1}^{\infty}\frac{\alpha_0^j}{j!}|s|^{Nj/(N-1)+1} \\
&\leqslant |s|^N\left[c + c\sum_{j=N-1}^{\infty}\frac{\alpha_0^j}{j!}A^{Nj/(N-1)+1-N}\right] \\
&\leqslant C(\alpha_0,A)|s|^N.
\end{aligned}$$

So we get

$$\begin{aligned}
\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}} \frac{F(x,v_k)}{|x|^{\beta}}\,dx &\leqslant \frac{C(\alpha_0,A)}{R^{\beta}}\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}} |v_k|^N\,dx \\
&\leqslant \frac{C(\alpha_0,A)}{R^{\beta}}\|v_k\|_E^N.
\end{aligned}$$

Again, note that $\|v_k\|_E$ is bounded, we can choose R such that

$$\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}} \frac{F(x,v_k)}{|x|^{\beta}}\,dx \leqslant \delta.$$

In conclusion, we can choose $R > 0$ such that

$$\int\limits_{|x|>R} \frac{F(x,v_k)}{|x|^{\beta}}\,dx \leqslant 3\delta.$$

Similarly, we can choose $R > 0$ such that

$$\int\limits_{|x|>R} \frac{F(x,u_0)}{|x|^{\beta}}\,dx \leqslant 3\delta.$$

Now, if $\beta = 0$, similarly, we have

$$\begin{aligned}\int\limits_{\substack{|x|>R\\|v_k|>A}} F(x,v_k)\,dx &\leqslant c\int\limits_{|x|>R}|v_k|^N\,dx + c\int\limits_{\substack{|x|>R\\|v_k|>A}} f(x,v_k)\,dx\\ &\leqslant \frac{c}{A}\int\limits_{|x|>R}|v_k|^{N+1}\,dx + c\frac{1}{A}\int\limits_{\mathbb{R}^N} f(x,v_k)v_k\,dx\\ &\leqslant \frac{c}{A}\|v_k\|_E^{N+1} + c\frac{1}{A}\int\limits_{\mathbb{R}^N} f(x,v_k)v_k\,dx\end{aligned}$$

so since $\|v_k\|_E$ is bounded and by (5.4), we can choose A such that

$$\int\limits_{\substack{|x|>R\\|v_k|>A}} F(x,v_k)\,dx \leqslant 2\delta.$$

Next, we have

$$\begin{aligned}\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}} F(x,v_k)\,dx &\leqslant C(\alpha_0,A)\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}}|v_k|^N\,dx\\ &\leqslant 2^{N-1}C(\alpha_0,A)\Bigg(\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}}|v_k-u_0|^N\,dx + \int\limits_{\substack{|x|>R\\|v_k|\leqslant A}}|u_0|^N\,dx\Bigg).\end{aligned}$$

Now, using the compactness of embedding $E \hookrightarrow L^q(\mathbb{R}^N)$, $q \geqslant N$ and noticing that $v_k \rightharpoonup u_0$, again we can choose R such that

$$\int\limits_{\substack{|x|>R\\|v_k|\leqslant A}} F(x,v_k)\,dx \leqslant \delta.$$

Combining all the above estimates, we have the fact that

$$\int\limits_{\mathbb{R}^N}\frac{F(x,v_k)}{|x|^\beta}\,dx \to \int\limits_{\mathbb{R}^N}\frac{F(x,u_0)}{|x|^\beta}\,dx$$

since δ is arbitrary and (5.3) holds. From this, we have

$$\lim_{k\to\infty}\|\nabla v_k\|_N^N = Nc_M - \lim_{k\to\infty}\int\limits_{\mathbb{R}^N}V(x)|v_k|^N\,dx + N\int\limits_{\mathbb{R}^N}\frac{F(x,u_0)}{|x|^\beta}\,dx + N\varepsilon\int\limits_{\mathbb{R}^N}hu_0\,dx. \tag{5.5}$$

Now, let

$$w_k = \frac{v_k}{\|\nabla v_k\|_N} \quad \text{and} \quad w_0 = \frac{u_0}{\lim_{k\to\infty}\|\nabla v_k\|_N}$$

we have $\|\nabla w_k\|_N = 1$ for all k and $w_k \rightharpoonup w_0$ in $D^{1,N}(\mathbb{R}^N)$, the closure of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm $\|\nabla\varphi\|_N$. In particular, $\|\nabla w_0\|_N \leqslant 1$ and $w_k|_{B_R} \rightharpoonup w_0|_{B_R}$ in $W^{1,N}(B_R)$ for all $R > 0$. We claim that $\|\nabla w_0\|_N < 1$.

