We give short proofs of the following binomial coefficient identities:

\[
\sum_{k \geq 0} \binom{a}{l-k} \binom{b}{m-k} \frac{1}{k} = \binom{a+m}{l} \binom{b+l}{m} \quad (1)
\]

\[
\sum_{k \geq 0} \frac{(-1)^k}{k} \frac{\binom{n}{k} \binom{m}{l-n+k} \binom{l}{l-m+k}}{\binom{l+m-n}{2r}} = \begin{cases} 
0 & \text{if } l+m-n \text{ is odd} \\
(-1)^{m-r} \binom{n+r}{n} \binom{n}{l-r} & \text{if } l+m-n = 2r.
\end{cases} \quad (2)
\]

Here \(l, m, n, \) and \(r\) are nonnegative integers, and \(a\) and \(b\) are arbitrary numbers. Formula (1) is a form of Saalschütz’s theorem [1, p. 9], and formula (2) is a terminating form of Dixon’s theorem [1, p. 13].

We shall work with Laurent series \(\sum_{m,n} a_{m,n} x^m y^n\) in which \(a_{m,n} \neq 0\) for only finitely pairs \((m, n)\) with \(m < 0\) or \(n < 0\). We write \(\text{CT} f(x, y)\) for the constant term of \(f(x, y)\).

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Our derivation of these identities is related to the method of Cartier and Foata [2]; however, instead of MacMahon’s master theorem, we use the following almost trivial lemma, which is essentially a special case of Jacobi’s multivariable Lagrange inversion formula [4]:

**Lemma.** For any Laurent series \( f(x, y) \),

\[
\text{CT} \left( \frac{x}{1+y}, \frac{y}{1+x} \right) = \frac{1}{1-xy} \text{CT} f(x, y).
\] (3)

**Proof.** By linearity, it suffices to prove the case \( f(x, y) = x^m y^n \):

\[
\text{CT} \left( \frac{x^l}{(1+y)^l}, \frac{y^m}{(1+x)^m} \right) = \begin{cases} 1 & \text{if } l = m \leq 0 \\ 0 & \text{otherwise}. \end{cases}
\] (4)

Since \((1+y)^{-l}(1+x)^{-m}\) has no negative powers of \(x\) or \(y\), (4) holds if \(l > 0\) or \(m > 0\). Now suppose \(l = -r\) and \(m = -s\), with \(r, s \geq 0\). Then the left side of (4) is

\[
\text{CT} \frac{(1+x)^r}{x} \frac{(1+y)^s}{y^s} = \left( \begin{array}{c} s \\ r \end{array} \right) = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{if } r \neq s, \end{cases}
\]

and the lemma is proved.

Now we apply the lemma to

\[
f(x, y) = \frac{(1+x)^a (1+y)^b (x-y)^n}{x^l y^m (1-xy)^{a+b+n}}.
\]

We obtain

\[
\text{CT} \frac{(1+x)^a+m (1+y)^{b+l} (x-y)^n}{x^l y^m} = \text{CT} \frac{(1+x)^a (1+y)^b (x-y)^n}{x^l y^m (1-xy)^{a+b+n+1}}.
\] (5)

If we take \(n = 0\) in (5) we obtain (1). Note that in this case (5) is equivalent to the formula

\[
\sum_{l, m=0}^{\infty} \left( \begin{array}{c} a+m \\ l \end{array} \right) \left( \begin{array}{c} b+l \\ m \end{array} \right) x^l y^m = \frac{(1+x)^a (1+y)^b}{(1-xy)^{a+b+1}}.
\] (6)

Some formulas closely related to (6) were recently studied by Evans et al. [3].

Now take \(a = b = 0\) in (5). The left side is easily seen to be the left side of (2).

To evaluate the right side, we may replace \(x\) by \(xz\) and \(y\) by \(yz\). Then the constant term (now in \(x, y, \) and \(z\)) will be unchanged, and is

\[
\text{CT} \frac{(x-y)^n}{x^l y^m z^{l+m-n} (1-xy)^{n+1}}.
\] (7)
If \( l + m - n \) is odd, this is clearly zero. If \( l + m - n = 2r \), then (7) is
\[
\text{CT} \left( \binom{n + r}{n} \right) \frac{(x - y)^n}{x^{l-r} y^{m-r}} = (-1)^{n-r} \binom{n + r}{n} \binom{n}{l-r},
\]
and this proves (2).

We can put (2) in a more symmetrical form when \( l + m - n \) is even: set \( m = q + r, l = r + p, \) and \( n = p + q \) (so that \( l + m - n = 2r \) as before). Then multiplying (2) by \((-1)^q\) and changing \( k \) to \( k + q \), we obtain
\[
\sum_k (-1)^k \binom{p + q}{k + q} \binom{q + r}{k + r} \binom{r + p}{k + p} = \frac{(p + q + r)!}{p! q! r!}. \tag{8}
\]

Some interesting generalizations of (5) can be obtained from the lemma. Take
\[
f(x, y) = (1 + x)^a (1 + y)^b \left[ 1 + \alpha x - (1 - \alpha) xy \right]^c \left[ \alpha + y + (1 - \alpha) xy \right]^d
\times \left[ x - \alpha y + (1 - \alpha) xy \right]^e (1 - xy)^{-a-b-c-d-e-n} x^{-l} y^{-m}. \tag{9}
\]

Then applying the lemma, we have
\[
\text{CT} (1 + x)^{a+m+e} (1 + y)^{b+l+e} \left( 1 + \alpha x \right)^c \left( \alpha + y \right)^d (x - \alpha y)^e (1 + x + y)^{-e} x^{-l} y^{-m}
= \text{CT} f(x, y)(1 - xy)^{-1}. \tag{10}
\]
(Here \( \text{CT} \) applies only to \( x \) and \( y \), not \( a \).)

As an example, for \( a = b = c = d = e = 0 \) we obtain
\[
\sum_{k \geq 0} (-\alpha)^k \binom{n}{k} \binom{m}{l-n+k} \binom{l}{l-m+k}
= (1 - \alpha)^{m+n-l} \sum_{j \leq (m+n-l)/2} \left[ -\frac{\alpha}{(1-\alpha)^2} \right]^j
\times \frac{(l+j)!}{j! (l-n+j)! (l-m+j)! (m+n-l-2j)!}, \tag{11}
\]
which reduces to (2) for \( \alpha = 1 \). This formula is equivalent to Whipple’s quadratic transformation [1, p. 97].

The lemma can easily be generalized to more variables. The 3-variable formula is
\[
\text{CT} f\left( \frac{x}{1+y}, \frac{y}{1+z}, \frac{z}{1+x} \right) = \text{CT} \frac{1}{1-xyz} f(x, y, z), \tag{12}
\]
and the general case follows the same pattern of variables arranged
cyclically. The proof is essentially the same as in the 2-variable case. As an application of (12), take

\[ f(x, y, z) = \frac{(1 + x + xy)^a (1 + y + yz)^b (1 + z + zx)^c}{x^l y^m z^n (1 - xyz)^{a+b+c}}. \]

We obtain a formula analogous to (6),

\[ \sum_{l, m, n=0}^{\infty} \binom{a+n}{l} \binom{b+l}{m} \binom{c+m}{n} x^l y^m z^n = \frac{(1 + x + xy)^a (1 + y + yz)^b (1 + z + zx)^c}{(1 - xyz)^{a+b+c+1}}. \]  

(13)

A similar formula holds for any number of variables. These formulas were found by D. Sturtevant [5], using a combinatorial argument.

REFERENCES

5. D. Sturtevant, personal communication.