Indeed, if $\|\nabla w_0\|_N = 1$, then we have $\lim_{k\to\infty}\|\nabla v_k\|_N = \|\nabla u_0\|_N$ and thus $v_k \to u_0$ in $W^{1,N}(\mathbb{R}^N)$ since $v_k \to u_0$ in $L^q(\mathbb{R}^N)$, $q \geqslant N$. So we can find $g \in W^{1,N}(\mathbb{R}^N)$ (for some $q \geqslant N$) such that $|v_k(x)| \leqslant g(x)$ almost everywhere in $\mathbb{R}^N$. From assumption $(f1)$, we have for some $\alpha_1 > \alpha_0$ that

$$\big|f(x,s)s\big| \leqslant b_1|s|^N + C\big[\exp\big(\alpha_1|s|^{N/(N-1)}\big) - S_{N-2}(\alpha_1,s)\big]$$

for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Thus,

$$\begin{aligned}\frac{|f(x,v_k)v_k|}{|x|^\beta} &\leqslant b_1\frac{|v_k|^N}{|x|^\beta} + C\frac{[\exp(\alpha_1|v_k|^{N/(N-1)}) - S_{N-2}(\alpha_1,v_k)]}{|x|^\beta}\\ &\leqslant b_1\frac{|v_k|^N}{|x|^\beta} + C\frac{[\exp(\alpha_1|g|^{N/(N-1)}) - S_{N-2}(\alpha_1,g)]}{|x|^\beta}\end{aligned}$$

almost everywhere in $\mathbb{R}^N$. Now, by Lebesgue's dominated convergence theorem,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}\frac{f(x,v_k)v_k}{|x|^\beta}\,dx = \int_{\mathbb{R}^N}\frac{f(x,u_0)u_0}{|x|^\beta}\,dx.$$

Similarly, since $u_k \to u_0$ in E, we also have

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}\frac{f(x,u_k)u_k}{|x|^\beta}\,dx = \int_{\mathbb{R}^N}\frac{f(x,u_0)u_0}{|x|^\beta}\,dx.$$

Now, noting that

$$DJ_\varepsilon(u_k)u_k = \|u_k\|_E^N - \int_{\mathbb{R}^N}\frac{f(x,u_k)u_k}{|x|^\beta}\,dx - \int_{\mathbb{R}^N}\varepsilon h u_k\,dx \to 0$$

and

$$DJ_\varepsilon(v_k)v_k = \|v_k\|_E^N - \int_{\mathbb{R}^N}\frac{f(x,v_k)v_k}{|x|^\beta}\,dx - \int_{\mathbb{R}^N}\varepsilon h v_k\,dx \to 0$$

we conclude that

$$\lim_{k\to\infty}\|v_k\|_E^N = \lim_{k\to\infty}\|u_k\|_E^N = \|u_0\|_E^N$$

and thus $J_\varepsilon(v_k) \to J_\varepsilon(u_0) = c_0 < 0$ and this is a contradiction.

So $\|\nabla w_0\|_N < 1$. Using Remark 5.1 we have

$$c_M - J_\varepsilon(u_0) < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

and thus

$$\alpha_0 < \frac{N-\beta}{N}\frac{\alpha_N}{[N(c_M - J_\varepsilon(u_0))]^{1/(N-1)}}.$$

Now if we choose $q > 1$ sufficiently close to 1 and set

$$L(w) = c_M - \frac{1}{N}\int\limits_{\mathbb{R}^N} V(x)|w|^N\,dx + \int\limits_{\mathbb{R}^N}\frac{F(x,w)}{|x|^\beta}\,dx + \varepsilon\int\limits_{\mathbb{R}^N} hw\,dx$$

then for some $\delta > 0$,

$$\begin{aligned} q\alpha_0\|\nabla v_k\|_N^{N/(N-1)} &\leqslant \frac{N-\beta}{N}\frac{\alpha_N\|\nabla v_k\|_N^{N/(N-1)}}{[N(c_M - J_\varepsilon(u_0))]^{1/(N-1)}} - \delta \\ &= \frac{N-\beta}{N}\frac{\alpha_N(NL(v_k))^{1/(N-1)} + o_k(1)}{[N(c_M - J_\varepsilon(u_0))]^{1/(N-1)}} - \delta. \end{aligned}$$

Note that

$$\lim_{k\to\infty} L(v_k) = c_M - \lim_{k\to\infty}\frac{1}{N}\int\limits_{\mathbb{R}^N} V(x)|v_k|^N\,dx + \int\limits_{\mathbb{R}^N}\frac{F(x,u_0)}{|x|^\beta}\,dx + \varepsilon\int\limits_{\mathbb{R}^N} hu_0\,dx + o_k(1)$$

and

$$\begin{aligned} &\left(c_M - \lim_{k\to\infty}\frac{1}{N}\int\limits_{\mathbb{R}^N} V(x)|v_k|^N\,dx + \int\limits_{\mathbb{R}^N}\frac{F(x,u_0)}{|x|^\beta}\,dx + \varepsilon\int\limits_{\mathbb{R}^N} hu_0\,dx\right)\left(1 - \|\nabla_{\mathbb{R}^N} w_0\|_N^N\right) \\ &\quad\leqslant c_M - J_\varepsilon(u_0) \end{aligned}$$

so for k, R sufficiently large,

$$q\alpha_0\|\nabla v_k\|_N^{N/(N-1)} \leqslant \frac{N-\beta}{N}\frac{\alpha_N}{[1 - \|\nabla_{\mathbb{R}^N} w_0\|_{L^N(B_R)}^N]^{1/(N-1)}} - \delta.$$

By Lemma 3.3, note that $\nabla w_k \to \nabla w_0$ almost everywhere since $\nabla v_k(x) \to \nabla u_M(x) = \nabla u_0(x)$ almost everywhere in $\mathbb{R}^N$:

$$\int\limits_{B_R}\frac{\exp(q\alpha_0\|\nabla v_k\|_N^{N/(N-1)}|w_k|^{N/(N-1)})}{|x|^\beta}\,dx \leqslant C. \tag{5.6}$$

By $(f1)$ and Holder's inequality,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \frac{f(x,v_k)(v_k-u_0)}{|x|^\beta}\,dx\right| \\
&\quad\leqslant b_1\int_{\mathbb{R}^N} \frac{|v_k|^{N-1}|v_k-u_0|}{|x|^\beta}\,dx + b_2\int_{B_R} \frac{|v_k-u_0|\exp(\alpha_0|v_k|^{N/(N-1)})}{|x|^\beta}\,dx \\
&\quad\leqslant b_1\biggl(\int_{\mathbb{R}^N} \frac{|v_k|^N}{|x|^\beta}\,dx\biggr)^{(N-1)/N}\biggl(\int_{\mathbb{R}^N} \frac{|v_k-u_0|^N}{|x|^\beta}\,dx\biggr)^{1/N} \\
&\qquad + b_2\biggl(\int_{\mathbb{R}^N} \frac{|v_k-u_0|^{q'}}{|x|^\beta}\,dx\biggr)^{1/q'}\biggl(\int_{B_R} \frac{\exp(q\alpha_0\|\nabla v_k\|_N^{N/(N-1)}|w_k|^{N/(N-1)})}{|x|^\beta}\,dx\biggr)^{1/q}
\end{aligned}$$

where $q' = q/(q-1)$. By (5.6), we have

$$\left|\int_{\mathbb{R}^N} \frac{f(x,v_k)(v_k-u_0)}{|x|^\beta}\,dx\right| \leqslant C_1\left\|\frac{v_k-u_0}{|x|^{\beta/N}}\right\|_N + C_2\left\|\frac{v_k-u_0}{|x|^{\beta/q'}}\right\|_{q'}.$$

Using the Holder inequality and the compact embedding $E \hookrightarrow L^q$, $q \geqslant N$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|v_k-u_0|^N}{|x|^\beta}\,dx &= \int_{|x|<1} \frac{|v_k-u_0|^N}{|x|^\beta}\,dx + \int_{|x|\geqslant 1} \frac{|v_k-u_0|^N}{|x|^\beta}\,dx \\
&\leqslant \biggl(\int_{|x|<1} \frac{1}{|x|^{\beta s}}\,dx\biggr)^{1/s}\biggl(\int_{|x|<1} |v_k-u_0|^{s'N}\,dx\biggr)^{1/s'} + \|v_k-u_0\|_N^N \\
&\to 0 \quad \text{as } k\to\infty
\end{aligned}$$

for some $s > 1$ sufficiently close to 1. Similarly,

$$\int_{\mathbb{R}^N} \frac{|v_k-u_0|^{q'}}{|x|^\beta}\,dx \xrightarrow{k\to\infty} 0.$$

Thus we can conclude that

$$\int_{\mathbb{R}^N} |\nabla v_k|^{N-2}\nabla v_k(\nabla v_k - \nabla u_0)\,dx + \int_{\mathbb{R}^N} V(x)|v_k|^{N-2}v_k(v_k-u_0)\,dx \to 0$$

since $DJ_\varepsilon(v_k)(v_k-u_0) \to 0$.

On the other hand, since $v_k \rightharpoonup u_0$

$$\int_{\mathbb{R}^N} |\nabla u_0|^{N-2}\nabla u_0(\nabla v_k - \nabla u_0)\,dx \to 0$$

and

$$\int_{\mathbb{R}^N} V(x)|u_0|^{N-2}u_0(v_k - u_0)\,dx \to 0$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v_k - \nabla u_0|^N\,dx + \int_{\mathbb{R}^N} V(x)|v_k - u_0|^N \\
&\quad \leqslant C_1 \int_{\mathbb{R}^N} \big(|\nabla v_k|^{N-2}\nabla v_k - |\nabla u_0|^{N-2}\nabla u_0\big)(\nabla v_k - \nabla u_0)\,dx \\
&\qquad + C_2 \int_{\mathbb{R}^N} V(x)\big(|v_k|^{N-2}v_k - |u_0|^{N-2}u_0\big)(v_k - u_0)\,dx
\end{aligned}$$

where we did use the inequality $(|x|^{N-2}x - |y|^{N-2}y)(x-y) \geqslant 2^{2-N}|x-y|^N$. So we can conclude that $v_k \to u_0$ in E. Thus $J_\varepsilon(v_k) \to J_\varepsilon(u_0) = c_0 < 0$. Again, this is a contradiction. The proof is thus complete. □

6. The existence result to the problem (1.4)

In this section, we deal with the problem (1.4). The main result of ours shows that we don't need a nonzero small perturbation in this case to guarantee the existence of a solution.

6.1. Proof of Theorem 2.3

It's similar to the proof of Theorems 2.1 and 2.2. We can find a sequence (v_k) in E such that

$$J(v_k) \to c_M > 0 \quad \text{and} \quad \big(1 + \|v_k\|_E\big)\big\|DJ(v_k)\big\| \to 0$$

where c_M is the mountain-pass level of J. Now, by Lemma 4.1, the sequence (v_k) converges weakly to a weak solution v of (1.4) in E. Now, suppose that $v = 0$. Similarly as in the proof of Proposition 5.2, we have that:

$$\int_{\mathbb{R}^N} \frac{F(x, v_k)}{|x|^\beta} \to 0. \tag{6.1}$$

So

$$\lim_{k\to\infty} \|v_k\|_E^N = \lim_{k\to\infty}\left(NJ(v_k) + N\int_{\mathbb{R}^N} \frac{F(x, v_k)}{|x|^\beta}\,dx\right) = NC_M.$$

Note that by Lemma 3.6, we have $0 < C_M < \frac{1}{N}(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0})^{N-1}$, so

$$\limsup_{k\to\infty}\|v_k\|_E < \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N}.$$

Thus by $(f1)$, we have

$$\frac{f(x,v_k)v_k}{|x|^\beta} \leqslant b_1\frac{v_k^N}{|x|^\beta} + b_2\frac{R(\alpha_0,v_k)v_k}{|x|^\beta}.$$

Note that

$$b_1\int\limits_{\mathbb{R}^N}\frac{v_k^N}{|x|^\beta} + b_2\int\limits_{\mathbb{R}^N}\frac{R(\alpha_0,v_k)v_k}{|x|^\beta} \to 0$$

since by Lemma 3.2 and by the compact embedding $E \hookrightarrow L^s(\mathbb{R}^N)$, $s \geqslant N$, $\int_{\mathbb{R}^N}\frac{R(\alpha_0,v_k)v_k}{|x|^\beta} \leqslant C(M,N)\|v_k\|_s \to 0$. Moreover, $\int_{|x|\geqslant 1}\frac{v_k^N}{|x|^\beta} \leqslant \|v_k\|_N^N \to 0$ again by the compact embedding $E \hookrightarrow L^N(\mathbb{R}^N)$ and $\int_{|x|\leqslant 1}\frac{v_k^N}{|x|^\beta} \leqslant C\|v_k\|_{Nr}^N \to 0$ by Holder's inequality and by the compact embedding $E \hookrightarrow L^s(\mathbb{R}^N)$, $s \geqslant N$. So we can conclude that

$$\int\limits_{\mathbb{R}^N}\frac{f(x,v_k)v_k}{|x|^\beta}\,dx \to 0$$

which thus $\lim_{k\to\infty}\|v_k\|_E^N = \lim_{k\to\infty}\int_{\mathbb{R}^N}\frac{f(x,v_k)v_k}{|x|^\beta}\,dx = 0$ and it's impossible. So we get the nontriviality of the solution.

7. Existence and multiplicity without the Ambrosetti–Rabinowitz condition

The main purpose of this section is to prove that all of the results of existence and multiplicity in Sections 5 and 6 hold even when the nonlinear term f does not satisfy the Ambrosetti–Rabinowitz condition. It is not difficult to see that there are many interesting examples of such f which do not satisfy the Ambrosetti–Rabinowitz condition, but satisfy our weaker conditions listed below. The existence of nontrivial solutions to a class of nonlinear equations of N-Laplace type [20] or polyharmonic operators [22] on bounded domains in $\mathbb{R}^N$ has been established when the nonlinear term satisfies the exponential growth but without satisfying the Ambrosetti–Rabinowitz condition.

In this section, instead of conditions $(f2)$ and $(f3)$, we assume that

$(f2')$ $H(x,t) \leqslant H(x,s)$ for all $0 < t < s$, $\forall x \in \mathbb{R}^N$ where $H(x,u) = uf(x,u) - NF(x,u)$.
$(f3')$ There exists $c > 0$ such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}^+$: $F(x,s) \leqslant c|s|^N + cf(x,s)$.
$(f4')$ $\lim_{u\to\infty}\frac{F(x,u)}{|u|^N} = \infty$ uniformly on $x \in \mathbb{R}^N$.

We should stress that $(f1)+(f3)$ will imply $(f3')$.

The key to establish the results in earlier sections is to prove that the Cerami sequence [12, 13] associated to the Lagrange–Euler functional is bounded. Once we will have proved this, the remaining should be the same as in previous sections. Therefore, we only include the proof of this essential ingredient in this section.

Lemma 7.1. *Let $\{u_k\}$ be an arbitrary Cerami sequence associated to the functional*

$$I(u)=\frac{1}{N}\|u\|^N-\int\limits_{\mathbb{R}^N}\frac{F(x,u)}{|x|^\beta}\,dx$$

such that

$$\frac{1}{N}\|u_k\|^N-\int\limits_{\mathbb{R}^N}\frac{F(x,u_k)}{|x|^\beta}\,dx\to C_M,$$

$$\big(1+\|u_k\|\big)\left|\int\limits_{\mathbb{R}^N}|\nabla u_k|^{N-1}\nabla u_k\nabla v\,dx+\int\limits_{\mathbb{R}^N}V(x)|u_k|^{N-1}u_kv\,dx-\int\limits_{\mathbb{R}^N}\frac{f(x,u_k)v}{|x|^\beta}\,dx\right|\leqslant\varepsilon_k\|v\|,$$
$$\varepsilon_k\to 0,$$

where $C_M\in(0,\frac{1}{N}((1-\frac{\beta}{N})\frac{\alpha_N}{\alpha_0})^{N-1})$. Then $\{u_k\}$ is bounded up to a subsequence.

Proof. Suppose that

$$\|u_k\|\to\infty. \tag{7.1}$$

Set

$$v_k=\frac{u_k}{\|u_k\|}$$

then $\|v_k\|=1$. We can then suppose that $v_k\rightharpoonup v$ in E (up to a subsequence). We may similarly show that $v_k^+\rightharpoonup v^+$ in E, where $w^+=\max\{w,0\}$. Thanks to the assumptions on the potential V, the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q\geqslant N$. So, we can assume that $\begin{cases} v_k^+(x)\to v^+(x) \text{ a.e. in } \mathbb{R}^N, \\ v_k^+\to v^+ \text{ in } L^q(\mathbb{R}^N),\ \forall q\geqslant N. \end{cases}$ We wish to show that $v^+=0$ a.e. $\mathbb{R}^N$. Indeed, if $S^+=\{x\in\mathbb{R}^N\colon v^+(x)>0\}$ has a positive measure, then in S^+, we have

$$\lim_{k\to\infty}u_k^+(x)=\lim_{k\to\infty}v_k^+(x)\|u_k\|=+\infty$$

and thus by $(f4')$:

$$\lim_{k\to\infty}\frac{F(x,u_k^+(x))}{|x|^\beta|u_k^+(x)|^N}=+\infty\quad\text{a.e. in } S^+.$$

This means that

$$\lim_{n\to\infty}\frac{F(x,u_k^+(x))}{|x|^\beta|u_k^+(x)|^N}\left|v_k^+(x)\right|^N=+\infty \quad \text{a.e. in } S^+ \tag{7.2}$$

and so

$$\int_{\mathbb{R}^N}\liminf_{k\to\infty}\frac{F(x,u_k^+(x))}{|x|^\beta|u_k^+(x)|^N}\left|v_k^+(x)\right|^N dx=+\infty. \tag{7.3}$$

However, since $\{u_k\}$ is the arbitrary Cerami sequence at level C_M, we see that

$$\|u_k\|^N=NC_M+N\int_{\mathbb{R}^N}\frac{F(x,u_k^+(x))}{|x|^\beta}dx+o(1)$$

which implies that

$$\int_{\mathbb{R}^N}\frac{F(x,u_k^+(x))}{|x|^\beta}dx\to+\infty$$

and then

$$\begin{aligned}
&\liminf_{k\to\infty}\int_{\mathbb{R}^N}\frac{F(x,u_k^+(x))}{|x|^\beta|u_k^+(x)|^N}\left|v_k^+(x)\right|^N dx\\
&\quad=\liminf_{k\to\infty}\int_{\mathbb{R}^N}\frac{F(x,u_k^+(x))}{|x|^\beta\|u_k\|^N}dx\\
&\quad=\liminf_{k\to\infty}\frac{\int_{\mathbb{R}^N}\frac{F(x,u_k^+(x))}{|x|^\beta}dx}{NC_M+N\int_{\mathbb{R}^N}\frac{F(x,u_k^+(x))}{|x|^\beta}dx+o(1)}\\
&\quad=\frac{1}{N}.
\end{aligned} \tag{7.4}$$

Now, note that $F(x,s)\geqslant 0$, by Fatou's lemma and (7.3) and (7.4), we get a contradiction. So $v\leqslant 0$ a.e. which means that $v_k^+\rightharpoonup 0$ in E.

Let $t_k\in[0,1]$ such that

$$I(t_ku_k)=\max_{t\in[0,1]}I(tu_k).$$

For any given $R\in(0,(\frac{(1-\frac{\beta}{N})\alpha_N}{\alpha_0})^{\frac{N-1}{N}})$, let $\varepsilon=\frac{(1-\frac{\beta}{N})\alpha_N}{R^{N/(N-1)}}-\alpha_0>0$, since f has critical growth $(f1)$ on $\mathbb{R}^N$, there exists $C=C(R)>0$ such that

$$F(x,s) \leqslant C|s|^N + \left| \frac{(1-\frac{\beta}{N})\alpha_N}{R^{N/(N-1)}} - \alpha_0 \right| R(\alpha_0+\varepsilon, s), \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R}. \tag{7.5}$$

Since $\|u_k\| \to \infty$, we have

$$I(t_k u_k) \geqslant I\left(\frac{R}{\|u_k\|} u_k\right) = I(Rv_k) \tag{7.6}$$

and by (7.5), $\|v_k\| = 1$ and the fact that $\int_{\mathbb{R}^N} \frac{F(x,v_k)}{|x|^\beta}\,dx = \int_{\mathbb{R}^N} \frac{F(x,v_k^+)}{|x|^\beta}\,dx$, we get

$$\begin{aligned} NI(Rv_k) &\geqslant R^N - NCR^N \int_{\mathbb{R}^N} \frac{|v_k^+|^N}{|x|^\beta}\,dx - N\left|\frac{(1-\frac{\beta}{N})\alpha_N}{R^{\frac{N}{N-1}}} - \alpha_0\right| \int_{\mathbb{R}^N} \frac{R(\alpha_0+\varepsilon, R|v_k^+|)}{|x|^\beta}\,dx \\ &\geqslant R^N - NCR^N \int_{\mathbb{R}^N} \frac{|v_k^+|^N}{|x|^\beta}\,dx - N\left|\frac{(1-\frac{\beta}{N})\alpha_N}{R^{\frac{N}{N-1}}} - \alpha_0\right| \int_{\mathbb{R}^N} \frac{R((\alpha_0+\varepsilon)R^{\frac{N}{N-1}}, |v_k^+|)}{|x|^\beta}\,dx \\ &\geqslant R^N - NCR^N \int_{\mathbb{R}^N} \frac{|v_k^+|^N}{|x|^\beta}\,dx - N\left|\frac{(1-\frac{\beta}{N})\alpha_N}{R^{\frac{N}{N-1}}} - \alpha_0\right| \int_{\mathbb{R}^N} \frac{R((1-\frac{\beta}{N})\alpha_N, |v_k|)}{|x|^\beta}\,dx. \end{aligned} \tag{7.7}$$

Since $v_k^+ \rightharpoonup 0$ in E and the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for all $p \geqslant N$, using the Holder inequality, we can show easily that $\int_{\mathbb{R}^N} \frac{|v_k^+(x)|^N}{|x|^\beta}\,dx \xrightarrow{k\to\infty} 0$. Also, by Lemma 1.1, $\int_{\mathbb{R}^N} \frac{R((1-\frac{\beta}{N})\alpha_N, |v_k(x)|)}{|x|^\beta}\,dx$ is bounded by a universal C.

Thus using (7.6) and letting $k \to \infty$ in (7.7), and then letting $R \to [(\frac{(1-\frac{\beta}{N})\alpha_N}{\alpha_0})^{\frac{N-1}{N}}]^-$, we get

$$\liminf_{k\to\infty} I(t_k u_k) \geqslant \frac{1}{N}\left(\left(1-\frac{\beta}{N}\right)\frac{\alpha_N}{\alpha_0}\right)^{N-1} > C_M. \tag{7.8}$$

Note that $I(0) = 0$ and $I(u_k) \to C_M$, we can suppose that $t_k \in (0,1)$. Thus since $DI(t_k u_k) t_k u_k = 0$,

$$t_k^N \|u_k\|^N = \int_{\mathbb{R}^N} \frac{f(x, t_k u_k) t_k u_k}{|x|^\beta}\,dx.$$

By $(f2')$:

$$\begin{aligned} NI(t_k u_k) &= t_k^N \|u_k\|^N - N \int_{\mathbb{R}^N} \frac{F(x, t_k u_k)}{|x|^\beta}\,dx \\ &= \int_{\mathbb{R}^N} \frac{[f(x, t_k u_k) t_k u_k - NF(x, t_k u_k)]}{|x|^\beta}\,dx \end{aligned}$$

$$\leqslant \int_{\mathbb{R}^N} \frac{[f(x,u_k)u_k - NF(x,u_k)]}{|x|^\beta}\,dx.$$

Moreover, we have

$$\int_{\mathbb{R}^N} \frac{[f(x,u_k)u_k - NF(x,u_k)]}{|x|^\beta}\,dx = \|u_k\|^N + NC_M - \|u_k\|^N + o(1)$$

$$= NC_M + o(1)$$

which is a contraction to (7.8). This proves that $\{u_k\}$ is bounded in E. $\square$

References

[1] S. Adachi, K. Tanaka, Trudinger type inequalities in R^N and their best exponents, Proc. Amer. Math. Soc. 128 (1999) 2051–2057.

[2] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian, Ann. Sc. Norm. Super. Pisa 17 (1990) 393–413.

[3] Adimurthi, K. Sandeep, A singular Moser–Trudinger embedding and its applications, NoDEA Nonlinear Differential Equations Appl. 13 (5–6) (2007) 585–603.

[4] Adimurthi, S.L. Yadava, Multiplicity results for semilinear elliptic equations in bounded domain of R^2 involving critical exponent, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 17 (1990) 481–504.

[5] Adimurthi, Yunyan Yang, An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications, Int. Math. Res. Not. IMRN 2010 (13) (2010) 2394–2426.

[6] Antonio Ambrosetti, Paul H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[7] F.V. Atkinson, L.A. Peletier, Ground states and Dirichlet problems for $-\Delta u = f(u)$ in R^2, Arch. Ration. Mech. Anal. 96 (1986) 147–165.

[8] F.V. Atkinson, L.A. Peletier, Elliptic equations with nearly critical growth, J. Differential Equations 70 (1987) 349–365.

[9] Haïm Brézis, Elliott Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (3) (1983) 486–490.

[10] Daomin Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in R^2, Comm. Partial Differential Equations 17 (3–4) (1992) 407–435.

[11] L. Carleson, S.Y.A. Chang, On the existence of an extremal function for an inequality of J. Moser, Bull. Sci. Math. 110 (1986) 113–127.

[12] Giovanna Cerami, An existence criterion for the critical points on unbounded manifolds, Istit. Lombardo Accad. Sci. Lett. Rend. A 112 (2) (1978) 332–336 (1979) (in Italian).

[13] Giovanna Cerami, On the existence of eigenvalues for a nonlinear boundary value problem, Ann. Mat. Pura Appl. (4) 124 (1980) 161–179 (in Italian).

[14] João Marcos do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. 2 (3–4) (1997) 301–315.

[15] João Marcos do Ó, Semilinear Dirichlet problems for the n-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range, Differential Integral Equations 9 (5) (1996) 967–979.

[16] João Marcos do Ó, Everaldo Medeiros, Uberlandio Severo, On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^n$, J. Differential Equations 246 (4) (2009) 1363–1386.

[17] Duong Minh Duc, Nguyen Thanh Vu, Nonuniformly elliptic equations of p-Laplacian type, Nonlinear Anal. 61 (8) (2005) 1483–1495.

[18] D.G. de Figueiredo, O.H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. Partial Differential Equations 3 (2) (1995) 139–153.

[19] Giovanni Gregori, Generalized solutions for a class of non-uniformly elliptic equations in divergence form, Comm. Partial Differential Equations 22 (3-4) (1997) 581–617.

[20] Nguyen Lam, Guozhen Lu, N-Laplacian equations in $\mathbb{R}^N$ with subcritical and critical growth without the Ambrosetti–Rabinowitz condition, arXiv:1012.5489.

[21] Nguyen Lam, Guozhen Lu, Existence and multiplicity of solutions to the nonuniformly subelliptic equations of Q-subLaplacian type with critical growth in $\mathbb{H}^n$, preprint, 2010.
[22] Nguyen Lam, Guozhen Lu, Existence of nontrivial solutions to polyharmonic equations with subcritical and critical exponential growth, Discrete Contin. Dyn. Syst., in press.
[23] Yuxiang Li, Bernhard Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^n$, Indiana Univ. Math. J. 57 (1) (2008) 451–480.
[24] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, Rev. Mat. Iberoam. 1 (1) (1985) 145–201.
[25] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1970/1971) 1077–1092.
[26] R. Panda, Nontrivial solution of a quasilinear elliptic equation with critical growth in R^n, Proc. Indian Acad. Sci. Math. Sci. 105 (4) (1995) 425–444.
[27] S.I. Pohozaev, On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR 165 (1965) 36–39 (in Russian).
[28] Bernhard Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$, J. Funct. Anal. 219 (2) (2005) 340–367.
[29] Mei-Chi Shaw, Eigenfunctions of the nonlinear equation $\Delta u + \nu f(x,u) = 0$ in $\mathbf{R}^2$, Pacific J. Math. 129 (1987) 349–356.
[30] A.S. Tersenov, On quasilinear non-uniformly elliptic equations in some non-convex domains, Comm. Partial Differential Equations 23 (11–12) (1998) 2165–2185.
[31] Hoang Quoc Toan, Quoc-Anh Ngo, Multiplicity of weak solutions for a class of nonuniformly elliptic equations of p-Laplacian type, Nonlinear Anal. 70 (4) (2009) 1536–1546.
[32] Neil S. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473–483.
[33] Yunyan Yang, Existence of positive solutions to quasilinear equations with exponential growth in the whole Euclidean space, arXiv:1106.4622v1.
[34] Huang Yisheng, Yuying Zhou, Multiple solutions for a class of nonlinear elliptic problems with a p-Laplacian type operator, Nonlinear Anal. 72 (7–8) (2010) 3388–3395.